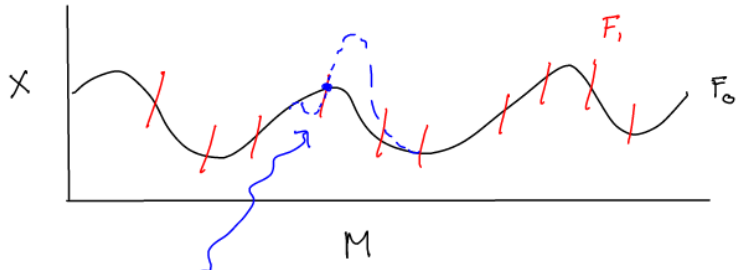
Holonomic approximation

Suppose we have a (non-holonomic section)

$$F: M \to X^{(\bullet)} \quad \text{into the } \bullet\text{-jet of } \begin{array}{c} X \\ \downarrow \\ M \end{array}$$

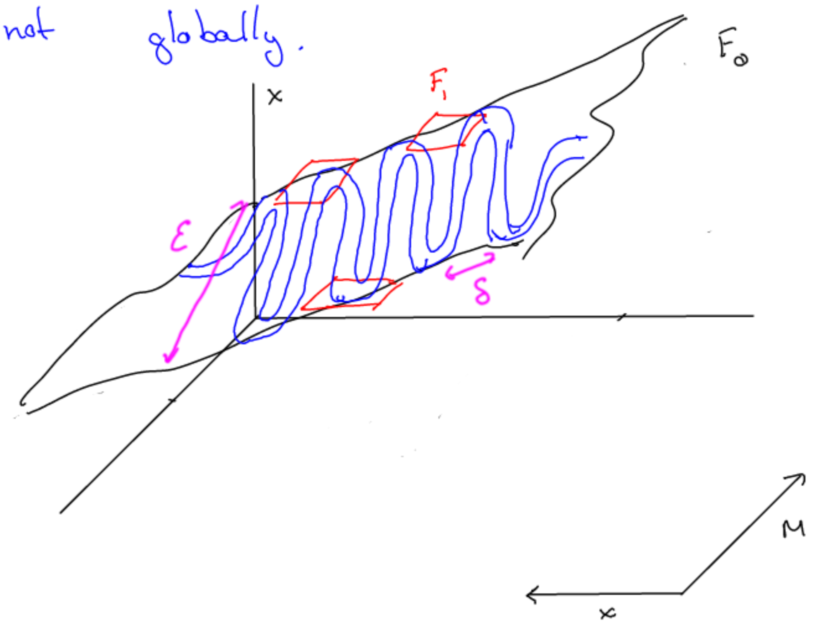
is there $s: M \to X$ st $\| j^{\bullet} s - F \|_{C^0} < \varepsilon$?

The answer is no:



F_1 ("the formal derivative of F_0") wants F_0 to be everywhere increasing with large slope, but F_0 is more or less constant.

At a point we can change F_0 to $\tilde{F}_0$ with $\partial \tilde{F}_0 = F_1$ but not globally.



F_1 wants F_0 to be flat but F_0 increases in the x axis

However in the neighbourhood of a 1-simplex, we can get $\tilde{F}_0$ with $\partial \tilde{F}_0 \simeq F_1$. This is achieved if $\varepsilon >> \delta$. The key point is that if δ is very small and ε is small but fixed, the change of F_0 in each δ-interval is small and can be performed in the ε interval "slowly" (with arbitrarily small derivative as $\delta \to 0$).

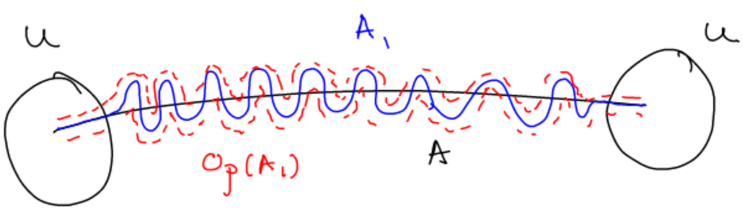
This is what we do in mountain trails: we make $\varepsilon >> \delta$ so that the slope differential becomes very small.

