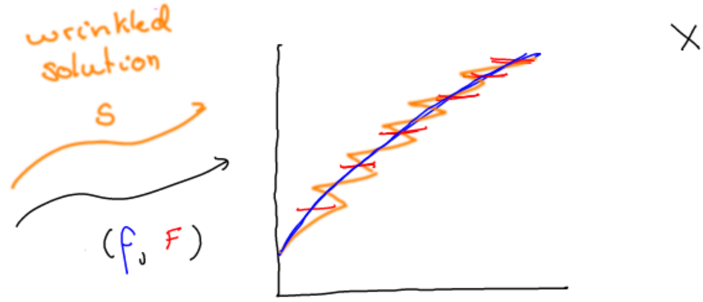
The wrinkling philosophy

We know we want to approximate some non-holonomic section $F: M \to X^{(\cdot)}$ by a holonomic one $j^{\cdot}s$.
If M is closed this is not possible (and we showed that if M is open then we can).

Consider:

• Ex.:



$M \simeq [0,1)$

That is, F wants f to be constant, but f is increasing. We cannot find s with $j^1 s \simeq (f, F)$, but we can find a "wrinkled solution". This solution is singular but has almost zero derivative (close to F) and is close to f.

Philosophy: since we cannot wiggle because we do not have enough dimensions, we "wiggle" the manifold upon itself.
This is called wrinkling.

Metatheorem:

$$\{\text{Solutions}\} \hookrightarrow \{\text{Formal solutions}\}$$

$$\{\text{Solutions}\} \downarrow \{\text{Wrinkled solutions}\} \dashrightarrow_{\simeq} \{\text{Formal solutions}\}$$

$$\{\text{Wrinkled solutions}\} \downarrow^{\simeq}_{\text{desing?}} \{\text{"Flexible" solutions}\} \dashrightarrow_{\simeq} \{\text{Formal solutions}\}$$

← Often this is not true because extra data must be fixed (o.b disc, loose chart ...)
This reflects the fact that one wrinkle (which desingularises to the

o.t disc, loose chart..) has to be used to swallow the others.

This metatheorem holds in a variety of cases. The most natural one is in the theory of submersions.

I. Submersions.

Let M^m, Q^q, $m \geq q$ manifolds. Define

$$\mathrm{Subm}(M,Q) = \{ f : M \to Q \mid Tf : TM \to TQ \text{ has rank } q \}$$

$$\mathrm{FSubm}(M,Q) = \left\{ (f,F) \;\middle|\; \begin{array}{l} f: M \to Q \\ \quad\;\, \uparrow \quad\; \uparrow \\ F: TM \to TQ \text{ of rank } q \end{array} \right\}$$

<u>Claim:</u> $\mathrm{Subm}(M,Q) \to \mathrm{FSubm}(M,Q)$

is not a w.h.e. in general.

For instance, if Q is parallelizable, any manifold M with q linearly independent sections formally submerses onto Q. Take $Q = \mathbb{R}^q$ and M closed, then an actual submersion is not possible (because the radius is maximal at some point of the image of the map).

<u>Aim:</u> allow submersions with the <u>simplest</u> singularities so that the inclusion into $\mathrm{FSubm}(M,Q)$ is a w.h.e.

<u>Note:</u> a singular map <u>does not</u> immediately define a formal submersion.

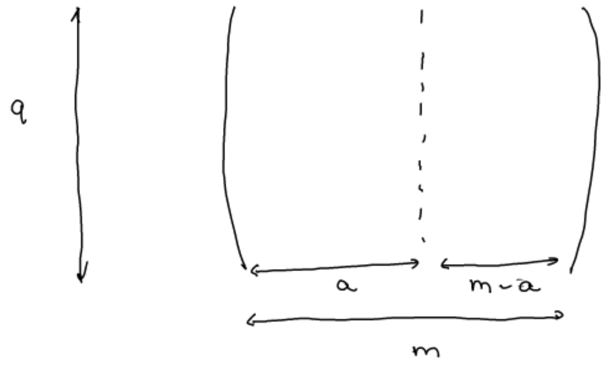
I.a. Singularities

Many singularities are actually stable: they cannot be removed by small perturbations. Let us discuss the hierarchy of singularities to see what simplest means.

We consider maps $M^m \to Q^q$ where m might be smaller than q.

Obs: The space of matrices $M_{m\times q}$ is naturally stratified by rank. The top (codimension zero) stratum are those matrices of maximal rank (q, if $m \geq q$, m otherwise).

The matrices $M^a_{m\times q}$ of rank a conform a stratum of dimension $a(m+q-a)$.



An open set of $M^a_{m\times q}$ are those matrices whose first a columns are l.i.

This open subset has dim

$$\underbrace{a \cdot q}_{\text{coefficients of the first } a \text{ columns}} + \underbrace{(m-a).\,a}_{\text{each of the last } (m-a) \text{ columns needs } a \text{ coefficients to be expressed as a linear combination of the first ones.}} = a.(m+q-a)$$

Equivalently, the space $\Sigma^a_{m\times n}$ of matrices of corank $a \leq q \leq m$ has codimension $m.q - (q-a)(m+q-(q-a)) = m.q - (q-a)(m+a) = a(m-q+a)$ $\boxtimes$

This stratification of $M_{m\times q}$ stratifies the space of 1-jets $(M\times Q)^{(1)}$; at every point $p \in M$ the fibre is

$$\begin{array}{ccc} (M\times Q)^{(1)} & & M_{m\times q} \to \mathrm{Hom}(\mathbb{R}^m, TQ) \\ \downarrow & & \begin{array}{c}\wr\wr \\ T_pM \\ \downarrow \\ Q\end{array} \\ M & & \end{array}$$

The Thom–Boardmann thm. says that, for a generic map $f: M \to Q$, the stratification on $(M\times Q)^{(1)}$ pulls back to a stratification on M, because $j^1 f$ is transverse to all the strata.

I.e. the points of M where Tf has corank a has the same codim in M as $\Sigma^a_{m\times n}$ has in $M_{m\times n}$.

We write $M^a = \left\{ p \in M \mid j^1 f \in \Sigma^a_{m\times n} \leftrightarrow \begin{array}{c} Tf: T_pM \to T_{f(p)}Q \\ \text{has corank } a \end{array} \right\}$

Ex: Let $Q = \mathbb{R}$. Since $\Sigma^1_{m\times 1}$ has codim m, this implies M^1 is made of isolated points (a generic function has isolated singularities).

Additionally, we said that $j^1 f$ is transverse to the stratification of $(M\times\mathbb{R})^{(1)}$. This is equivalent to the fact that generically singularities are Morse.

Ex: Observe that $\operatorname{codim}(\Sigma^2_{m\times q}) = 2(m-q+2) \leq m$ iff $m - 2q + 4 \leq 0$ iff $m + 4 \leq 2q$. In particular, if $m >> q$ there are no higher singularities (singularities where the corank is greater than 1).

However, if m and q are similar, there are higher singularities generically.

Goal: construct maps without higher singularities i.e. $M^a = \emptyset$ if $a > 1$. Also M^1 with a special model.

Last remark about singularities: Thom-Boardmann works parametrically. I.e. if $f_k : M \to Q$ is a generic family $k\in K$ of maps, there is a stratification $(M\times K)^a$ of $M\times K$

$$(M\times K)^a = \{(p,k) \mid p \in (M\times\{k\})^a\}$$

Additionally, each M^a is itself stratified by repeating the process for $f: M^a \to Q$.

Ex: Consider a K-family of functions:

$$\begin{tikzcd} (M\times K)^{(1)} \arrow[r] \arrow[dr, dashed, "\varphi"'] & M\times K \arrow[r] \arrow[d, "\pi_2"] & \mathbb{R} \\ & K & \end{tikzcd}$$

Each of them has isolated sing, so $(M\times K)^1$ is a manifold of $\dim = \dim(K)$.

