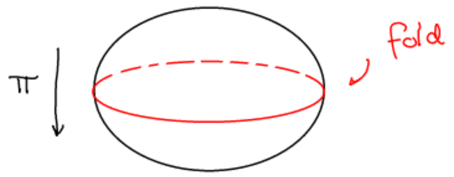
S-immersions

We studied maps between manifolds $M^m \to Q^m$ assuming $FSubm(M,Q) \neq \emptyset$ and concluded that, although $Subm(M,Q)$ might be empty; $WSubm(M,Q) \simeq FSubm(M,Q)$.

However, even if $FSubm(M,Q)$ is empty there might be maps $M \to Q$ with reasonable singularities (which cannot all be wrinkles due to regularisation). Consider

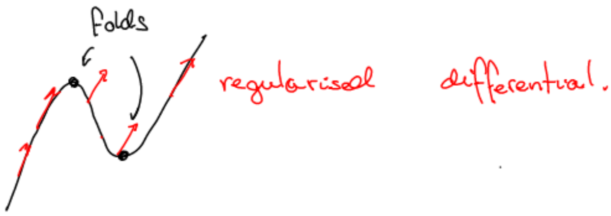
$$\begin{tikzcd} S^2 \arrow[r, phantom, "\subset"] \arrow[dr, dashed, "\pi"'] & \mathbb{R}^3 \arrow[d] \\ & \mathbb{R}^2 \end{tikzcd}$$



the map π has a single fold along the equator (so it is a "simple" singularity) but there are no formal submersions $S^2 \to \mathbb{R}^2$.

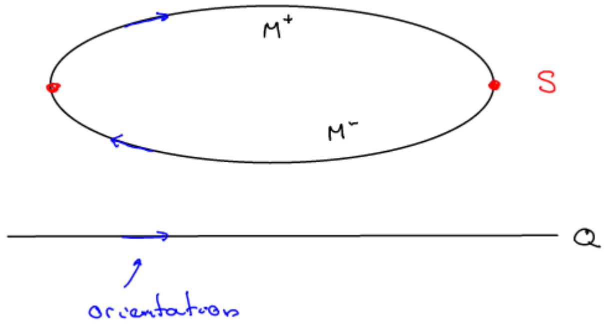
<u>Definition</u>: Let $S^{m-1} \subset M$ be a closed submanifold. Then $SImm(M,Q;S)$ is the space of S-immersions (maps that are immersions in $M \setminus S$ and S is the set of folds)

<u>Obs</u>: two folds in a row can be regularised, but one fold cannot.

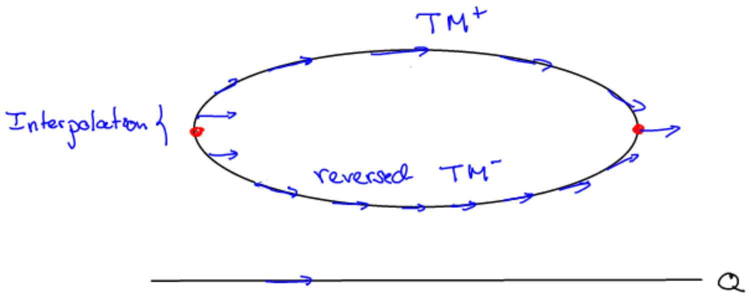


We have to find the formal analogue of $SImm$. The complement of S is divided in two regions*: M^+ and M^-, depending on whether TM is sent to TQ preserving orientation or not.

* I will assume here that everything is orientable for simplicity, but it is not essential.



So we want to glue TM^+ and TM^- in such a way that the resulting bundle maps fibrewise surjectively onto TQ.



Basically, take TM^+, "reverse" TM^- and interpolate using the normal direction close to S.

<u>Formally</u>: Let $\nu \subset TM|_S$ be the direction normal to S in M. Glue TM^+ and TM^- over S by setting:

$$TM^+|_S = TS \oplus \nu \xrightarrow[\mathrm{Id}\oplus(-\mathrm{Id})]{} TS \oplus \nu = TM^-|_S$$

Denote the resulting bundle by $T^S V$, the <u>folded tangent</u> to V.

<u>Obs</u>: It is still an orientable bundle and if we orient it with TM^+, the orientation on TM^- is reversed.

<u>Ex</u>: For $S^n \xrightarrow{\pi} \mathbb{R}^n$ projection, $T^{S^{n-1}} S^n = \pi^* T\mathbb{R}^n$.