Now a precise statement. Let M be a manifold, K a compact mnfld (the parameter space), $\mathcal{U}, A \subset M \times K$ with $\mathcal{U}$ a closed set and A a CW-complex of positive codim.

<u>Thm</u>: Let $F_K : M \to X^{(\bullet)}$, $K \in K$. Assume $F_K(p) = j^{(\bullet)} f_K(p)$ if $(p,K) \in Op(\mathcal{U})$ for some family $f_K : M \to X$.

Then there is an isotopy $A_t \subset M \times K$, $t \in [0,1]$, fixed on $\mathcal{U}$ and sections $S_K : M \to X$ st:

- $\| j^{(\bullet)} S_K - F_K \|_{C^0} < \varepsilon$ if $(p,K) \in Op(A_1)$.
- $S_K(p) = f_K(p)$ if (p,K) $Op(\mathcal{U})$.
- the isotopy A_t is C^0-small but its C^1-norm is inversely proportional to ε. A_t is fixed in $Op(\mathcal{U})$.



This process of isotopies is called wiggling.

<u>Claim</u>: it is sufficient to prove it in the euclidean setting:

$M = \mathbb{D}^m$, $K = \mathbb{D}^k$, $A = \mathbb{D}^a \times \mathbb{D}^b \subset M \times K$ (where $a \neq 0$ iff $b = k$)

$Op(\mathcal{U}) = Op(\partial \mathbb{D}^a \times \mathbb{D}^k) \cup Op(\mathbb{D}^m \times \partial \mathbb{D}^b)$

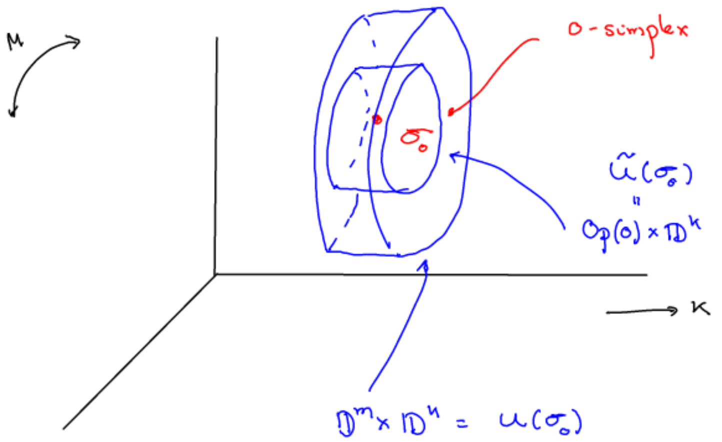
$X = \begin{array}{c} \mathbb{D}^m \times \mathbb{R}^q \\ \downarrow \\ \mathbb{D}^m \end{array}$, $X^{(\bullet)} = J^{\bullet}(\mathbb{D}^m, \mathbb{R}^q)$

Let us assume that we have proven it in this case and see how this implies the theorem.

I. Reducing to the Euclidean setting

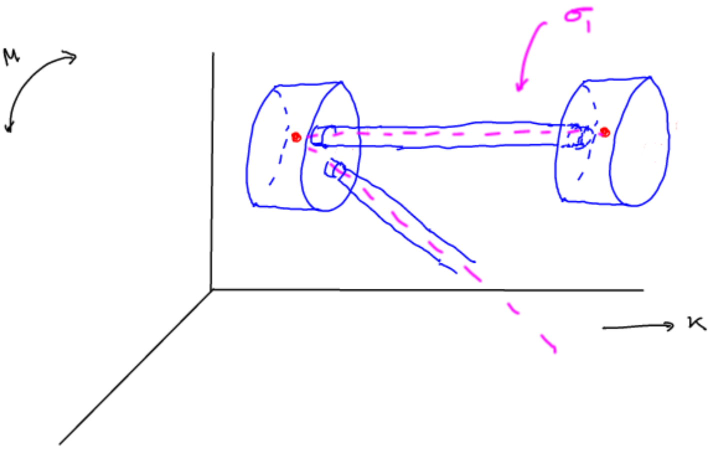
The idea: triangulate $A \subset M \times K$. Simplices have a euclidean neighbourhood as described. Solve on each of them inductively in the dimension of the simplices.