Now, $(M\times K)^{(1)} \to K$ is a map between manifolds of the same dim which, by Thom-Boardmann, is generic.

In particular, higher singularities can appear if $\dim(K) \geqslant 4$.

The corank 1 singularities include the usual birth – death.

III. Simple singularities: folds, cusps, wrinkles

From now on we write Σ^i for the sing. of corank i. $\Sigma^{i,j}$ are the sing of corank j of $\Sigma^i \to Q$, and so on.

(The reference for what follows is Wrinkling of smooth mappings I.)

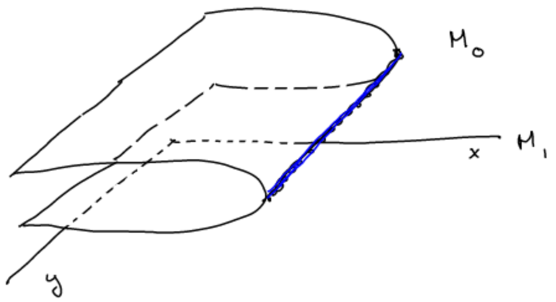
A. Folds have the local form:

$$\mathbb{R}^{q-1} \times \mathbb{R}^{m-q+1} \qquad\qquad \mathbb{R}^{q-1} \times \mathbb{R}$$

$$(y, x) \longrightarrow \left(y, -\sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{m-q+1} x_i^2\right)$$

where k is called the index.

- Critical points $M \to \mathbb{R}$ of a Morse function are folds.
- If $M_0^m \to M_1^m$ a fold is



Obs: points in a fold belong to Σ^{10} (the smooth part of Σ^1)

B. Cusps

$$\mathbb{R}^{q-1} \times \mathbb{R}^{m-q} \times \mathbb{R} \longrightarrow \mathbb{R}^{q-1} \times \mathbb{R}$$

$$(y, x, z) \longrightarrow \left(y, z^3 - 3y_1 z - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{m-q} x_i^2\right)$$

Let us compute the singular locus by computing the derivative:

$$\left(\begin{array}{cc|cccc} \multicolumn{2}{c|}{Id} & & 0 & & \\ \hline -3z & 0 & -2x_1 & \cdots & 2x_{m-q} & 3(z^2-y_1) \end{array}\right)$$

So the singular locus is the set

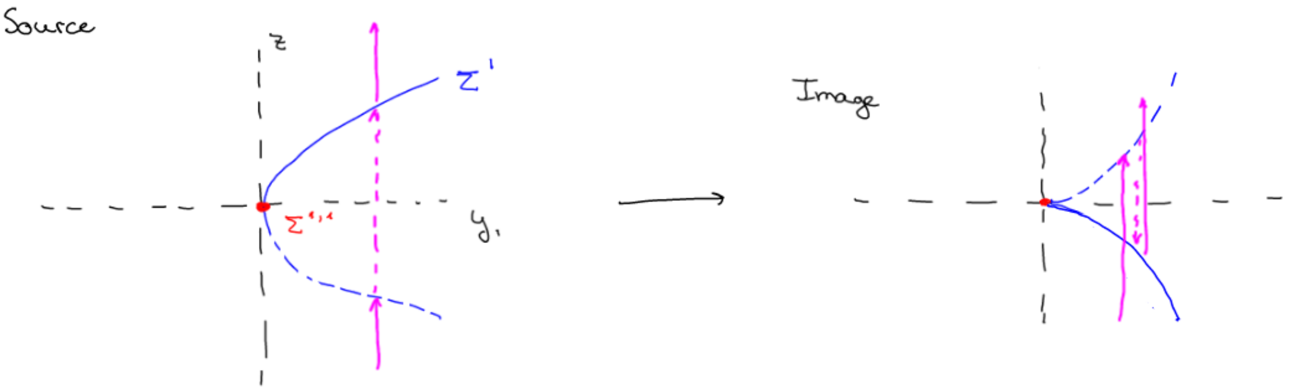
$$\Sigma^1 = \left\{ \begin{array}{l} z^2 = y_1 \\ x = 0 \end{array} \right\}$$

which is a smooth submanifold of dimension $q-1$.

However, its image is cuspidal:

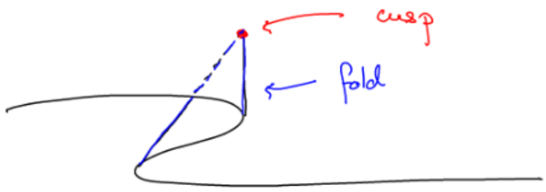
$$\begin{array}{ccc} \Sigma^1 & \longrightarrow & \mathbb{R}^{q-1} \times \mathbb{R} \\ (y_1 = z^2, \dots y_{q-1}, z) & \longrightarrow & (z^2, y_2 \cdots y_{q-1}, -2z^3) \end{array}$$

where the cusps take place in $\Sigma^{11} = \left\{ \begin{array}{l} z = 0 \\ y_1 = 0 \\ x = 0 \end{array} \right\} \subset \Sigma^1$



The lower branch on the source goes to the upper one in the target.

We can add a vertical direction to see this better:

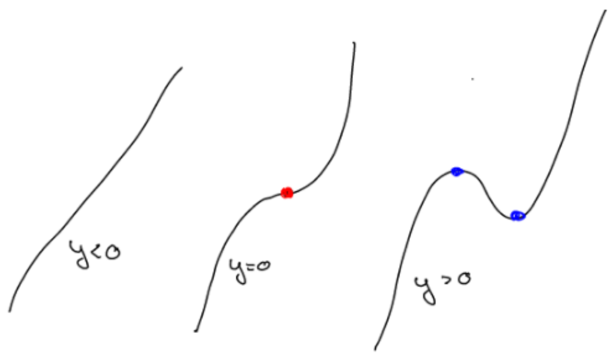


The model can be thought of as a family of functions

$$(y, x, z) \longrightarrow (y, f_{x,y}(z))$$

In the 2-dimensional case:

$$(y, z) \longrightarrow (y, f_y(z) = z^3 - 3yz)$$

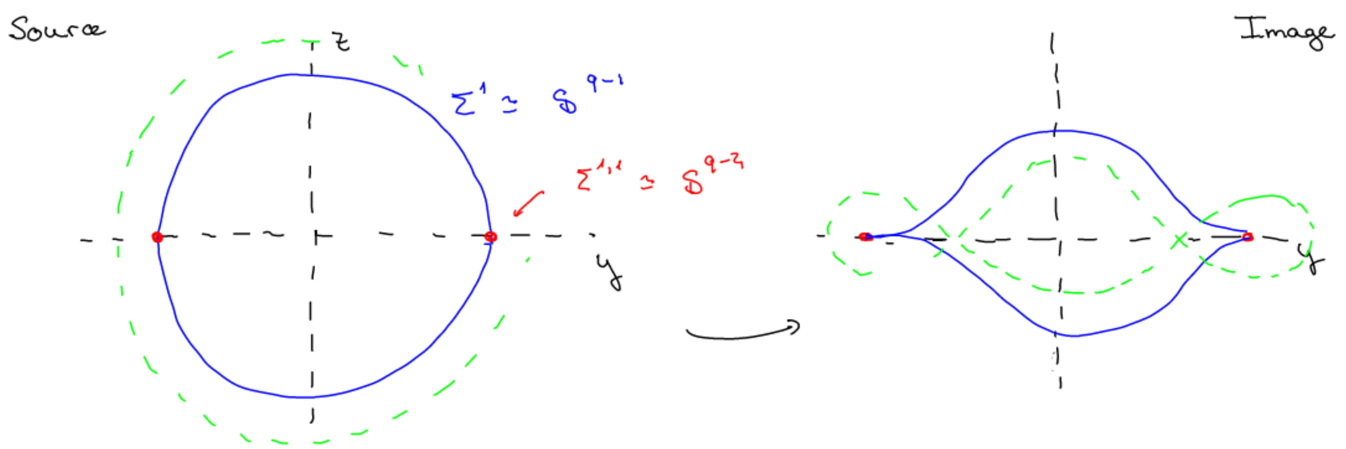


Now we define the wrinkle, which is a "self-cancelling" family of cusps / folds.