<u>Definition</u>: We define the space of formal S-immersions to be

$$FSImm(M,Q;S) = \left\{ (f,F) : \begin{array}{l} F: T^S M \to TQ \text{ surjective} \\ \qquad\quad \downarrow \qquad\;\; \downarrow \\ f: \quad M \;\to\; Q \end{array} \right\}.$$

<u>Q</u>: There is a natural inclusion

$$SImm(M,Q;S) \hookrightarrow FSImm(M,Q;S)$$

is it a w.h.e.?

I. Statement of the result.

Theorem: Assume M closed and connected, $S \neq \phi$, $\dim(Q) \geq 3$.

Then, $\mathcal{S}Imm(M, Q; S)$ can be subdivided by connected components in two sets: $\mathcal{S}Imm^{soft}(M, Q; S)$ and $\mathcal{S}Imm^{taut}(M, Q; S)$.

It holds that the h-principle holds for the first:

$$\mathcal{S}Imm^{soft}(M, Q; S) \underset{w.h.e}{\simeq} F\mathcal{S}Imm(M, Q; S)$$

And the homotopy type of $\mathcal{S}Imm^{taut}(M, Q; S)$ can be easily described (because it also satisfies a form of h-principle).

Observations:

a) We should think soft $\simeq$ overtwisted, taut $\simeq$ tight.

b) Both the h-principle for overtwisted and the h-principle for soft rely on the existence of a flexible model (overtwisted disc in one case, zig-zag in the other).

c) The h-principle for $\mathcal{S}$-immersions is stronger than the contact one (where it is isomorphism in π_0 only). The reason is that, in the contact case, it is not always possible to select a continuous family of o.t discs given a family of o.t structures. For $\mathcal{S}$-immersions there is always a coherent choice of zig-zags.

d) Key idea: o.t discs cannot always be displaced from one another but zig-zags can. $\Rightarrow$ c).

e) In the contact case, studying tight contact structures requires machinery. h-principle is sufficient to completely understand taut $\mathcal{S}$-immersions.

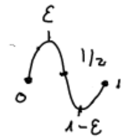
f) This h-principle is closely related to the one for loose legendrians. See below.

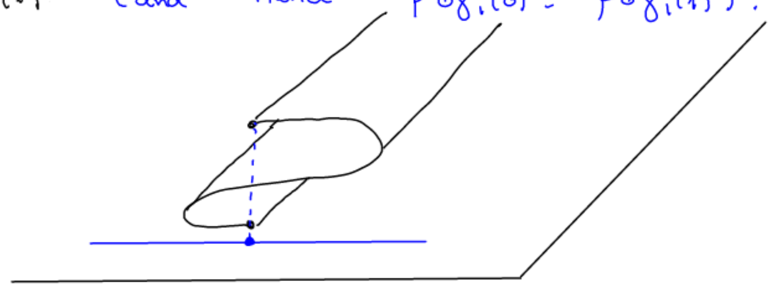
Let us first explain what soft and taut are.

<u>Definition</u>: An S-immersion $M \xrightarrow{f} Q$ is said to be soft if there are paths

- $\gamma_1 : I \to M$ embedded and transverse to S so $\gamma_1(0) \neq \gamma_1(1)$
- $\gamma_2 \; I \to Q$ immersed.

st: $$f \circ \gamma_1 = \gamma_2 \circ \omega$$

where $\omega : I \to I$ is the 1-dimensional wrinkle  with $\omega(0) = \omega(1)$ (and hence $f \circ \gamma_1(0) = f \circ \gamma_1(1)$).

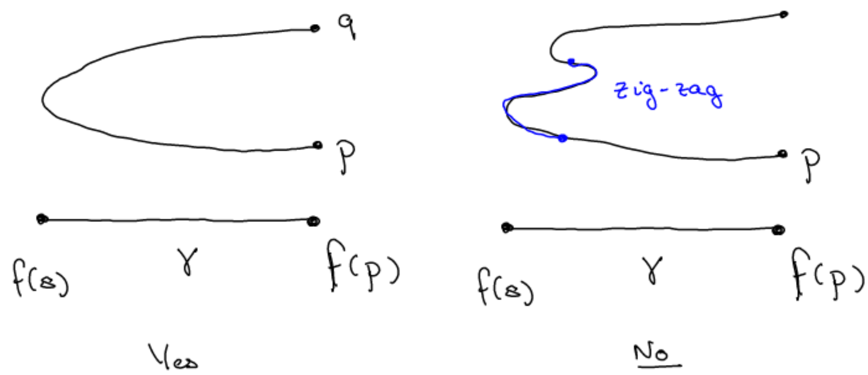


The path γ_1 is called a <u>zig-zag</u>.

