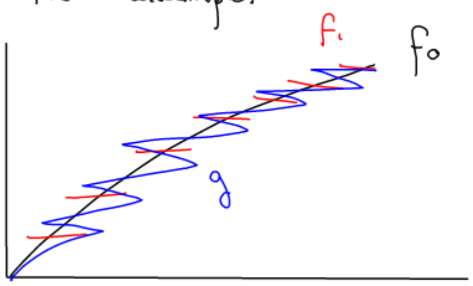
Wrinkled embeddings

Our aim now is to study embeddings $M^m \to Q^q$, $m < q$. We already know that $\mathrm{Imm}(M,Q)$ satisfies the h-principle but, as we saw in the proof of hol. approx., this h-principle is <u>not</u> C^1-close*, since wiggling is required.

* to clarify: the immersions produced are C^0-close to the desired map but the C^1-part is not C^0-close to the original formal derivative.

Recall the example:



g is C^0-close to f_0 and the tangent lines to g are C^0-close to the lines defined by f_1.

We want to show: (below we explain the concepts involved)

Thm: Let $f_k : M \to Q$ be a family of embeddings, $k \in K$. Let

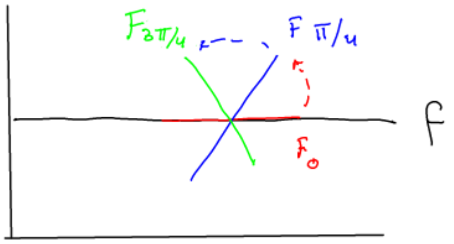
$$\begin{array}{ccccc} F_{k,s} : & M & \to & Gr(Q,m) \\ & \downarrow & & \downarrow \\ f_k : & M & \to & Q \end{array}$$

be a family of m-plane fields lifting f_k, $s \in [0,1)$, with $F_{k,0} = G(Tf_k)$.

Then there is $g_{k,s} : M \to Q$ <u>wrinkled</u> embeddings with
$$\| (f_k, F_{k,s}) - (g_{k,s}, G(Tg_{k,s})) \| < \varepsilon, \quad g_{k,0} = f_k.$$

Additionally, if $F_{k,s}(p) = G(Tf_k)(p)$ in $\{p \in U \subset M\} \cup \{k \in K' \subset K\}$ then $g_{k,s} = f_k$ there.

The point of the theorem is: we take the tangent planes to M in the image (i.e. $G(Tf_k)$) and we move them ($F_{k,s}$). As we do so we would like to move f so that it remains tangent (or close to tangent) to $F_{\kappa,s}$. This in general cannot be done:



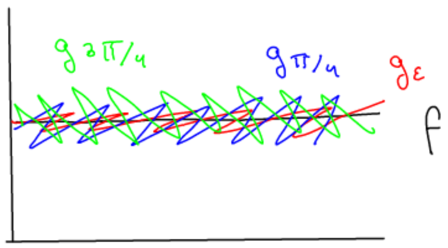
in this example $K = \{*\}$

$Q = \mathbb{R}^2$; $M = \mathbb{R}$;

$f(x) = (x, 1)$

GF_κ = line at angle κ.

But we can allow topological embeddings with singularities s.t $G(Tg_\kappa)$ is well defined (even if Tg_κ drops in rank). Namely:

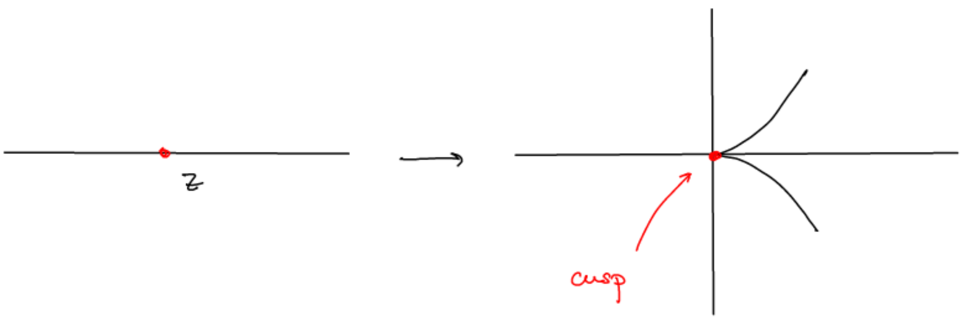


Let us define now what a wrinkled embedding is. It is good to define first a couple of concepts.

- <u>Cusps</u>:

$$\mathbb{R}^{m-1} \times \mathbb{R} \longrightarrow \mathbb{R}^{m-1} \times \mathbb{R}^2 \times \mathbb{R}^{q-m-1}$$

$$(x, z) \longrightarrow (x, z^2, z^3, 0)$$



For instance, if w is a wrinkle, the restriction of w to Z^1 has cusps along Z^{11}.