- Suppose we have triangulated A and σ_0 is a 0-simplex. There is a nbhd $\mathcal{U}(\sigma_0) \cong \mathbb{D}^m \times \mathbb{D}^k \subset M \times K$. Apply the euclidean case in $\mathcal{U}(\sigma_0)$ with $A = \{0\} \times \{0\} = \sigma_0$, $\mathcal{U} = \phi$. This constructs a section s_K in $Op(\sigma_0)$ with $|j^0 s_K - F_K| < \varepsilon$.



[Note: in this case we can just take s_K = polynomial whose Taylor expansion is $F_K(0)$.]

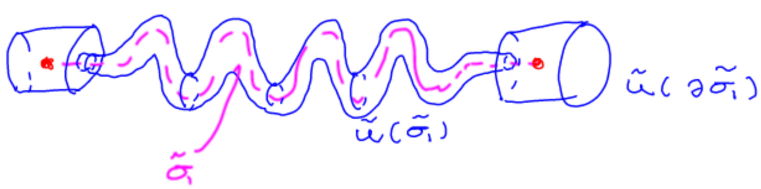
- Suppose now σ_1 is a 1-simplex and sections have been constructed close to $\partial\sigma_1$.



In the picture the 1-simplices are transverse to the fibration $\begin{matrix} M \times K \\ \downarrow \\ K \end{matrix}$.

It allows us to assume that there is an open set $\mathcal{U}(\sigma_1) \cong \mathbb{D}^m \times \mathbb{D}^k$ where $\mathcal{U}(\sigma_1) \cap \sigma_1 \cong \{0\} \times \mathbb{D}^1 \equiv A$ and we can take $\mathcal{U} = \mathbb{D}^m \times \partial \mathbb{D}^1$.

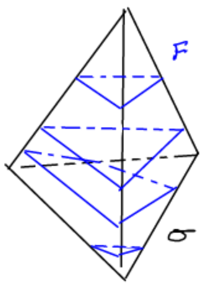
$\rightarrow$ This yields a wiggling $\tilde{\sigma}_1$ of σ_1 (rel. to $\partial\sigma_1$) and a section over $\tilde{\mathcal{U}}(\tilde{\sigma}_1) \cup \tilde{\mathcal{U}}(\partial\tilde{\sigma}_1)$.



Let us formalise and generalise these two cases:

- $M \times K$ is foliated by the fibres of $\begin{array}{c} M \times K \\ \downarrow \\ K \end{array}$.

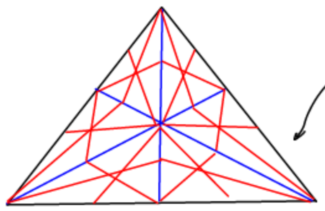
<u>Claim:</u> we can triangulate A and then perturb this triangulation T so that it is transverse to the foliation. In particular, $T|_\sigma$ with σ a simplex is a linear fol.:



Transversality was important so that neighbourhoods $U(\sigma)$ had the desired form.

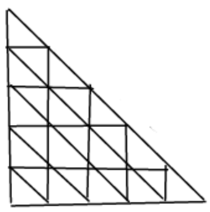
<u>Sketch of proof:</u>

Triangulate A and then subdivide in a crystalline fashion. The point is: barycentric subdivision makes simplices smaller but not in a uniform manner:



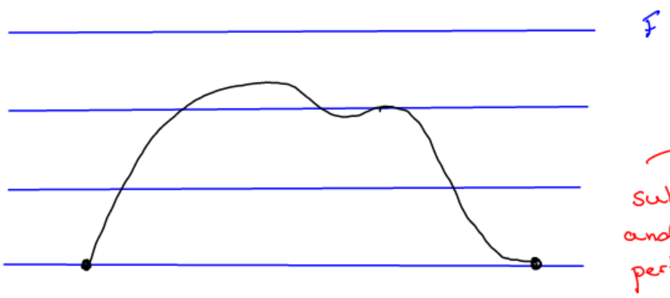
it produces "thin" simplices, so if the original one was very "deformed" within M, the subdivisions remain so.