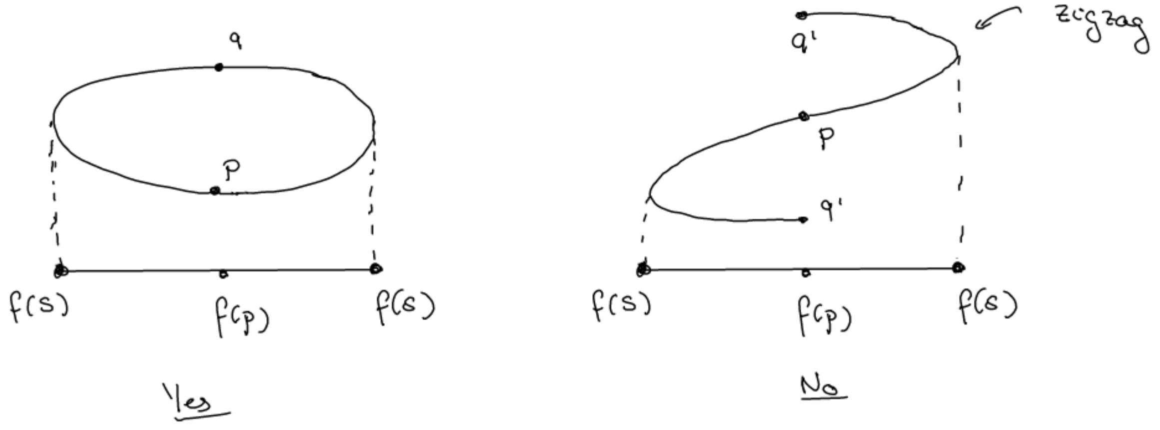
<u>Lemma</u>: $f : M \to Q$ is taut iff there is an involution $\alpha \neq Id : M \to M$ fixing S pointwise with $f \circ \alpha = f$.

<u>Pf</u> / f is taut iff it has no zig-zags. Take $p \in M$ and denote M_0 for the connected component of p in $M \setminus S$.

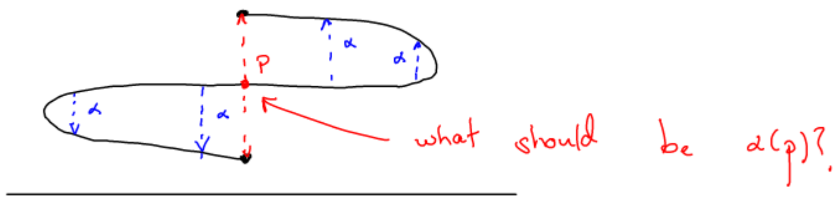
Take $\gamma : I \to f(\overline{M_0}) \subset Q$ connecting $f(p)$ with $f(S)$. There is a single point $q \neq p \in f^{-1}(f(p))$ in the connected component of p in $f^{-1}(\gamma(I))$ (or otherwise there would be a zig zag)



If now we have γ and $\tilde{\gamma}$ as above, they yield the same q (or there would be a zig-zag).



Same reasoning shows that a zig-zag and an involution are incompatible:



$\boxed{/\!/}$

<u>Lemma</u>: being soft is both an open and a close condition. (so being soft / taut depends only on the connected component).

<u>Pf</u> / Having an involution is open.

Having a zigzag is open $\boxed{/\!/}$

II. Proof of h-principle for soft S-immersions

Each connected component of $M \setminus C$ is called a chamber. Let C be a chamber. We define:

$$WSImm(M,Q;S,C) = \left\{ \begin{array}{l} f|_{M \setminus C \,\cup\, Op(S)} \text{ is an } SImm \text{ into } Q \\ f|_C \text{ is a wrinkled submersion} \end{array} \right\}$$

Sketch of proof: (in the case $f_0, f_1 \in SImm^{soft}$, $f_t \in FSImm$)

- the zig-zags of f_0, f_1 are different $\to$ move them to be the same
- modify f_t so that they lie in WSImm and have the same zig-zag.
- use zig-zags to eliminate the wrinkles introduced.