C. Wrinkle

$$\mathbb{R}^{q-1} \times \mathbb{R}^{m-q} \times \mathbb{R} \longrightarrow \mathbb{R}^{q-1} \times \mathbb{R}$$

$$(y, x, z) \longmapsto \left(y,\ z^3 + 3(|y|^2-1)z - \sum_{i=1}^{k} x_i^2 + \sum_{k+1}^{m-q} x_i^2\right)$$



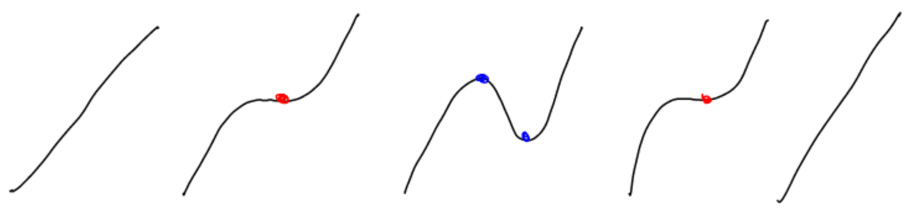
Let us compute the image of each of the hemispheres:

$$\left(y, x, z = \pm\sqrt{1-|y|^2}\right) \longrightarrow \left(y, \mp 2\,(1-|y|^2)^{3/2} + \ldots\ldots\right)$$

So again, they are "reversed" in the image.

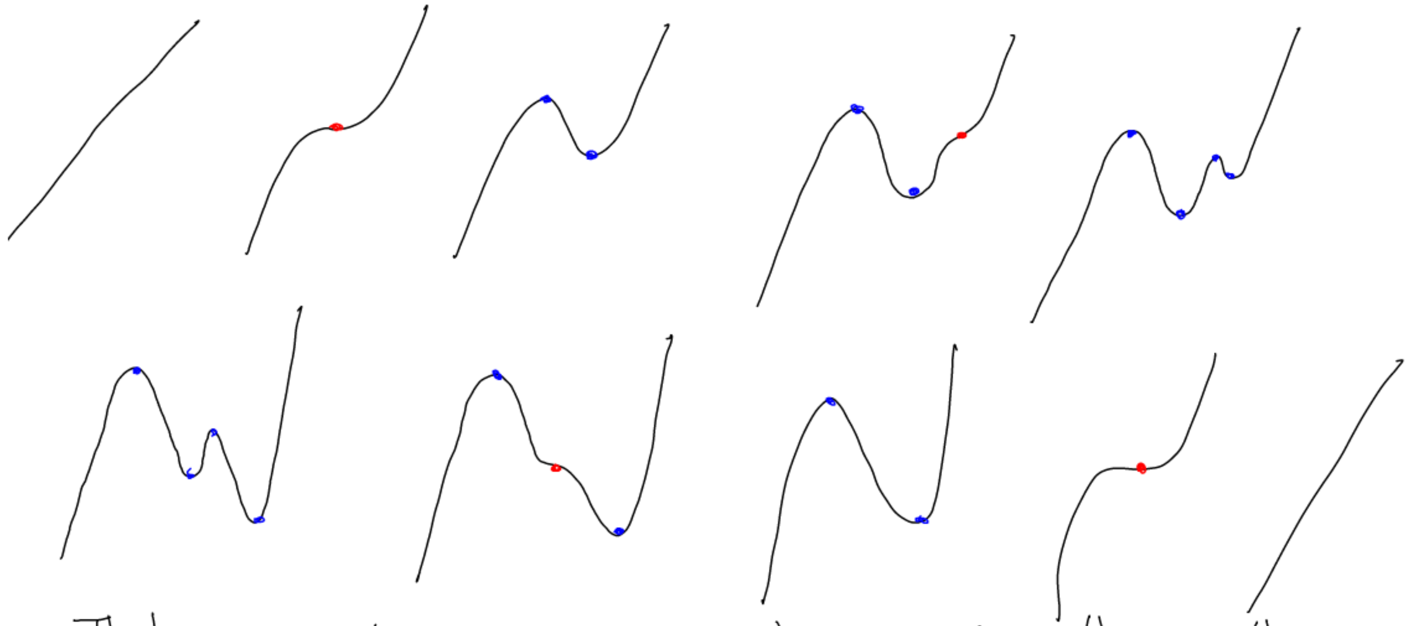
<u>Philosophy:</u> a wrinkle is the "simplest" singularity in the sense that it lives in Σ^1 and is topologically uncomplicated. Indeed, consider the following example:

$(y, z) \longrightarrow (y, f_y(z))$ with $f_y : \mathbb{R} \to \mathbb{R}$ the functions. A wrinkle is

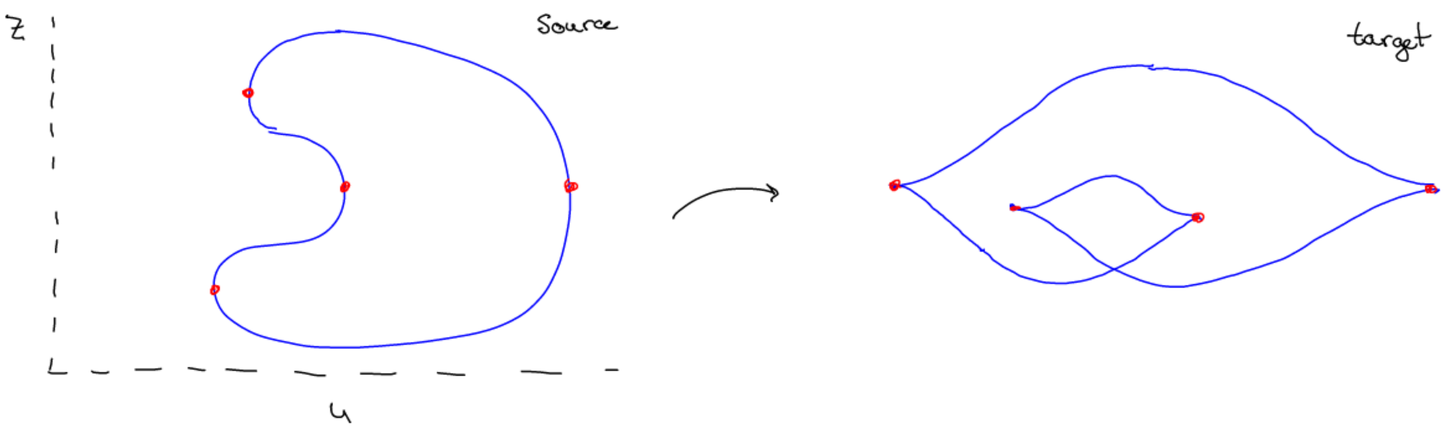


that is, is a pair of critical points that are born, then remain in cancelling position, and then die.