<u>Remark</u>: a cusp can be regularised as follows:

$$(x, z) \longrightarrow (x, z^2, z^3 + \varepsilon z, 0 \ldots)$$

- Swallow tails

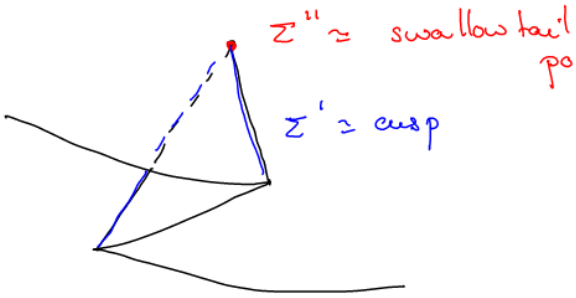
$$\mathbb{R}^{m-1} \times \mathbb{R} \longrightarrow \mathbb{R}^{m-1} \times \mathbb{R}^2 \times \mathbb{R}^{q-m-1}$$

$$(x, z) \longrightarrow \left(x,\ z^3 - 3x_1 z,\ \int_0^z (s^2 - x_1)^2 \, ds,\ 0 \dots 0\right)$$

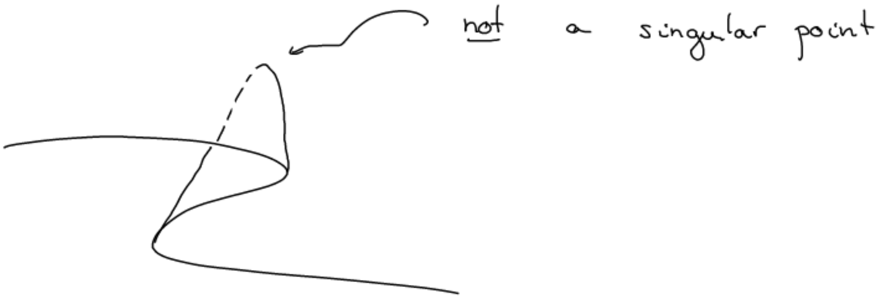
Just like two folds died in a cusp (as we saw for submersions), two cusps die in a swallow tail:

$\Sigma' = \{ z^2 = x_1 \}$

$\Sigma'' = \{ z = x_1 = 0 \}$



As before, swallowtails can be regularised:



- Wrinkled embeddings

Definition: A wrinkled embedding is a topological embedding $f : M \to Q$ that is smooth away from a collection of annuli $A_i \cong S^{m-1} \times I \subset M$. In each annuli, the map is diffeomorphic to

$$\mathcal{O}p(S^{m-1}) \longrightarrow \mathbb{R}^q$$

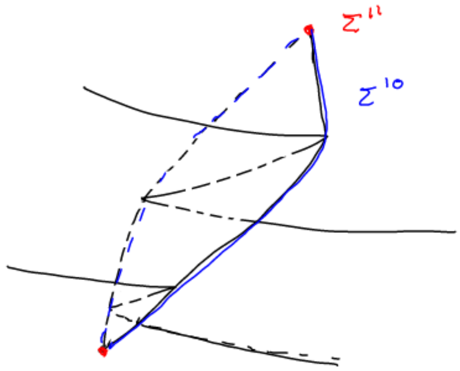
$$(x_1, \dots, x_{m-1}, z) \xrightarrow{\omega} \underbrace{(x_1, \dots x_{m-1},\ z^3 + 3(|x|^2 - 1) z}_{\text{this is a wrinkle (as in submersions)}},\ \underbrace{\int_0^z (s^2 + |x|^2 - 1)^2 ds}_{\text{this is the unfolding of the wrinkle.}},\ 0, \dots, 0)$$

Much like a wrinkle was a "self-cancelling sphere of cusps", this model is a self-cancelling sphere of swallowtails.

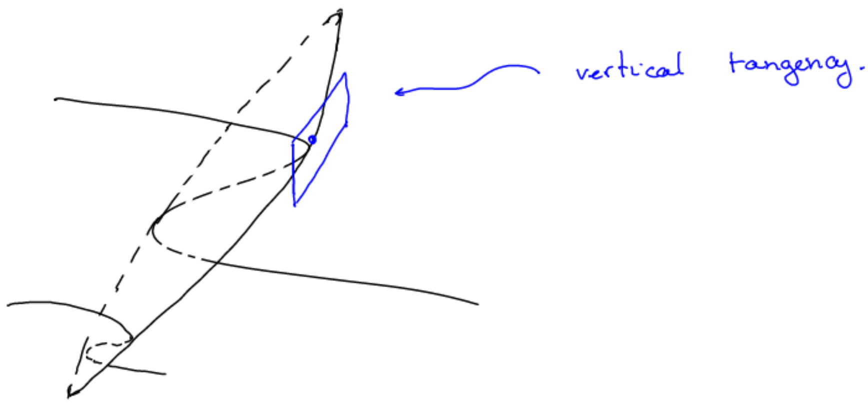
We have that $S^{m-1} \setminus \{z=0\}$ correspond to cuspidal points (Σ^{10}) and $S^{m-1} \cap \{z=0\}$ correspond to swallowtails (Σ^{11}). Indeed, by restricting ω to S^{m-1}:

$$(x_1, \dots, x_{m-1}, \pm\sqrt{1-|x|^2}) \xrightarrow[\omega]{} (x_1, \dots x_{m-1}, \mp 2(1-|x|^2)^{3/2}, 0 \dots 0)$$

it is smooth away from $\{z=0\}$, where the map develops cusps.



As in the case of cusps and swallowtails, the map can be regularised. Indeed, the unfolding function is increasing but singular along S^{m-1}, by a small perturbation we obtain a smooth embedding.

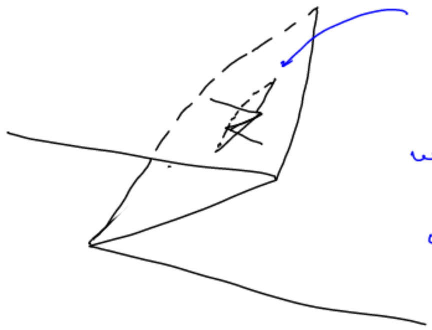


When we desingularise the singularities get replaced by vertical tangencies (that we precisely want to avoid when approximating the planes F_k).

<u>Motivation</u>: we will later apply this theorem in the context of foliation theory. This trading-off between vertical tangencies and singularities will allow us to construct embeddings with only fold type singularities with the leaves of a foliation.

Key remark: a wrinkled embedding has a well defined tangent plane at every point. This is obvious, since this is true for cusps and swallowtails.

Remark: we defined the model wrinkled embedding with domain $Op(S^{m-1})$. The point is that we allow for nesting of wrinkled embeddings (which we did not for submersions):



Little wrinkle inside a big one. We can define the depth of a wrinkle depending on how many wrinkles it sits inside.

- Embryos

We have to allow for wrinkled embeddings to be born and die (as it does in families). As such, we define:

$$Op(\{0\}) \longrightarrow \mathbb{R}^q$$

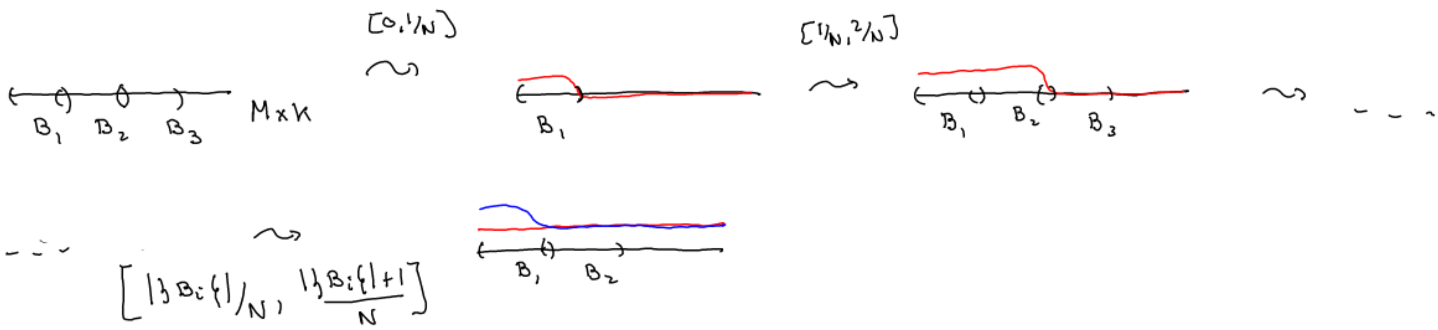
$$(x_1,\dots,x_{m-1},z) \xrightarrow[\omega]{} \left(x_1,\dots,x_{m-1},\ z^3+3|x|^2z,\ \int_0^z (s^2+|x|^2)^2\,ds,\ 0,\dots,0\right)$$

II Proof of the theorem

It can be structured in three steps:

1. $F_{k,s}$ is decomposed into elementary moves. Over each time interval $[i/N, i+1/N]$ $F_{k,s}$ will perform a rotation along an $(m-1)$-dim. axis and supported on a ball $B_i \subset M \times K$.

   Remark $\{B_i\}$ is a cover of $M\times K$ with cardinality $<< N$, Ensuring that this decomposition is C^0-close to the original $F_{k,s}$.