Instead:

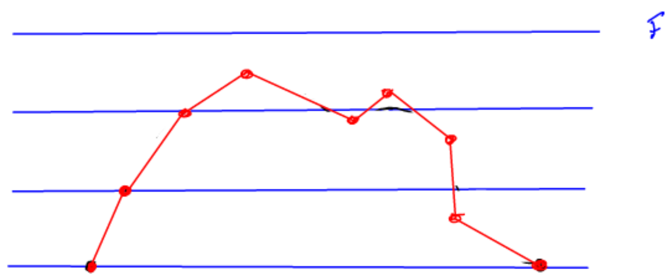


← the simplices get uniformly smaller.

Then, we inductively perturb the zero simplices to achieve transversality:



subdivide and perturb



This is called Thurston's jiggling.

• Once we have triangulated, we want to assign open sets $\mathcal{U}(\sigma)$ to each σ. The order in which they are constructed is as follows:

$i+1$ step

→ Construct $\mathcal{U}(\sigma)$ for all i-simplices.

→ Apply euclidean case to construct hol. section over $\tilde{\mathcal{U}}(\tilde{\sigma}) \subset \mathcal{U}(\sigma)$, corresponding to the wiggling $\tilde{\sigma}$ of σ.

→ the isotopies $\sigma \to \tilde{\sigma}$ induce an isotopy of A.

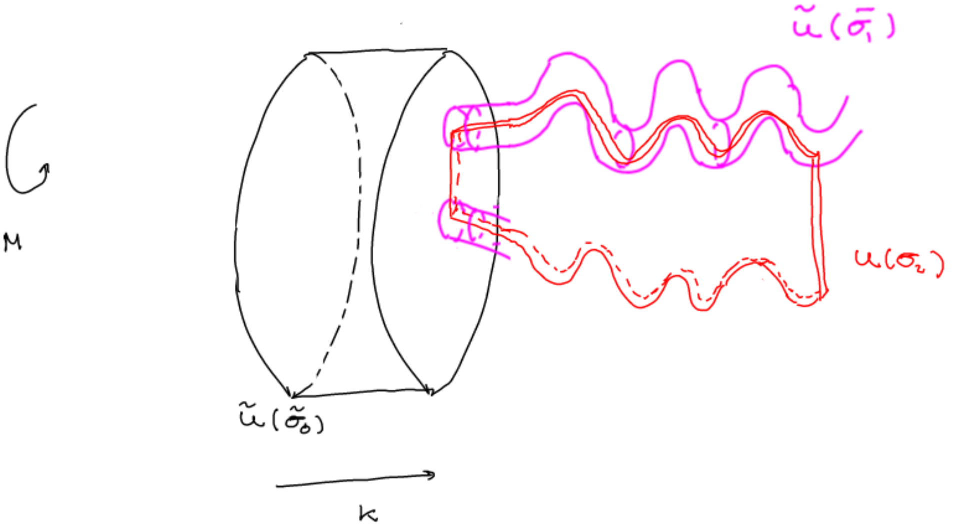
Hence, the $\mathcal{U}(\sigma)$ should satisfy:

→ $\mathcal{U}(\sigma)$ only intersects $\tilde{\mathcal{U}}(\tilde{\tau})$ if $\sigma \supset \tilde{\tau}$.

→ $\mathcal{U}(\sigma) \cup \bigsqcup_{\tilde{\tau} \subset \sigma} \tilde{\mathcal{U}}(\tilde{\tau})$ covers σ.

→ $\mathcal{U}(\sigma)$ intersects $\tilde{\mathcal{U}}(\tilde{\tau})$ in their vertical boundary $\mathbb{D}^m \times \partial\mathbb{D}^k$.

$\Rightarrow (\bigsqcup \tilde{\mathcal{U}}(\tilde{\tau})) \cap \mathcal{U}(\sigma) \subset \mathcal{U}(\sigma)$ is of the form $\mathcal{O}p(\mathbb{D}^m \times \partial\mathbb{D}^b \cup \partial\mathbb{D}^a \times \mathbb{D}^k)$.



These properties hold if $\tilde{\mathcal{U}}(\tilde{\tau})$ is thin in the k-directions, due to transversality.