However, a more general example could be:



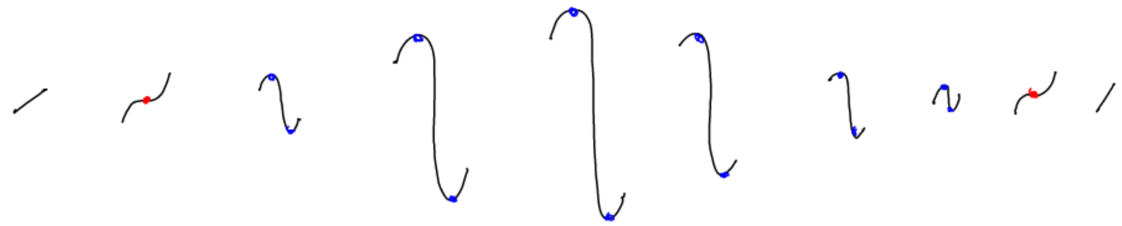
That is, two pairs are born and then they cancel in a different way. The corresponding Cerf diagram (just the map $(y,z) \to (y, f_y(z))$) is:



That is, wrinkles do not allow bifurcation.

! Important remark: a wrinkle is a map that agrees with the given model up to diffeo in domain and target. In particular, it agrees with the model in an arbitrarily small nbhd. of $D = \{z^2 + |y|^2 \leq 1\}$ (D is usually called the membrane)

This implies that an accurate picture of a wrinkle (as a family of functions) is:

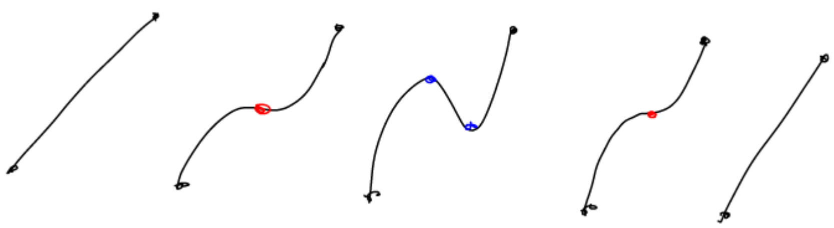


The key point is that although the wrinkle looks like a configuration of critical points in cancelling position, they cannot be cancelled because of the boundary conditions.

That is, given the graph of a function , we cannot eliminate the maximum and minimum relative to the ends, because there is no increasing function agreeing with  at the endpoints.

D. Embryos

Some wrinkles can be eliminated, for instance:



but this requires us to find a "large neighbourhood" of the wrinkle allowing for cancelling. Conversely, we can introduce a wrinkle if there was none:

$$\mathbb{R}^{q-1} \times \mathbb{R}^{m-q+1} \times \mathbb{R} \longrightarrow \mathbb{R}^{q-1} \times \mathbb{R}$$

$$(y, x, z) \longrightarrow \left(y,\ z^3 - 3|y|^2 z - \sum^{k} x_i^2 + \sum_{k+1} x_i^2\right)$$

If we perturb this map to yield

$$(y, x, z) \longrightarrow \left(y,\ z^3 - 3(|y|^2 - \varepsilon) z \ldots \right)$$

there are two options:

- if $\varepsilon > 0$ a wrinkle appears
- if $\varepsilon < 0$ the map becomes regular.

Remark: wrinkles appear in individual maps and embryos in families as wrinkles are born and die.

IV. The h-principle for wrinkled submersions

Let us define $W\mathrm{Subm}(M,Q)$, the space of wrinkled submersions:

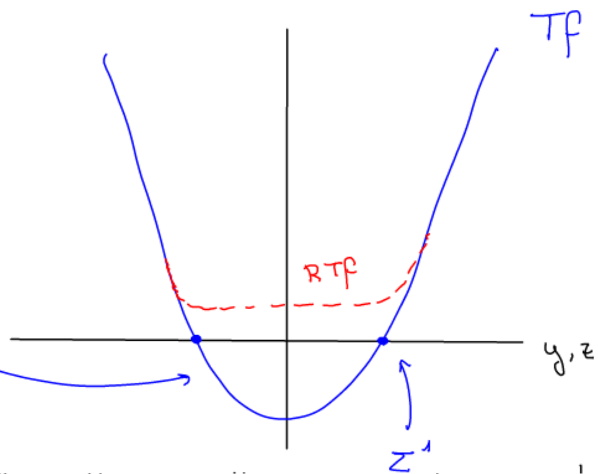
$$W\mathrm{Subm}(M,Q) = \left\{ f: M \to Q \;\middle|\; \begin{array}{l} f|_{M \setminus (\cup B_i)} \text{ is a submersion} \\ f|_{B_i} \text{ is a wrinkle} \end{array} \right\}$$

$\mathrm{Subm}(M,Q)$ is a subspace of both $W\mathrm{Subm}(M,Q)$ and $F\mathrm{Subm}(M,Q)$. We claim that $W\mathrm{Subm}(M,Q)$ can naturally be regarded as a subspace of $F\mathrm{Subm}(M,Q)$ called <u>regularisation</u>.

$$\begin{tikzcd} \mathrm{Subm}(M,Q) \arrow[r, hook] \arrow[d, hook] & F\mathrm{Subm}(M,Q) \\ W\mathrm{Subm}(M,Q) \arrow[ur, dashed, "R"'] & \end{tikzcd}$$

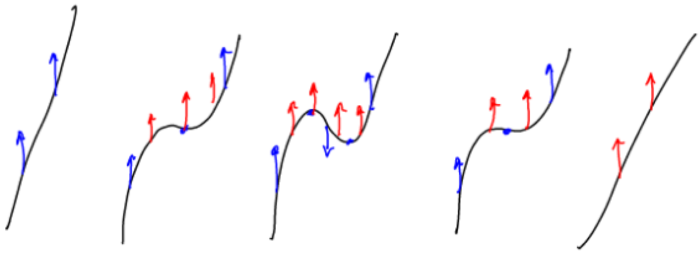
Consider the differential of a wrinkle:

$$\left(\begin{array}{ccc|cccc} & \mathrm{Id} & & & 0 & & \\ \hline 6y_1 z & \cdots & 6y_{q-1} z & -2x_1 & \cdots & 2x_{m-q} & 3(z^2+|y|^2-1) \end{array} \right)$$



We can simply modify this term around the membrane to obtain a map $RTf: TM \to TQ$ corresponding to a formal submersion.

A similar depiction as a parametric family of functions:



This process of regularising the differential is exactly what would happen if we eliminate the wrinkle.

<u>Thm 1</u>: $WSubm(M,Q) \xhookrightarrow{R} FSubm(M,Q)$ is a weak homotopy equivalence.

What we will prove is actually stronger. Consider the following situation:

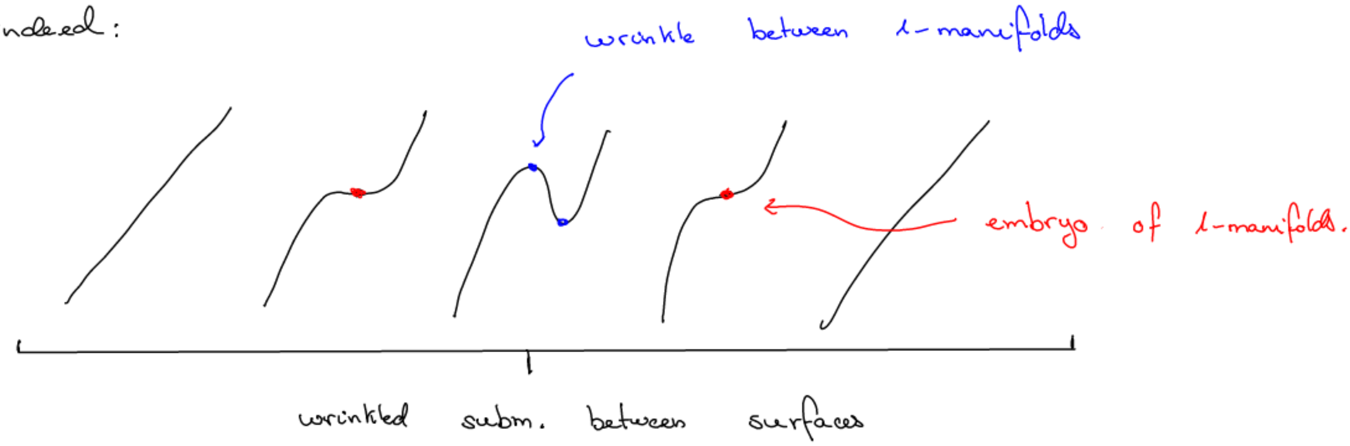
- given $\varphi : S^l \to FSubm(M,Q)$ the theorem says that there is $\tilde{\varphi} : S^l \to WSubm(M,Q)$ homotopic to it. Regard

  $\tilde{\varphi} : M \times S^l \to Q \times S^l$.