2. We approximate $F_{k,s}|_{B_1\times[0,1/N]}$ by $F_{0,1/N}(0)$, with $(0,0)\in B_1$ the center of B_1.* $F_{0,1/N}(0)$ is a foliation and we can triangulate $B_1 \times [0,1/N]$ in a manner suited to $F_{0,1/N}(0)$.

2b. using wrinkles it is easy to define $g_{k,s}|_{B_1\times[0,1/N]}$ and $g_{k,s} = g_{k,0}$ away from B_1.

3. we use holonomic approx to define $g_{k,s}|_{[1/N,2/N]}$ along the wrinkles of $g_{k,1/N}$ (so the wrinkles stay but get wiggled). Now we repeat 2.b in $B_2 \times [1/N, 2/N]$, relative to the wrinkles.

* If the covering is fine enough and N is very large, this is a C^0-close approximation.

Step 1 is formalised by the following lemma:

Lemma 1: Let f_K, $F_{K,s}$, $\mathcal{U}$, K' as in the theorem. There is $\tilde{F}_{K,s}$ C^0-close to $F_{K,s}$ such that:

- for each $i = 0 \dots N$ there is
$$L^i : B_i \subset M \times K \longrightarrow Gr(Q, m+1)$$
with B_i a ball in $M \times K$ and $L^i(p,K) \supset \tilde{F}_{K,s}(p)$ $\forall s \in [i/N, i+1/N]$,
- $\tilde{F}_{K,s}$ is graphical over $\tilde{F}_{K,i}$ $\forall s \in [i/N, i+1/N]$.

Pf/ We are simply rectifying $F_{K,s}$. Given $F \in Gr(q,m)$, the nearby planes are parametrised by $n_3 = m(q-m)$ functions.
Let $n_2 = |\{B_i\}|$,* Let $n_1 >> n_2 . n_3$.*

On the interval $\left[\frac{i + n_2 . c}{n_1 . n_2}, \frac{i+1 + n_2 . c}{n_1 . n_2}\right]$, $i \in 0 \dots n_2 - 1$, we interpolate linearly between $F_{K, c/n_1}$ and $F_{K, c+1/n_1}$ but only over B_i.

Being over balls, $F_{K, c+1/n_1}$ is seen as a map $\mathbb{D}^m \times \mathbb{D}^k \to \mathbb{D}^{m(q-m)}$ so this linear interpolation can be done sequentially over each component of $\mathbb{D}^{m(q-m)}$, so the interval is further subdivided in n_3 smaller intervals where $F_{K,s}$ is an elementary rotation within some $(m+1)$-plane and having a fixed $(m-1)$ axis.

* This is the key for C^0-closeness

* The covering we will choose to be very fine so n_2 is potentially very large.

$\boxtimes$

For Step 3 we will need to manipulate the map close to the wrinkles. This is a consequence of hol. approx:

Lemma 2: Let f_κ, $F_{\kappa,s}$, $\mathcal{U}$, K' as in the Theorem with $F_{\kappa,s}$ graphical. Let $A \subset M \times K$ be a CW-complex of positive codim.

Then, there is a family of isotopies $h_s : A \to M \times K$ a family of embeddings $g_{\kappa,s} : M \to Q$, graphical over f_κ, satisfying $|(g_{\kappa,s}, T g_{\kappa,s}) - (f_\kappa, F_{\kappa,s})| < \varepsilon$ over $Op(h_s(A))$.

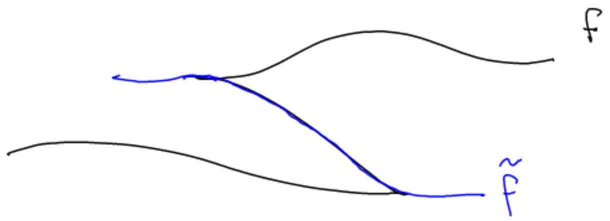
Additionally, $g_{\kappa,s}(p) = f_\kappa(p)$, $h_s(p,\kappa) = (p,\kappa)$ if $s=0$, $p \in \mathcal{U}$, $\kappa \in K'$.

Pf/

We regard $(f_\kappa, F_{\kappa,s})$ as formal maps into the normal bundle of M.