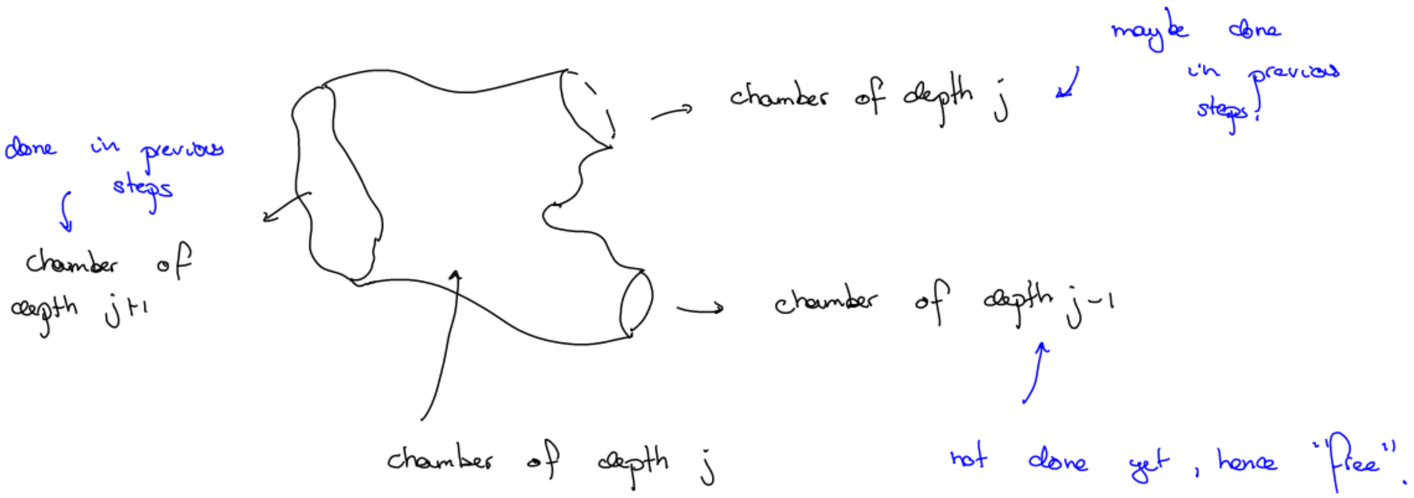
In this proof the zig-zag plays the role of the loose chart for loose legendrians. We will explain the last step using the language of twist-markings (as in Murphy's paper) and Eliashberg and Mischacher' proof (the difference between the two approaches is cosmetic).

Lemma 1: the inclusion given by regularisation is a w.h.e.

$$WSImm(M,Q;S,C) \overset{R}{\hookrightarrow} FSImm(M,Q;S)$$

Pf/ The chambers complementary to C can be assigned a depth depending on how many times one must cross S, starting from C, to get there. Order the chambers by decreasing depth.

Given a family $(\mathbb{D}^k, \partial \mathbb{D}^k) \overset{\varphi}{\hookrightarrow} (FSImm, WSImm)$, representing an element in $\pi_k(FSImm, WSImm)$, we deform φ inductively over the chambers. This can be achieved using the h-principle for immersions in open manifolds (rel. to previous chambers), because at least one of the ends of the chamber has less depth so the deformation is not defined there yet.



In the last chamber (namely C), we use the wrinkled submersions thm. to conclude. $\square$

(The soft part of) the main theorem has been reduced to showing:

$\rightarrow$ $\pi_k(WSImm, SImm) = 0$, i.e. Every family $(\mathbb{D}^k, \partial\mathbb{D}^k) \xrightarrow[\varphi]{} (WSImm, SImm^{soft})$ can be homotoped to lie completely in $SImm^{soft}$.

This can be done in 3 steps:

1) Construct a "good" collection of zig-zags for φ (by modifying φ suitably).
2) Move this collection so that it touches C.
3) Having the zig-zags at C we make them "grow tentacles" to swallow the wrinkles in C.