  Then, even though each $\tilde{\varphi}(-,s)$ is wrinkled, the theorem says nothing about $\tilde{\varphi}$ as a whole. We will show that $\tilde{\varphi}$ is a wrinkled submersion too.

<u>Fact</u>: the observation is that a wrinkle is itself a parametric families of wrinkles and embryos:

We keep depicting wrinkles as families of functions and indeed:



Indeed, the variable y can be regarded as a parameter in all previous constructions.

Thm 2 Let $M \times K \xrightarrow{\varphi} Q \times K$ be a K-family of formal submersions. Then there is a wrinkled map $M \times K \xrightarrow{\tilde{\varphi}} Q \times K$ that restricts to a wrinkled map (potentially with embryos) on each $M \times \{x\}$. $\tilde{\varphi}$ lifts the identity on K.

If φ is already a submersion over $Op(U \times K \cup M \times K')$ then $\tilde{\varphi} = \varphi$ over there.

- General clarification about h-principles:

Statements like Theorem 2 imply w.h.equivs. like Theorem 1. Theorem 1 gives a <u>global</u> result, but Theorem 2 says that the h-principle is also <u>relative</u> in domain and parameter.

Namely:

- Given $\varphi_0 : S^k \to FSubm$ representing $[\varphi_0] \in \pi_k(FSubm)$ we want to show that there is a map

  $\varphi_1 : S^k \longrightarrow WSubm$ whose class $[\varphi_1]$ is taken to $[\varphi_0]$ under the map $\pi_k(WSubm) \to \pi_k(FSubm)$.

  (showing π_k - surjectivity).

  Then: we define

$$\begin{array}{ccc} \phi_0 : M \times S^k & \longrightarrow & Q \times S^k \\ \downarrow & & \downarrow \\ S^k & \xrightarrow[Id]{} & S^k \end{array} \qquad \text{by } \underbrace{\phi_0(-,k) = \varphi_0(k)}_{} \in FSubm(M,Q).$$

  that is, we regard the family φ_0 as a single map.

Theorem 2 implies that ϕ_0 is homotopic to some

through fibered submersions

$$\begin{tikzcd} M \times S^k \arrow[r, "\phi_1"] \arrow[d] & Q \times S^k \arrow[d] \\ S^k \arrow[r, "Id"'] & S^k \end{tikzcd}$$

with $\phi_1 \in W\mathrm{Subm}(M \times S^k, Q \times S^k)$. with $\phi_1(-,k) \in W\mathrm{Subm}(M,Q)$

$\rightsquigarrow$ the family $\varphi_1(k) = \phi_1(-,k)$ is what we want $\boxtimes$

- A similar argument shows π_k - injectivity. For this we use $K = \mathbb{D}^k$, $K' = \partial \mathbb{D}^k$. $\boxtimes$

V. Proof of the h-principle

- Just like we did for holonomic approximation, we can assume $M = I^m$, $Q = I^q$, $K = I^k$, and φ is a submersion over $\partial(I^m \times I^k)$.

Indeed, triangulate $M \times K$ relative to $U \times K$ and $M \times K'$; since submersions are an open and Diff - invariant partial differential relation we can construct a submersion $\varphi_{1/2}$ on a nbhd of the codim-1 skeleton (homotopic to φ rel $U \times K \cup M \times K'$).

Our problem reduces to balls $I^m \times I^k$ as stated before (the top-dimensional cells). To extend $\varphi_{1/2}$ to the interior we will use wrinkling.

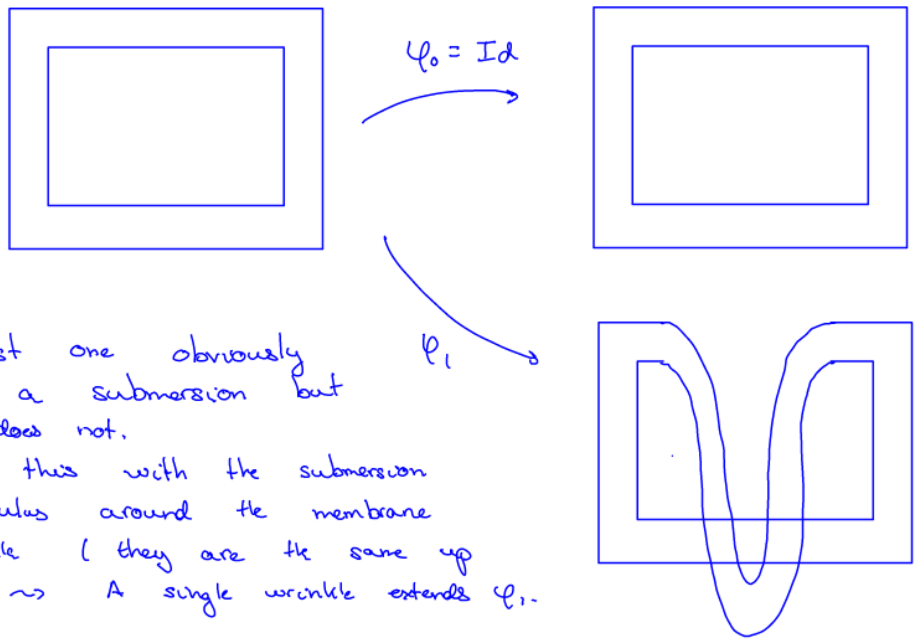
- The proof is simpler in the equidimensional case $m = q$ and actually the general case follows from this one by adding extra dimensions and then projecting.

- <u>Case $m = q$</u>

As stated after we proved holonomic approximation, the idea now is to "wiggle the manifold M onto itself". This gives us the room we need to interpolate:

Let us write $\varphi_k : I^m \to I^m$, $k \in I^k$, with φ_k a submersion over ∂I^k that is a submersion everywhere if $k \in \partial I^k$.

Ex $\quad \varphi_0, \varphi_1 : Op(\partial I^2) \to I^2$



The first one obviously extends as a submersion but the second does not.
Compare this with the submersion of an annulus around the membrane of a wrinkle (they are the same up to diffeo). $\rightsquigarrow$ A single wrinkle extends φ_1.

Now. We apply hol. approx. to find a family of submersions:

$$\varphi_{y,k} : Op(I^{m-1} \times \{y\}) \longrightarrow I^m$$

with $\quad \varphi_{y,k}(p) = \varphi_k(p)$ if $y \in Op(\partial I)$ or $k \in \partial I^k$, or $p \in Op(\partial I^m)$.

That is, over each slice $I^{m-1} \times \{y\}$ we find a submersion but these do not glue to a global submersion.

Let $y_i = i/N$, $i = 0 \ldots N$.

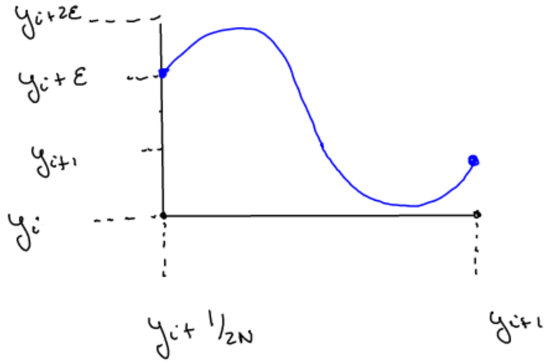
Fix some ε s.t $Op(I^{m-1} \times \{y\}) \supset I^{m-1} \times [y-\varepsilon, y+\varepsilon]$ and φ is a submersion on an ε-nbhd of $\partial(I^m \times I^k)$.