Holonomic approximation yields $g_{\kappa,s}$, h_s with the desired properties. ☒

Remark: a wrinkled embedding is not an embedding. So we cannot just apply Lemma 2 with $A = \{$singular locus of the wrinkle$\}$. However, given f a wrinkled embedding, there is $\tilde{f}$ extending the embedding of the membrane. We apply Lemma 2 to it and then we extend to a wrinkled embedding:



For <u>Step 2</u> the setting is:

$f_\kappa : \mathbb{D}^m \longrightarrow \mathbb{D}^{m+1}$, $\kappa \in \mathbb{D}^k$

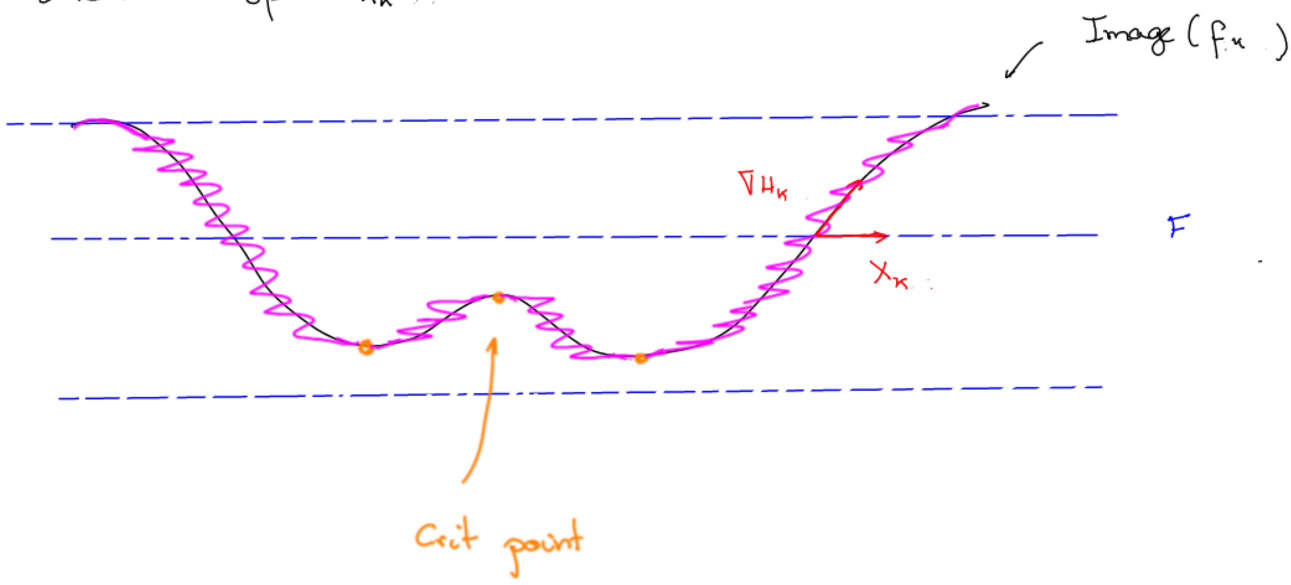
$F_{\kappa,s} : \mathbb{D}^m \longrightarrow Gr(m+1, m)$ with $F_{\kappa,s}(p) = Tf_\kappa(p)$ if $p \in \partial\mathbb{D}^m$ or $\kappa \in \partial\mathbb{D}^k$.

We let $F = F_{0,1}(0) \in Gr(m+1, m)$ and regard it as a linear codim-1 fol in $\mathbb{D}^m$ with height function H. Let

- $H_\kappa = H \circ f_\kappa$, ∇H_κ its gradient.
- $Crit_\kappa = \{$critical points of $H_\kappa\}$
- $X_\kappa \in F$ a vector field normal to $Image(f_\kappa)$.

And we will define $g_{\kappa,s} : \mathbb{D}^m \to \mathbb{D}^{m+1}$, $s \in [0,1)$ wrinkled embedding with $Tg_{\kappa,s}$ approximating F, relative to $\partial\mathbb{D}^m$, $\partial\mathbb{D}^k$ and $\{s=0\}$.