□

<u>One last example:</u>

$M = \mathbb{D}^2$, $k = \mathbb{D}^1$. Then

$$\mathcal{U}(\sigma_0) = \begin{cases} A = \{0\} \times \{0\} \subset \mathbb{D}^2 \times \mathbb{D}^1 \\ \mathcal{U} = \emptyset \end{cases}$$

$$\mathcal{U}(\sigma_1) = \begin{cases} A = \{0\} \times \mathbb{D}^1 \\ \mathcal{U} = \mathbb{D}^2 \times \partial\mathbb{D}^1 \end{cases}$$

$$\mathcal{U}(\sigma_2) = \begin{cases} A = \mathbb{D}^1 \times \mathbb{D}^1 \\ \mathcal{U} = \partial\mathbb{D}^1 \times \mathbb{D}^1 \cup \mathbb{D}^2 \times \partial\mathbb{D}^1 \end{cases}$$

II. The euclidean case

Now we have to prove the theorem in the special case from above. I.e:

Thm Let $F_k : \mathbb{D}^m \to \mathcal{J}^1(\mathbb{R}^m, \mathbb{R}^q)$ $k \in \mathbb{D}^k$

with $F(p) = j^1 f_k(p)$ if $p \in Op(\partial \mathbb{D}^a)$ or $k \in Op(\partial \mathbb{D}^k)$

Then there is $s_k : \mathbb{D}^m \to \mathbb{R}^q$, $h_{t,k} : \mathbb{D}^m \to \mathbb{D}^m$ isotopy.

- $\| j^1 s_k - F_k \| < \varepsilon$ if $p \in Op(h_1(\mathbb{D}^a))$.
- $s_k = f_k$ if $p \in Op(\partial \mathbb{D}^a)$ or $k \in Op(\partial \mathbb{D}^k)$
- $h_{t,k}$ is C^0-small and fixed over $Op(\partial \mathbb{D}^a \times \mathbb{D}^k \cup \mathbb{D}^m \times \partial \mathbb{D}^b)$

The key ingredient is:

Inductive lemma: (here $I = [-1,1]$)

Let $F_k : I^m \to \mathcal{J}^1(I^m, I^q)$ which is holonomic over $Op(\partial I^a)$ or if $k \in Op(\partial I^b)$.

Let $F_{y,k} : Op(I^{a-l} \times \{y\}) \to \mathcal{J}^1(I^m, I^q)$, $y \in I^l$ a family of holonomic sections with $F_{y,k}(p) = F_k(p)$ if $p \in Op(\partial I^a)$ or $k \in Op(\partial I^b)$.

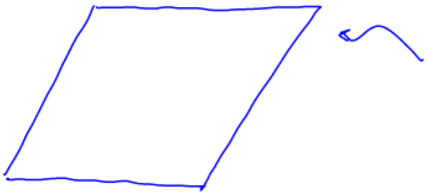
Then, there is a family of isotopies $h_k : I^a \hookrightarrow I^m$ graphical over Id_{I^a}, with $h_k(p) = p$ if $p \in Op(\partial I^a)$ or $k \in Op(\partial I^b)$ and a family of hol. sec.

$$G_{(y_2 \dots y_l),k} : Op(I^{a-l+1} \times \{(y_2 \dots y_l)\}) \to \mathcal{J}^1(I^m, I^q)$$

with $G_{\tilde{y},k}(p) = F_k(p)$ if $p \in Op(\partial I^a)$ or $k \in Op(\partial I^b)$

and $\| G_{\tilde{y},k} - F_k \|$ C^0-small.

The lemma says:



$A \ni I^2$

We want a hol. sec. over A.