• Let $\beta: I \to I$ a wrinkled submersion of the form

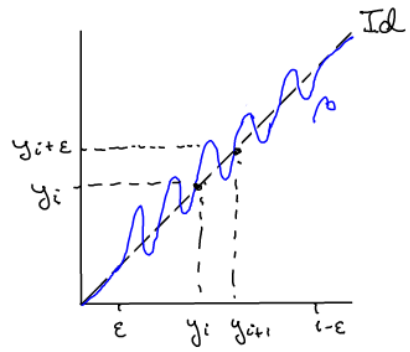
$\beta|_{[y_i, y_i + 1/2N]}$ is an increasing submersion into $[y_i, y_i + \varepsilon]$

$\beta|_{[y_i + 1/2N, y_{i+1}]}$ is a single wrinkle into $[y_i, y_i + 2\varepsilon]$

of the form



β is the identity in $[0, \varepsilon] \cup [1-\varepsilon, 1]$



Define $\alpha: I^m \to I^m$ wrinkled submersion by:

$$\alpha(x, y) = (x, \rho(x)\beta(y) + (1 - \rho(x))y)$$

with ρ a cutoff function

that is 1 in $I^{m-1}_{1-\varepsilon}$ and 0 outside of $I^{m-1}_{1-\varepsilon/2}$

and whose level sets are convex. $\rightsquigarrow$ α is a wrinkled map

with a wrinkle in $I^{m-1} \times [y_i + 1/2N, y_{i+1}]$

The point is that $\alpha|_{I^{m-1}_{1-\varepsilon} \times [\varepsilon, 1-\varepsilon]}$ identifies the interval

$[y_i, y_i + 1/2N]$ with $[y_i, y_i + \varepsilon]$. The former is very short

but the latter has fixed size. This gives us "extra space"

to interpolate between the $\varphi_{y,x}$.

We define;

- $\tilde{\varphi}_k\Big|_{I^{m-1}\times[y_i,\, y_i+1/2N]}(x,y) = \varphi_{y,k}(\alpha(x,y))$

that is, in the interval $[y_i, y_i+1/2N]$ $\tilde{\varphi}_k$ performs what $\varphi_{y_i+s,k}$ does in $[y_i, y_i+\varepsilon]$. Since ε is fixed but $1/N$ will be arbitrarily small, $\tilde{\varphi}_k$ is C^1- large.

Since $\varepsilon >> 1/N$, we have

$$\left.\frac{\partial\tilde{\varphi}_k}{\partial y}(x,y) \doteq \frac{\partial\alpha}{\partial y}\cdot\frac{\partial\varphi}{\partial y}(\alpha(x,y)) \underset{\leftarrow \text{ in the domain}}{} \approx \varepsilon.2N\cdot\frac{\partial\varphi}{\partial y}\right\} \simeq O(2N)$$

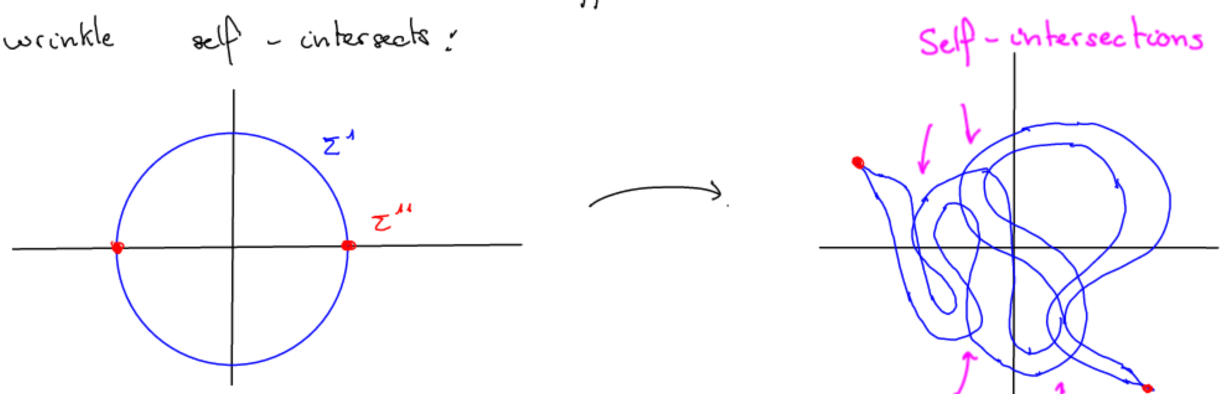
$$\left.+\ \frac{\partial\varphi}{\partial y}_{y,k}(\alpha(x,y)) \underset{\leftarrow \text{ in the parameter}}{}\right\} \simeq O(1)$$

So we obtain a submersion if N is large.

- $\tilde{\varphi}_k\Big|_{I^{m-1}\times[y_i+1/2N,\, y_{i+1}]}(x,y) = \varphi_{y_{i+1},k}(\alpha(x,y))$

which is the composition of a submersion $\varphi_{\underbrace{y_{i+1}}_{\text{fixed}},k}$ and the wrinkle α.

We are <u>almost</u> done. The issue is that $\varphi_{y_{i+1},k}\circ\alpha$ might not be quite a wrinkle. Composing a submersion with a wrinkle is problematic if the submersion is not equidimensional (as we shall see), but even in the present case $m=q$ it may happen that the image of the wrinkle self-intersects!



In the domain what we have already looks like a wrinkle

In the image, since the maps $\varphi_{y,k}$ were wild, there may be self-intersections.

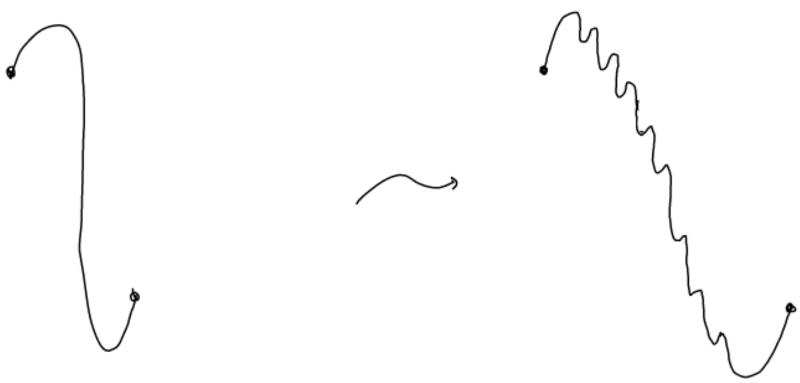
Remark: In some sense this is fine, at this point of the proof we have already simplified the singularity locus to be "wrinkle-like".

However, it is convenient to obtain actual wrinkles. In particular we will need it for the case $m > q$.

Chopping wrinkles

The idea now is that we subdivide the "wrinkle-like" singularities from above into small pieces that will be actual wrinkles.

- The 1-dimensional case contains the idea:



Big wrinkle   Small wrinkles

"any function can be C^0-approximated by arbitrarily small wrinkles"

- Our wrinkle-like singularities are of the form

  $\varphi \circ \alpha$ with φ a submersion $I^m \to I^m$ and

  α a wrinkle. It is sufficient to subdivide α then. Since φ is an immersion, the composition $\varphi \circ \bar{\alpha}$ with $\bar{\alpha}$ a "small" wrinkle will have no self-intersections.