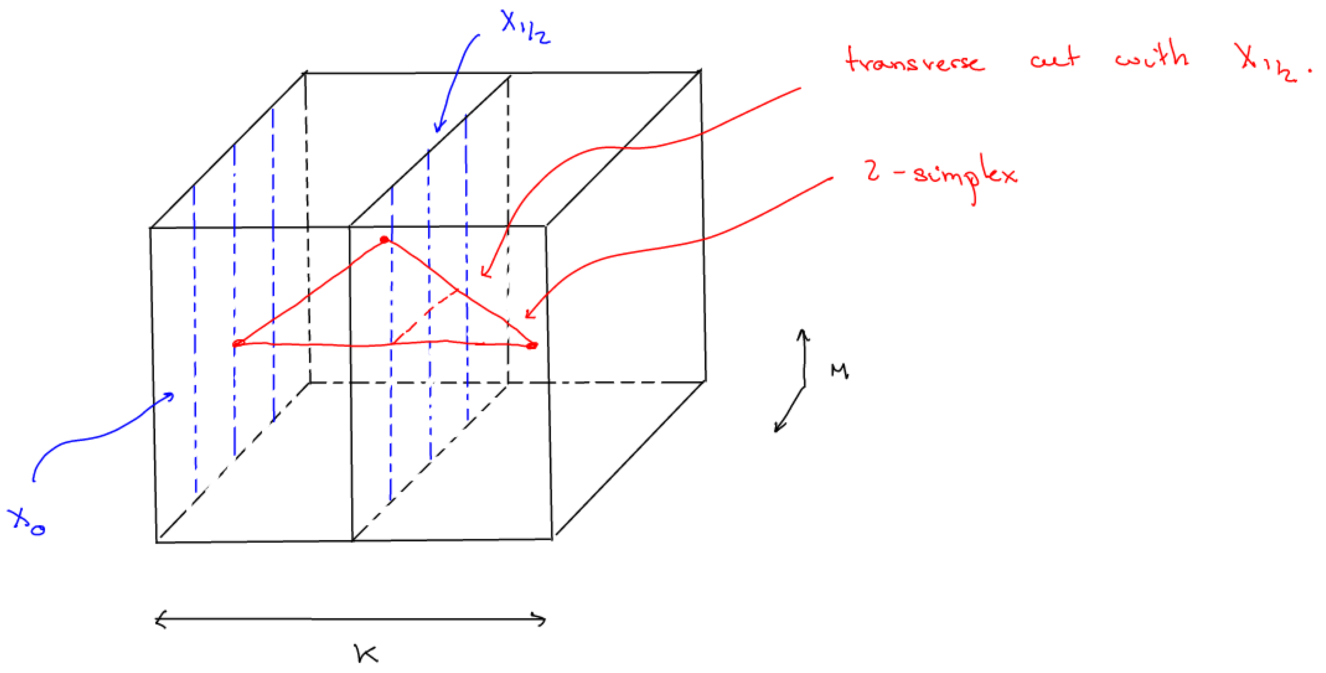
The idea is to wrinkle along the flowlines of ∇H_κ in the direction of X_κ.



In the 1-dimensional case it is clear (we leave the embedding as is near $Crit_\kappa$ and replace the direction $Tf_\kappa(\nabla H_\kappa)$ by X_κ) but the general case is more subtle.

Let us assume, after a small perturbation, that H_k is generic, so Crit_k are isolated points $\forall k$ (generally not of Morse type).

We triangulate $\mathbb{D}^k \times \mathbb{D}^m \setminus (\coprod_k \mathrm{Crit}_k)$ so the simplices are transverse to $\{k\} \times \mathbb{D}^m$ and $\langle X_k \rangle$. (this follows from Thurston's jiggling).



Since the triangulation is transversal to $\langle X_k \rangle$, every simplex has a neighbourhood:

$$O_p(\sigma) \simeq \mathbb{D}^m \times \mathbb{D}^k$$

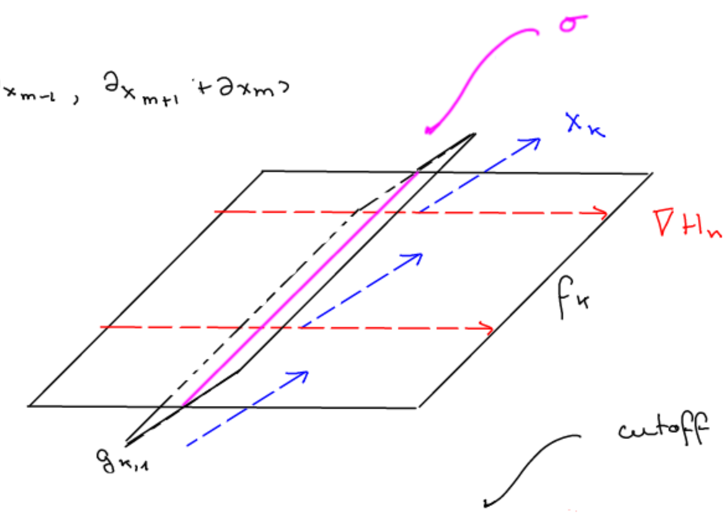
$$\sigma \simeq \{0\} \times \mathbb{D}^a \quad \text{or} \quad \mathbb{D}^a \times \mathbb{D}^k$$

$$f_k|_{O_p(\sigma)}(x_1 \dots x_m, k) = (x_1 \dots x_m, 0) \in \mathbb{D}^{m+1}$$

$$F|_{O_p(\sigma)} = \langle \partial x_1 \dots \partial x_{m-1}, \partial x_{m+1} + \partial x_m \rangle$$