First we observe:

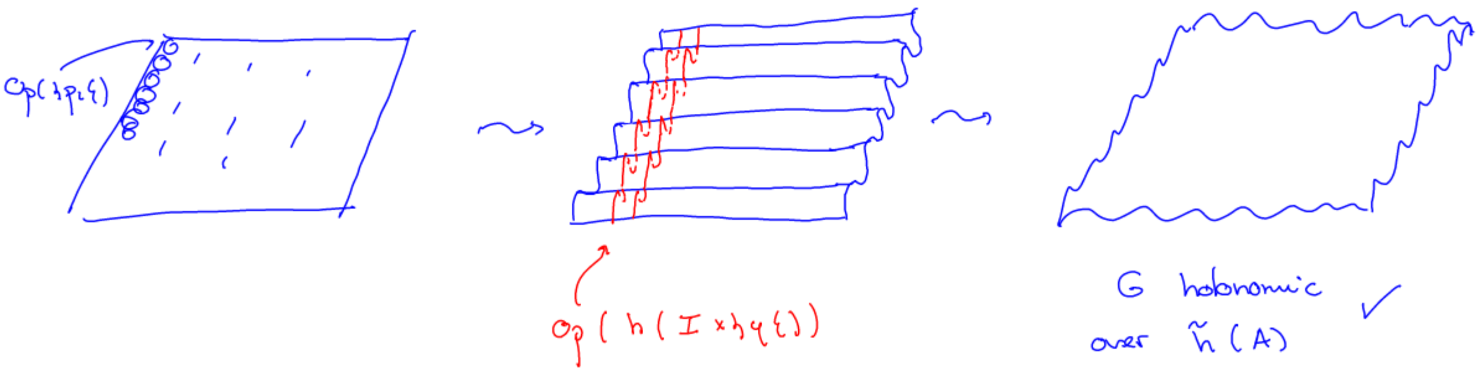
- over each point $p \in I^2$ we can find a hol. section $F_p : Op(p) \to J^1(\mathbb{R}^m, \mathbb{R}^q)$ with $F_p(p) = F(p)$. This is true because $F(p)$ are the Taylor coefficients of some polynomial F_p.
- The inductive lemma glues these solutions to obtain hol. sections $F_y : Op(I \times \{y\}) \to J^1(\ )$, i.e., from the solutions around points we obtain solutions along each segment $I \times \{y\}$.

  Observe that this is not quite true, since the construction of hol. secs. requires wiggling, so we obtain rather:

  $F_y : Op(h(I \times \{y\})) \to J^1(\ )$, with h an isotopy of I^2.

- The F_y defined in nbhds of segments are glued by wiggling to yield G holonomic over the whole of $\tilde{h}(A)$ (a wiggled version of the original A).

- Graphically:



Now we can prove the inductive lemma. Let us focus on the case $l=1$. The proof is essentially the same since these coordinates : $y = (y_1, y_2 \dots y_l)$ behave as parameters.

Pf / Let us make a few remarks:

Fact 1: $Op(I^{a-1} \times \{y\}) \supset I^{a-l} \times (y-\varepsilon, y+\varepsilon) \times (-\varepsilon, \varepsilon)^{m-a}$

ε not depending on y. That is, we have neighbourhoods of uniform size, due to compactness.

Fact 2: $|F_{y,\kappa}(p) - F_{y+\delta,\kappa}(p)| \to 0 \quad \forall \ \delta \in (-\varepsilon, \varepsilon)^l$,

uniformly on y, σ, κ, p. I.e. if δ is very small, the hol. solutions $F_{y,\kappa}$ and $F_{y+\delta,\kappa}$ are very close.

Naively: we would want to set $G_\kappa(z,y) = F_{y,\kappa}(z,y)$

However this does not work. Let g_κ = zero order part of G_κ

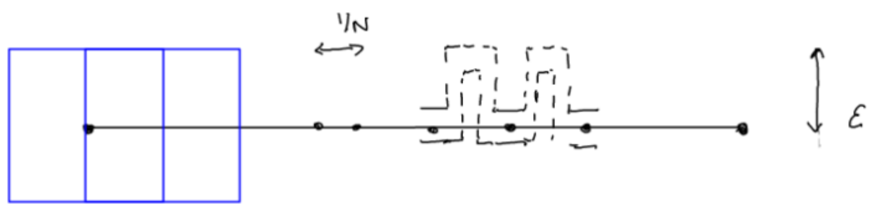
$f_{y,\kappa}$ = zero order part of $F_{y,\kappa}$

$$\frac{\partial g_\kappa}{\partial y}(z,y) = \underbrace{\frac{\partial f_{y,\kappa}}{\partial y}(y,z)}_{\text{this is uncontrolled}} + \underbrace{\frac{\partial f_{y,\kappa}}{\partial y}(y,z)}_{\text{this is the } 1^{st} \text{ order part of } F_y.}$$