- "Vertical" chopping is easy. α was of the form

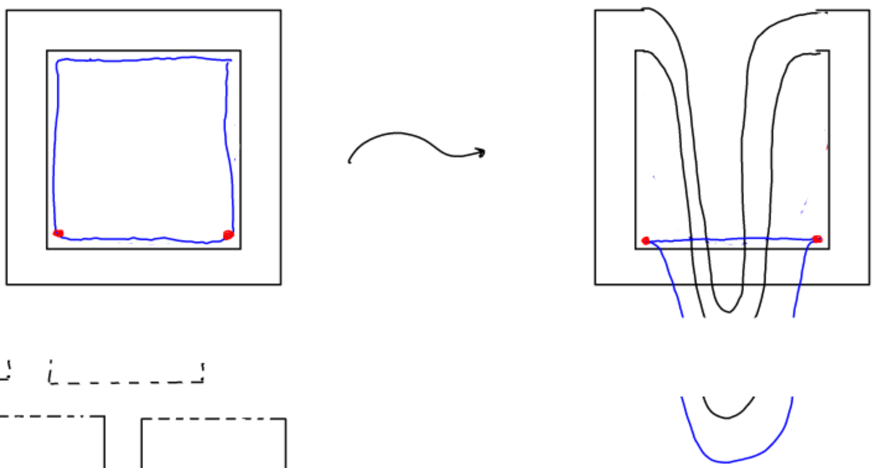
  $$\alpha(x,y) = \rho(x)\beta(y) + (1-\rho(x))y$$

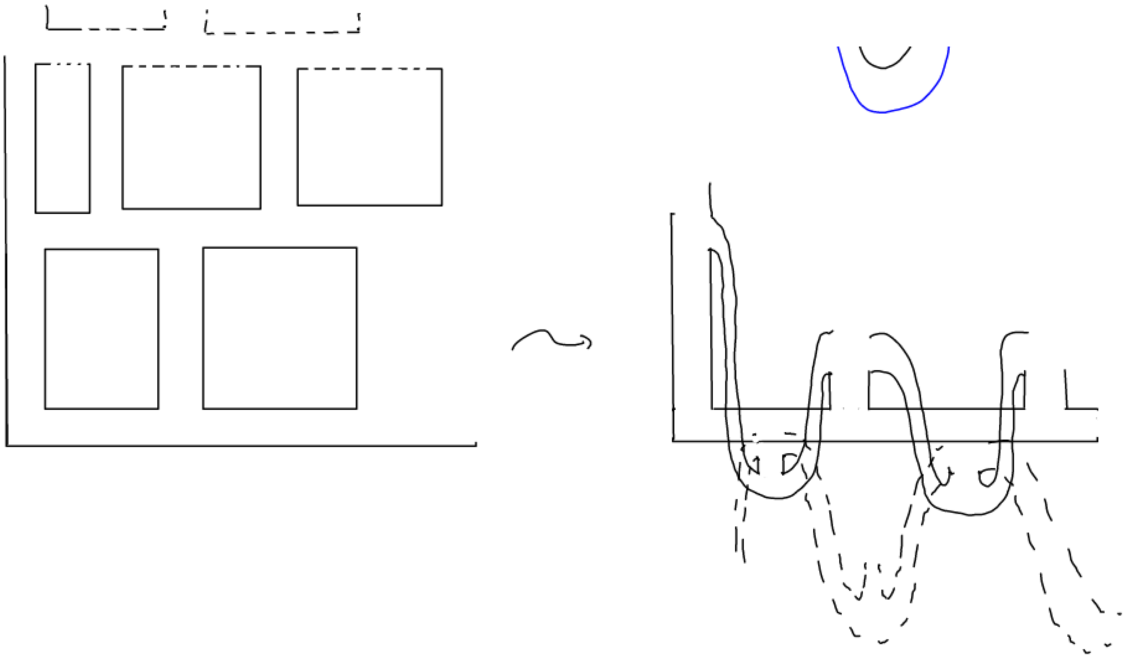
  with β a large wrinkle  which we replace by $\hat{\beta}$, many small ones.

So: $\tilde{\alpha}(x,y) = \varphi(x)\tilde{\beta}(y) + (1-\varphi(x))y$ has many wrinkles that are small in y but wide in x.

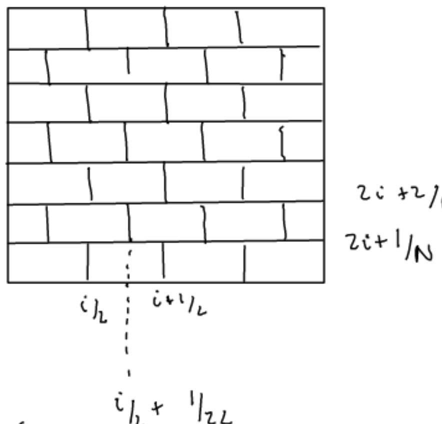
- To chop in x we need to simultaneously chop in y.

Consider





The idea is that we chop the domain in the y-direction first and then we subdivide I^{m-1} using the planes $\{x_j = i/L\}_{j=1..m-1,\ i=0..L}$ in the intervals $\{y \in [2i/N, 2i+1/N]\}$ and using $\{x_j = i/L + 1/2L\}$ over $\{y \in [\frac{2i+1}{N}, \frac{2i+2}{N}]\}$.



Denote by A the subset of I^m:

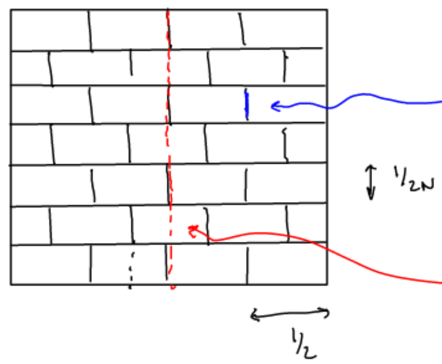
$$\bigsqcup_{0\le i\le N} \{y = i/N\} \cup$$

$$\bigsqcup_{\substack{0\le j\le m-1;\\ 0\le 2i\le N;\ k=0..L}} \{y \in [2i/N, 2i+1/N]\} \cap \{x_j = k/L\} \cup \bigsqcup \{y \in [2i+1/N, 2i+2/N]\} \cap \{x_j = k/L + 1/2L\}$$

Then we claim that if we leave L fixed but N is made large we can C^0-approximate $\alpha|_{Op(A)}$ by a submersion of the form $(x,y) \to (x, \tilde{\tilde{\alpha}}(x,y))$ with $\frac{\partial \alpha}{\partial y} > 0$.

As before, $\tilde{\tilde{\alpha}}$ extends to the complement of A using wrinkles that are now small $\boxtimes$

The point is that if $N >> L$ then:



these intervals are very short, so even if we impose $\partial_y \tilde{\tilde{\alpha}} > 0$, we can preserve C^0-closeness to α.

if we subdivided like this, with long vertical planes, this wouldn't be possible because potentially $\alpha(x,0) >> \alpha(x,1)$.

- $\underline{\text{Case } m > q}$:

The idea of the proof is: Given the family

$$\begin{tikzcd} \varphi_k : I^m \arrow[r] \arrow[dr, dashed, "\tilde{\varphi}_k"'] & I^q \\ & I^m \arrow[u, "\pi"'] \end{tikzcd}$$

we construct some formal submersion $\tilde{\varphi}_k$ into I^m lifting φ_k.

(observe that $\operatorname{Ker}(\varphi_k : TI^m \to TI^q)$ is a trivial bundle because it is over I^m, so it can be mapped to $\operatorname{Ker}(\pi)$ to indeed yield a formal submersion).

Now, by the previous arguments we can homotope $\tilde{\varphi}_k$ to some wrinkled submersion $\phi_k : I^m \to I^m$. The question now is: how far is $\pi \circ \phi_k$ from being wrinkled?

Let us start by understanding the issue in the case of the model wrinkle.