$$\nabla H_k = \partial x_m.$$

$$X_k = \partial x_{m+1} + \partial x_m$$

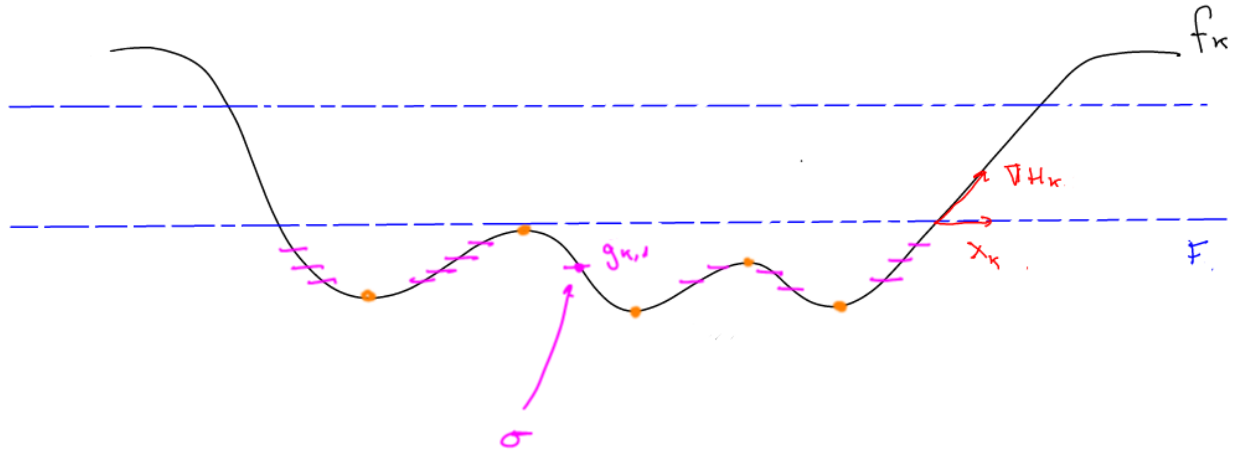


We simply define

$$g_{k,s}(x_1, \dots x_m, k) = (x_1 \dots x_{m-1}, x_m, \rho(x,k) \cdot s x_m) \quad (*)$$

this is done inductively over $\dim(\sigma)$ and relatively to previous steps so little adjusting is needed in the formula $(*)$.

This yields:



This can be done up until the top-dim cells. Now we need to introduce wrinkling. Now we have:

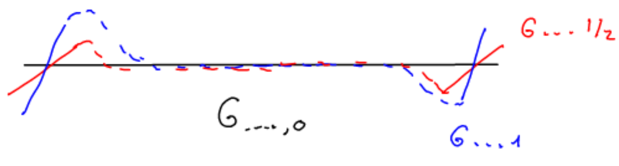
$g_{k,s}$ defined over $\mathbb{D}^{m+k} \times [0,1)$

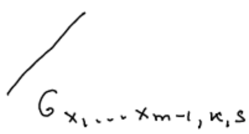
$X_k = \partial_{x_m} + \partial_{x_{m+1}}$; $\nabla H_k = \partial_{x_m}$

$g_{k,s}|_{\partial \mathbb{D}^{m+k}}$ is tangent to $\langle \partial_{x_1}, \dots \partial_{m-1}, \partial_{x_m} + s\,\partial_{x_{m+1}} \rangle$.

As such, we can regard $g_{k,s}$ as a family of 1-dimensional functions (with domains intervals of differing sizes):

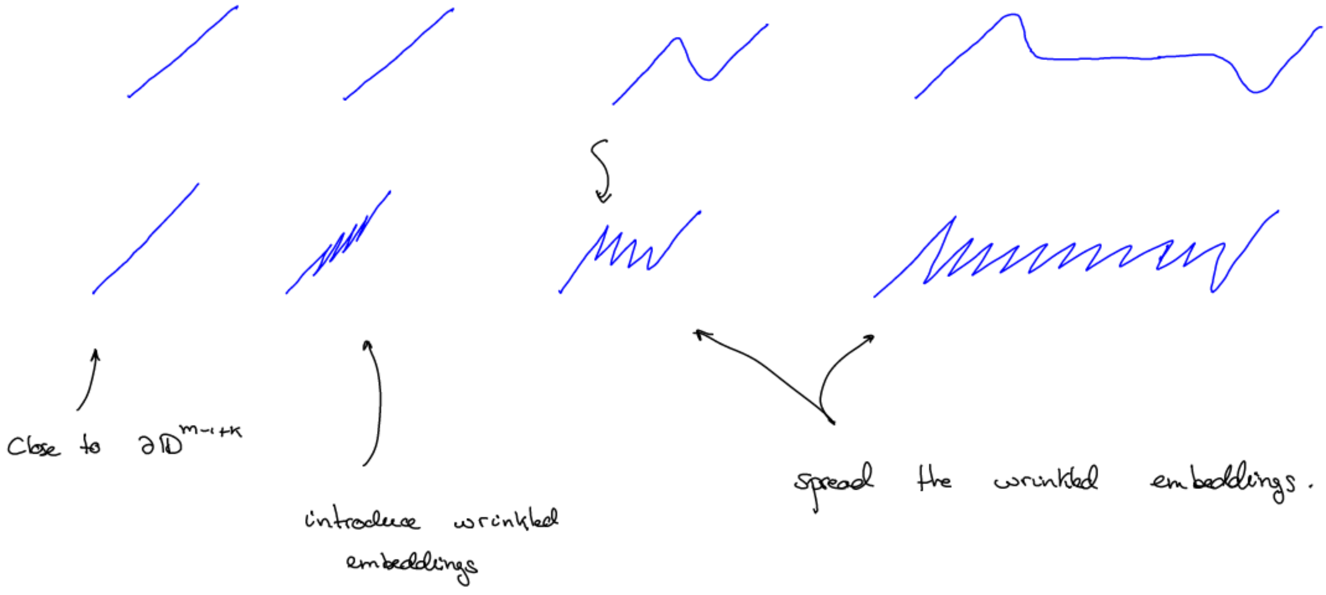
$$G_{x_1,\dots x_{m-1},k,s}(x_m) = g_{k,s}(x_1 \dots x_m),$$