<u>Lemma 2:</u> there is a collection of disjoint intervals

$$\{\gamma_i : I \to M\}$$

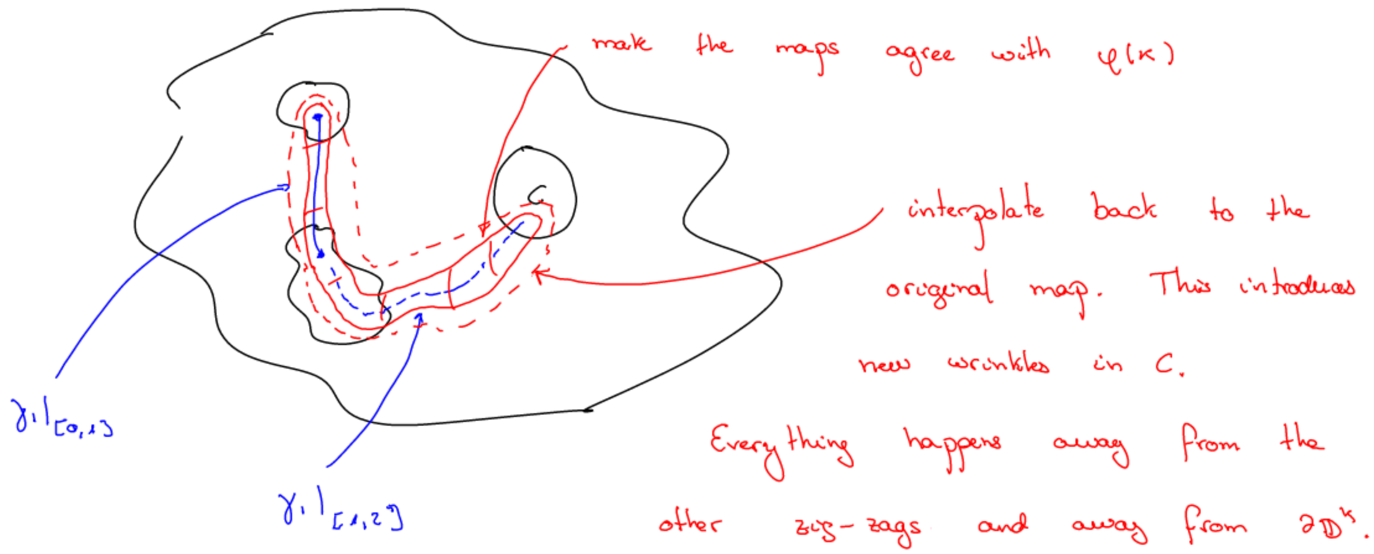
with γ_i a zig-zag for each $\varphi(k)$, $k \in \mathcal{U}_i$, with $\{\mathcal{U}_i\}_{\underset{0}{=}}^{N}$ a cover of $\mathbb{D}^k$.

Pf/ the statement is clearly true for a collection $\{\mathcal{U}_i \subset \mathbb{D}^k\}_{\underset{1}{=}}^{N}$ covering $\partial\mathbb{D}^k$, since each $\varphi(k) \in SImm^{soft}$ has a zig-zag and therefore we can deform the nearby $\varphi(k')$ to have the same one.

Since $\dim(Q) = \dim(M) \geq 3$, we can assume intervals (in particular zig-zags) to be disjoint.

Now consider $\mathcal{U}_0$ so that $\{\mathcal{U}_i\}_0^N$ is a covering of $\mathbb{D}^k$ and $\mathcal{U}_0 \cap \mathcal{U}_1 \neq \phi$. We deform $\varphi|_{\mathcal{U}_0}$ as follows:

Take the zigzag γ_1 and extend it to an immersion $[0,2] \to M$ touching C. Modify $\varphi|_{\mathcal{U}_0}$ to agree with $\varphi(k)$, $k \in \mathcal{U}_1$, in $Op(\gamma_1)$. Since γ_1 touches C, we can use Lemma 1 to make this modification agree with the original $\varphi|_{\mathcal{U}_0}$ outside of $Op(\gamma_1)$.



$\boxed{//}$

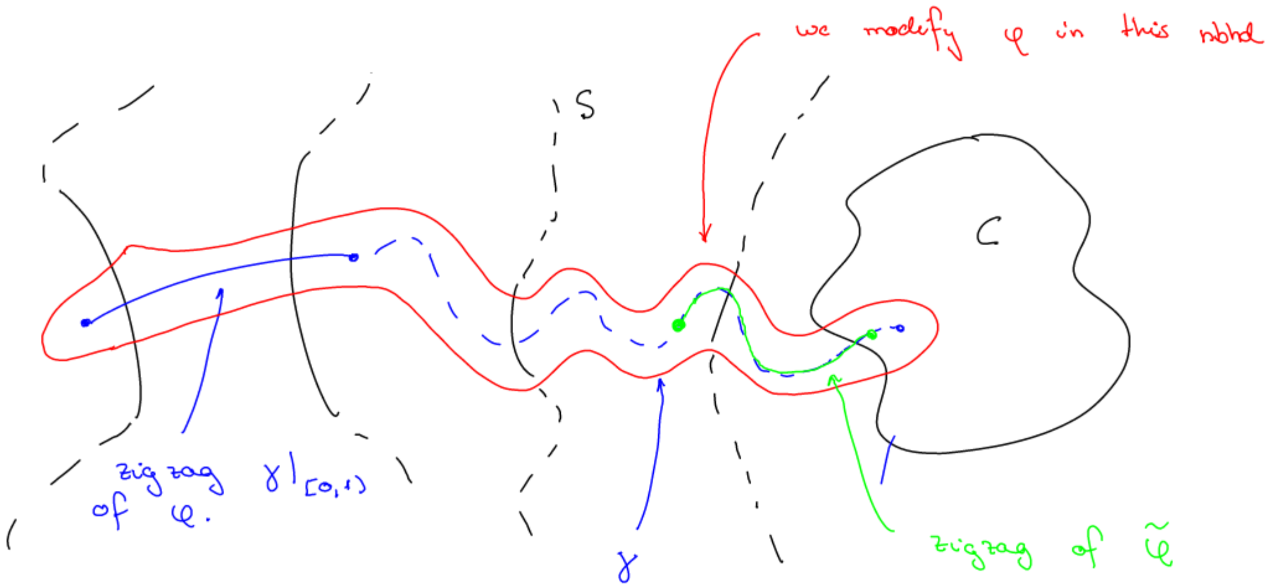
<u>Lemma 3</u>: Given a covering $\{\mathcal{U}_i\}$ with zig-zags $\{\gamma_i\}$, any subcover $\{\tilde{\mathcal{U}}_i\}$ has a choice of zig-zags $\{\tilde{\gamma}_i\}$.