- Observe that a wrinkle from $\mathbb{R}^3$ to $\mathbb{R}^3$ projected to $\mathbb{R}^2$ is <u>not</u> a itself:

$$
\begin{array}{ccc}
(y,x,z) & \xrightarrow{\omega} & (y,\ x,\ z^3+3(y^2+x^2-1)z) \\
& \searrow & \downarrow \pi_{\mathbb{R}^3\to\mathbb{R}^2} \\
& & (y,\ \underbrace{z^3+3(y^2-1)z \ +3zx^2}_{\approx})
\end{array}
$$

(wrinkle $\mathbb{R}^3 \to \mathbb{R}^3$)
↓
project to $\mathbb{R}^2$

$$(y,x,z) \longrightarrow (y,\ z^3+3(y^2-1)z \ \pm x^2)$$

(wrinkle $\mathbb{R}^3 \to \mathbb{R}^2$)

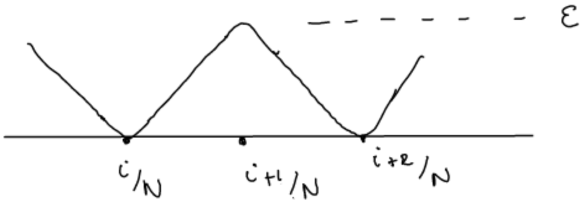
The issue is that zx^2 is very degenerate at $z=0$. Let us compute $T(\pi\circ\omega)$

$$T(\pi\circ\omega)(y,x,z) = \begin{pmatrix} 1 & 0 & 0 \\ 6yz & \underbrace{6xz} & \overbrace{3(z^2+y^2+x^2-1)} \end{pmatrix}$$

this term vanishes on S^2 and yields cuspidal behaviour in $S^1 = S^2 \cap \{z=0\}$

the zero locus of this term should avoid S^1 and cut S^2 transversely. It does not.

Let $g: \mathbb{R} \to \mathbb{R}$ be a function of x of the form:



- $g|_{Op(i/N)} = \pm M x^2$
- outside of $Op(i/N)$, g is $\simeq$ linear
- g is $2/N$ periodic;
- $|g| \leq \varepsilon$.

If ε is small but fixed and N, M are large, then g' is large away from the critical points and g'' is large at the critical points.

Then we replace our wrinkle by:

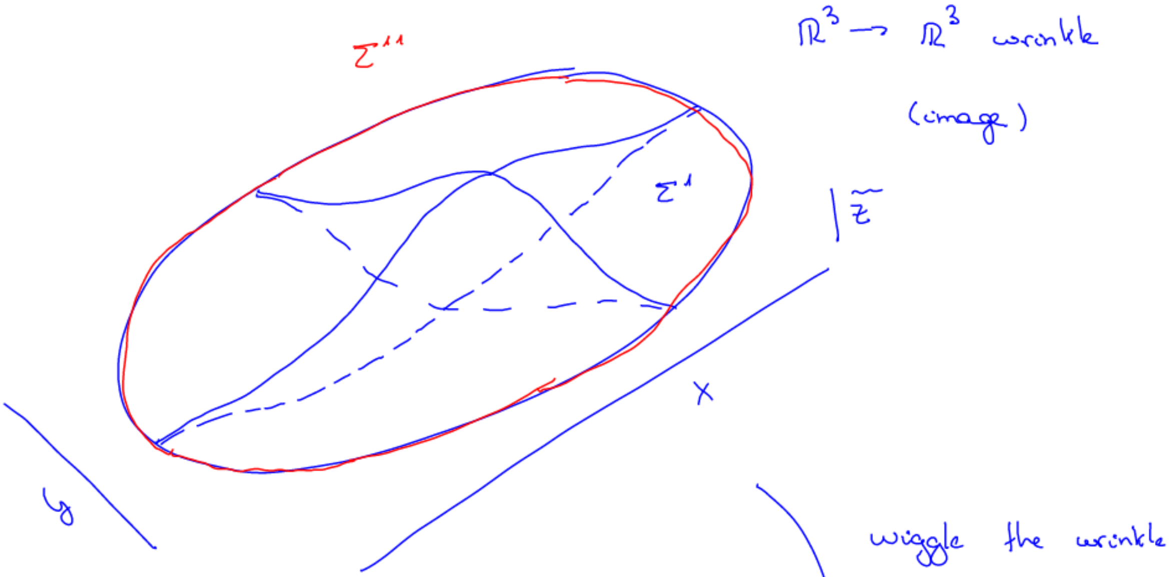
$$(y,x,z) \longrightarrow (y,\ x,\ z^3+3(|y|^2+|x|^2-1)z + \rho(y,x,z)(g(x_1)+g(x_2)\ldots+g(x_n)))$$

with ρ some cutoff function that is 1 in a neighbourhood of the membrane. Since ε is small the image remains within the model. Additionally, since N is large, the critical points of the projection concentrate along the simultaneous critical points of $(g(x_1) \dots g(x_{m-q}))$. Since g'' is large there, the projection looks like an actual wrinkle:

$$(y,x,z) \to \left(y,\ z^3 - (|y|^2+|x|^2-1)z \underbrace{- M\sum x_i^2 + M\sum x_i^2}\right)$$

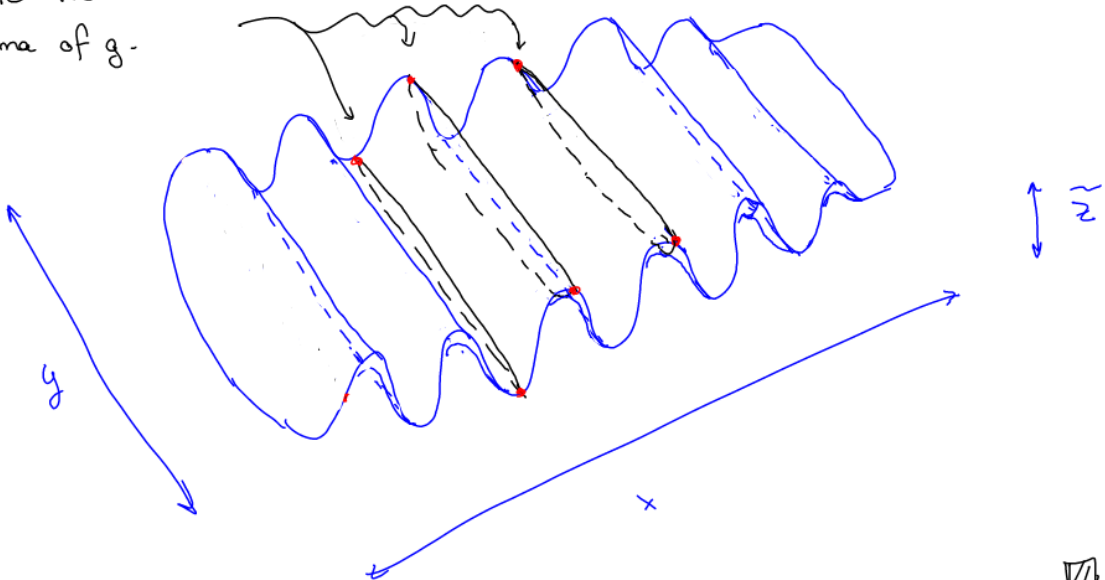
these terms dominate this one.

A very exaggerated picture: (there should be <u>more</u> wiggles that are smaller)



wiggle the wrinkle as we move in x

now, when we project to $(y,\tilde{z})$, wrinkles appear corresponding to the maxima/minima of g.



$\boxtimes$

<u>Remark</u>: by construction, the function $g(x_1)+\dots+g(x_{m-q})$ has critical points of every index. In many cases it is interesting to have control of the index (for instance to

say there are no maxima or minima) but this proof provides no such control,