with $(x_1 \dots x_{m-1}, k) \in \mathcal{O}p(\partial \mathbb{D}^{m-1+k})$.

We think of G as a movie of $I \to I^2$ maps that we want to turn into wrinkled embeddings of fixed slope. Observe that this can easily be done with piecewise smooth pieces which we then turn into cusps and swallowtails.



$\square$

This finishes the proof in $[0, 1/N]$. For $[1/N, 2/N]$ first we use Lemma 2 to extend the embedding of the wrinkles to the new interval (so they are wiggled). Then, when we triangulate B_2 we do it <u>relatively</u> to the wrinkles of the previous steps.

For $[i/N, i+1/N]$ the proof is the same. At every step the <u>depth</u> of the wrinkles increases by 1. $\square$

III. Corollaries.

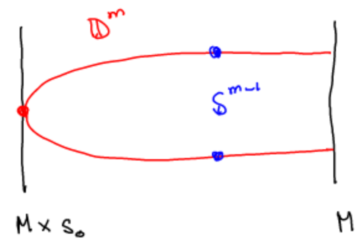
Let us explain now some consequences of our result:

Corollary 1. Let $f: M \to Q$ be an embedding. Let $F_s: M \to Gr(Q, m)$ lifting f. Then there is $g_s: M \to Q$ a topological embedding only with cusp singularities s.t $\| Tg_s - F_s \| < \varepsilon$.

Pf/

Note: we allow for birth and death of pairs of cusps.

The idea is that the swallowtails $\simeq S^{m-2}$ of each wrinkled embedding can be cancelled. Suppose there is a wrinkle that is born at $s = s_0$. Then we regard it as a $\mathbb{D}^m \subset M \times [s_0, 1)$ that cuts $M \times \{s_0\}$ in an embryo and $M \times \{s\}$, $s \geq s_0$, in a wrinkle embedding.



We find $\mathbb{B}$, another $\mathbb{D}^m \subset M \times [s_0, 1)$ cutting $M \times \{s\}$ in a disc whose boundary is the swallowtail of the family of wrinkles. Additionally, $f|_{Op(\mathbb{B})}$ is a submersion away from $\partial \mathbb{B}$. ($\mathbb{B}$ is a little pushoff of the lower hemisphere of the wrinkles).

Then the cusps can be enlarged along $\mathbb{B}$ and get cancelled.

☑

Corollary 2: Let $(Q, \mathcal{F})$ be a foliated manifold. Let $f: M \to Q$ embedding and $F_s: M \to Gr(Q, m)$ with $F_0 = Tf$ and $F_1 \pitchfork \mathcal{F}$. Then there is $g_s: M \to Q$ an embedding with g_1 having only wrinkle type* singularities (or fold type).*

* these wrinkles have domain $Op(S^{m-1})$ and not $Op(\mathbb{D}^m)$, unlike the ones used in the other lecture.

* what this means is that locally in a foliated chart of $\mathcal{F}$ we can project to the leaf space and the singularities are wrinkles / folds.

Pf/ By applying our theorem we obtain $\tilde{g}_1$ wrinkle embedding transverse to $\mathcal{F}$. The wrinkles can be assumed to be small, so their image lies in a fol chart.

The subtlety is (as for wrinkled maps) that projections of wrinkles are not wrinkles. Locally wiggling the wrinkled embeddings yields the result.

Applying Corollary 1 and wiggling the folds gives the other claim. $\boxtimes$