(parameter) (domain)

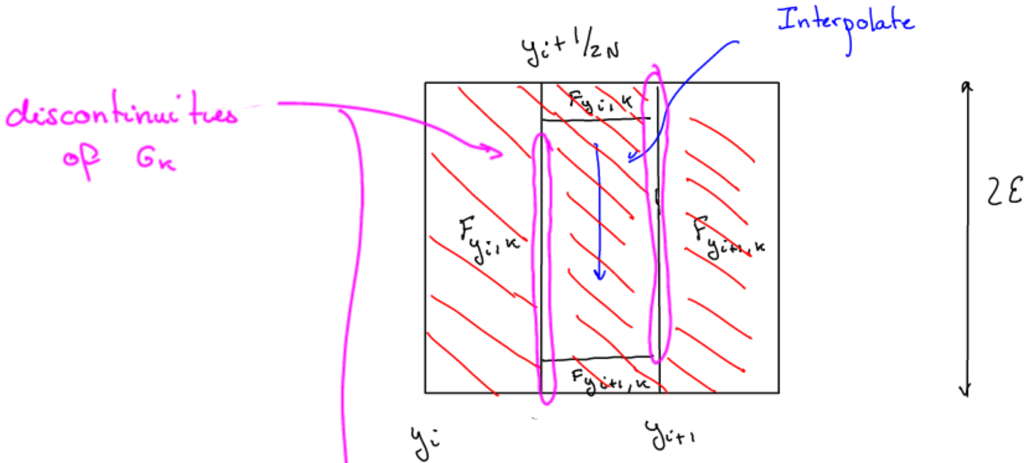
this part prevents $\frac{\partial g_\kappa}{\partial y}$ from being the 1^{st} order part of G_κ (which by definition is just the 1^{st} order part of $F_{y,\kappa}$, varying y.)

The idea is to use the extra directions to make the interpolation slowly, so $\frac{\partial f_{y,\kappa}}{\partial y \text{ (parameter)}}$ is small. Namely:

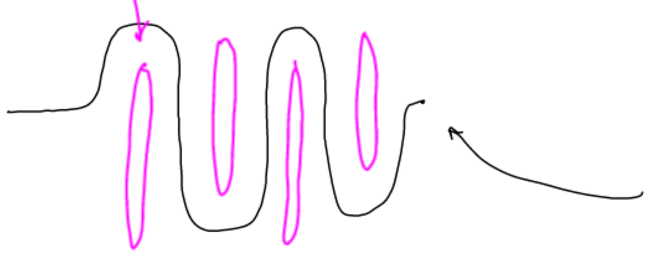


$I^a = I$

- We space points y_i in $\overset{\delta}{\overset{\shortparallel}{1/N}}$ intervals.
- Over $I^{a-1} \times (y_1, y_1 + 1/2N) \times (-\varepsilon, \varepsilon)^{m-a}$ we take $G_\kappa = F_{y_i,\kappa}$, so our solution there is holonomic.
- In $\times \underbrace{Op(\varepsilon)}$ call this variable z_m we also set $G_\kappa = F_{y_i,\kappa}$, so it glues with the previous step.
- In $I^{a-1} \times (y_i + 1/2N, y_{i+1}) \times (-\varepsilon, \varepsilon)^{m-n-1} \times \{-\varepsilon, \varepsilon\}$, as we move z_m from ε to $-\varepsilon$ we set $G_\kappa(y \ldots z_m) = \frac{(\varepsilon - z_m)}{2\varepsilon} F_{y_{i+1},\kappa} + \frac{(z_m + \varepsilon)}{2\varepsilon} F_{y_i,\kappa}$
- Since $\delta = 1/N \ll \varepsilon$ this linear interpolation contributes very little and the derivatives of the zeroth order part g_κ of G_κ are very close to the higher order part of G_κ. The desired hol. sec is $j^0 g_\kappa \simeq G_\kappa$.