Pf/ take parallel copies of each zig-zag. $\boxed{//}$

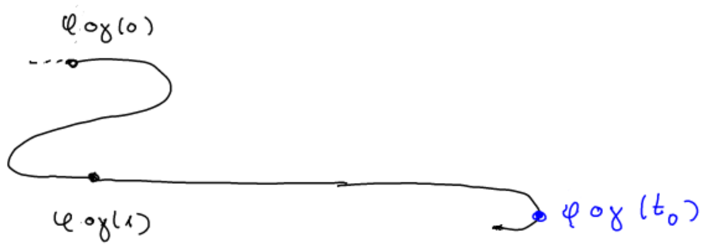
This settles step 1. For step 2:

<u>Lemma 4</u>: Let $\varphi \in WSImm$ and $\gamma : [0,2] \to M$ disjoint from wrinkles with $\gamma(2) \in C$ and $\gamma|_{[0,1]}$ a zigzag. There is $\tilde{\varphi}$ homotopic to φ with $\tilde{\varphi}|_{M \setminus Op(\gamma)} = \varphi$ having a zigzag touching C and with domain contained in γ.

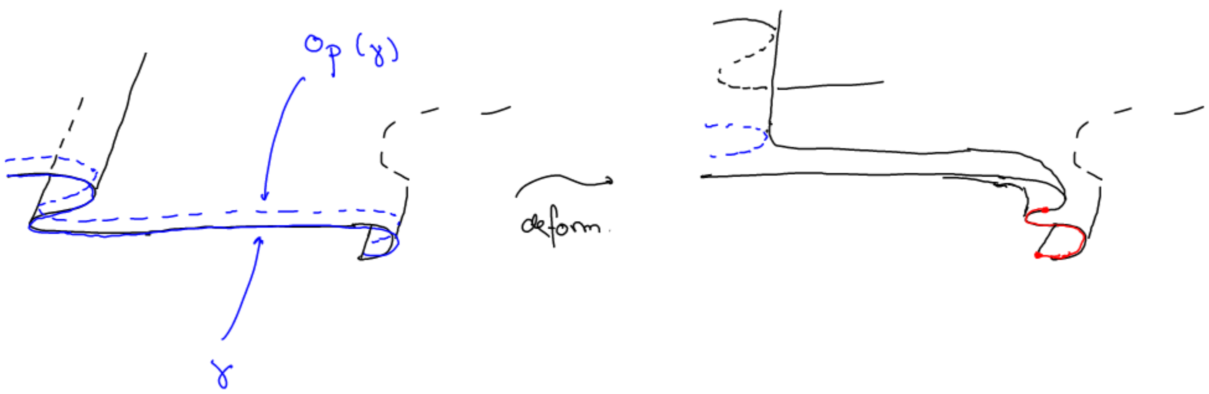
<u>Pf</u> /



Let $t_0 > 1$ be the first time where $\gamma(t_0) \in S$. Then, the map $\varphi \circ \gamma|_{[0, t_0+\varepsilon]}$ onto its image looks like:



If we thicken $\varphi\circ\gamma([0, t_0+\varepsilon])$ to a $\dim(M)-1$ family of immersed curves, it looks like this over each of them. Modify φ as follows, relative to the boundary of the model:



Iterating this argument yields the claim. $\boxtimes$

Combining Lemmas 2, 3, 4 we have that, after a deformation, our family $\varphi : (\mathbb{D}^k, \partial\mathbb{D}^k) \to (WSImm, SImm)$ admits a covering $\{\mathcal{U}_i\}$ of $\mathbb{D}^k$ with disjoint intervals $\{\gamma_i : I \to M\}$ touching C st γ_i is a zig-zag for each $\varphi(k)$, $k \in \mathcal{U}_i$.