Note that is <u>not</u> <u>defined everywhere</u>.



wiggling allows us to avoid the singularities of G_κ.

$\square$

Punchline: one would want to interpolate linearly between Taylor expansions. This works using the extra dimensions to make the derivatives of the interpolation small.

The question now is ... what if we do not have extra dimensions? $\longrightarrow$ we wrinkle.

III. Main application: h-principle for open manifolds

Def: A differential relation $R \subset J^{\bullet}(X)$ is said to be Diff-invariant if it is preserved by the natural action of $\mathrm{Diff}(M)$ on $\begin{matrix} X \\ \downarrow \\ M \end{matrix}$

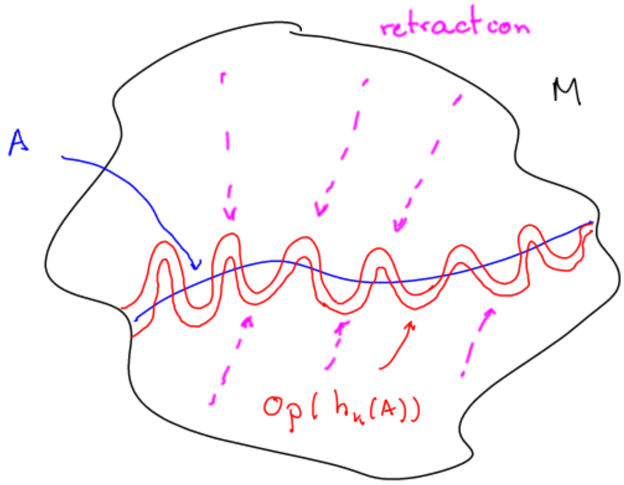
so X should be something like TM, T^*M, $\Lambda^k M$, $M \times Q$ ...

This says that R is defined intrinsically and not chart by chart.

Thm (Gromov): Let R be an open, Diff-invariant relation. Let M be open. Then, the complete h-principle holds for R.

Pf. Since M is open, it retracts onto a core A.
Consider $\varphi : (\mathbb{D}^k, \partial\mathbb{D}^k) \longrightarrow (\{\text{Formal solutions}\}, \{\text{solutions}\})$.
Apply hol. approx. to modify it to $\tilde{\varphi}$ with $\tilde{\varphi}(k)$ holonomic over $h_k(A)$ (a wiggled version of A), which is still a core.
R being open means that $\tilde{\varphi}$ is a solution. Find now a retraction $\phi_k : M \longrightarrow \mathcal{O}p(h_k(A))$ with $\phi_k = \mathrm{Id}$ if $k \in \partial\mathbb{D}^k$.

Then Diff-invariance shows that $\tilde{\varphi}\circ\phi_k : \mathbb{D}^k \longrightarrow \{\text{solutions}\}$ $\square$



Examples: contact, symplectic, immersions, submersions.

Obs: For immersions $M^m \to Q^q$ $m<q$, we think of M as the core of its normal bundle $\nu(M)$*, which is open. The h-principle then applies to M (even if M is closed) because immersions restrict to immersions. $\square$

* We call this the microextension trick

Obs. Let M^{2m+2}. An even-contact str. is a distribution ξ^{2m+1} s.t $\forall$ α with $\ker(\alpha)=\xi$ we have $\alpha\wedge d\alpha^m \neq 0$.

A pair (ξ, ω) with $\omega^n|_\xi \neq 0$ is called a formal even-contact str.

Now, given a formal even-contact str. (ξ,ω) on M, we can give a formal contact str. $(\xi\oplus\mathbb{R}, \omega+\beta\wedge dt)$ on $M\times\mathbb{R}$ with $\beta\wedge\omega^n|_\xi > 0$. Apply holonomic approx. to obtain a contact str in $h(M\times\{0\})$.

It can be assumed that the wiggling is transverse to $\ker(\omega|_\xi)$ by triangulating $M\times\{0\}$ transversely to $\ker(\omega|_\xi)$. This ensures that the resulting contact structure restricts to $h(M\times\{0\})$ as an even contact str. Similarly: the complete h-principle holds for even contact (even in closed manifolds). $\square$