(Lemma 3 was necessary so that the deformation from Lemma 4 could be applied in little balls $\mathcal{U}_i$ where φ was almost constant.)

Let us describe now how the wrinkles are eliminated. The following lemma corresponds to the approach in EM:

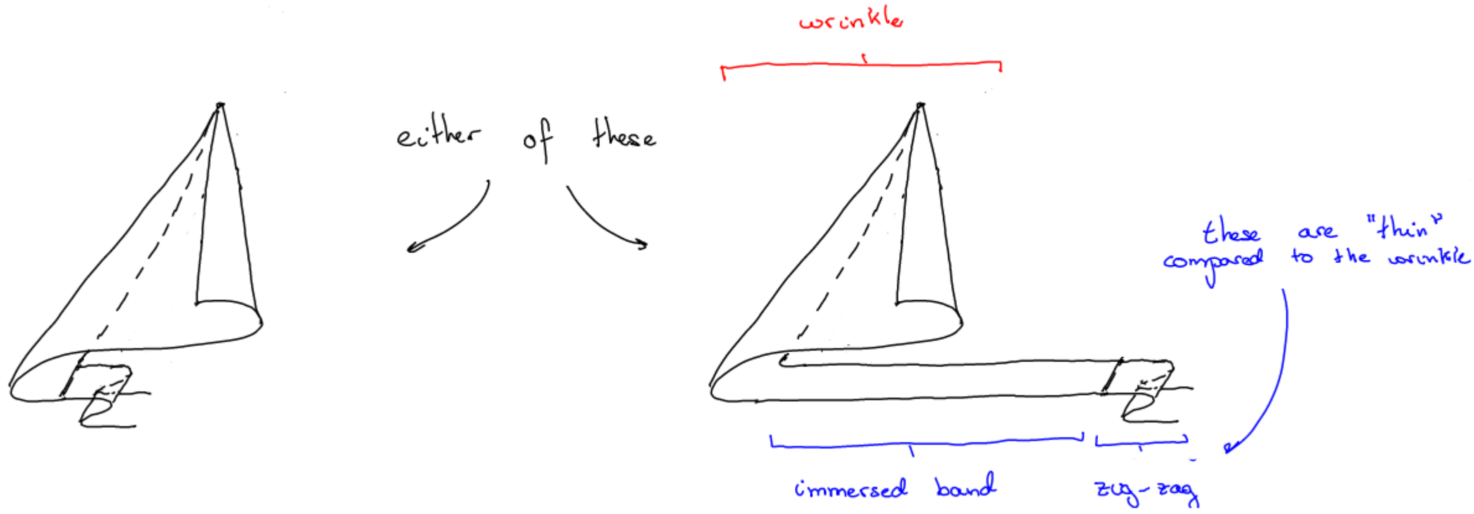
<u>Lemma 5</u> (Swallowing of wrinkles)

Let $\varphi \in WSImm(M, Q; S, C)$. Let $D \subset C$ be the membrane of a wrinkle of φ. Let $\gamma : [0,2] \to M$ be a path st:

- $\gamma|_{[0,1)}$ is a zig-zag, $\gamma|_{[1,2]} \subset C$.
- $\gamma(2) \in D$ and $\gamma|_{[0,2)}$ is disjoint from the wrinkles.

$\to$ φ can be deformed to $\tilde{\varphi} \in WSImm(M, Q; S, C)$ in $Op(D \cup \gamma)$ so that $\tilde{\varphi}$ has one less wrinkle.

Pf/ This is very much like Lemma 2. A neighbourhood of $Op(D \cup \gamma)$ looks as follows:



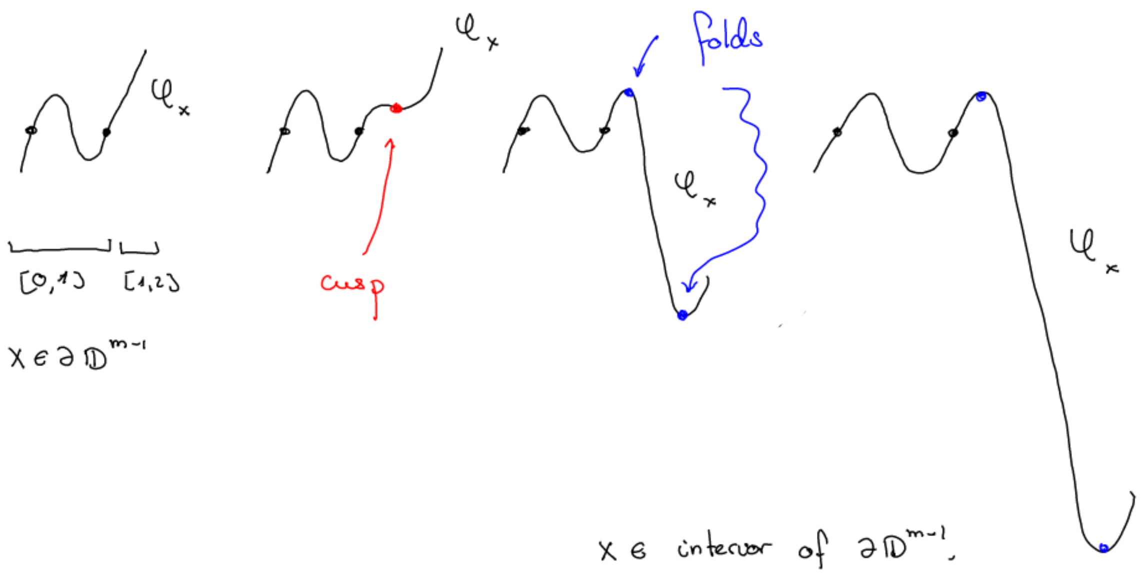
Our claim is that, after a deformation, we can obtain a model where we have that φ can be written as:

$$\mathbb{D}^{m-1} \times [0,2) \longrightarrow \mathbb{D}^{m-1} \times [0,1]$$

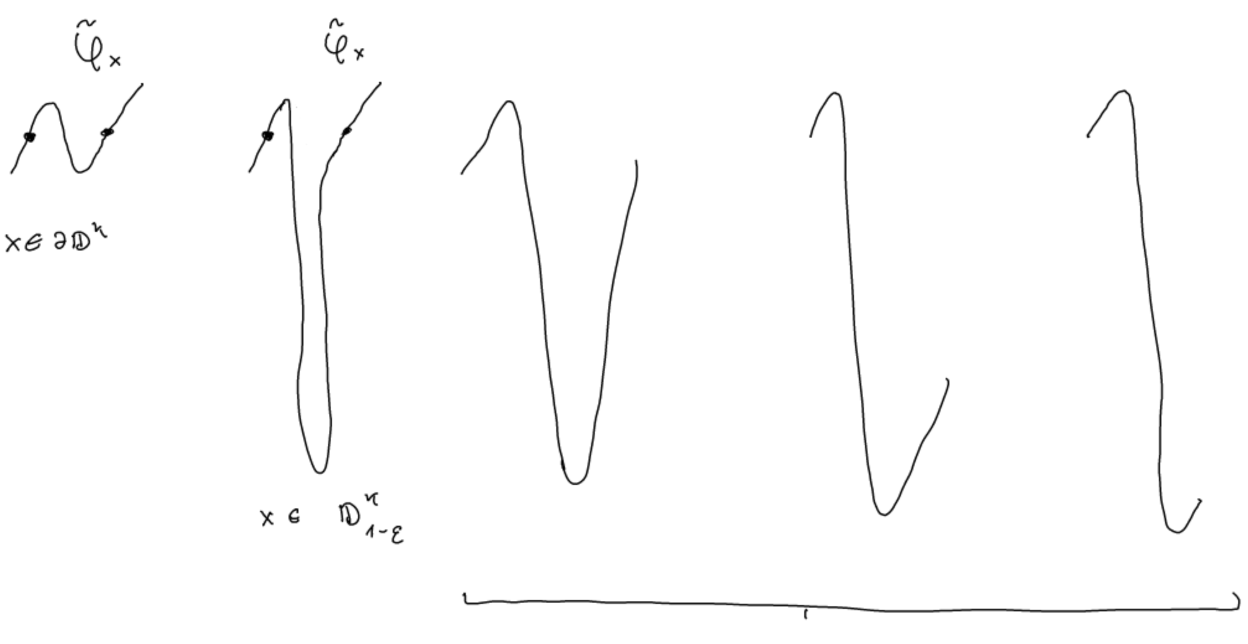
$$\varphi(x_1 \dots x_{m-1}, z) = (x_1 \dots x_{m-1}, \varphi_x(z))$$

with

- $\varphi|_{[0,1)}$ containing a zig-zag
- $\varphi|_{[1,2)}$ a wrinkle:

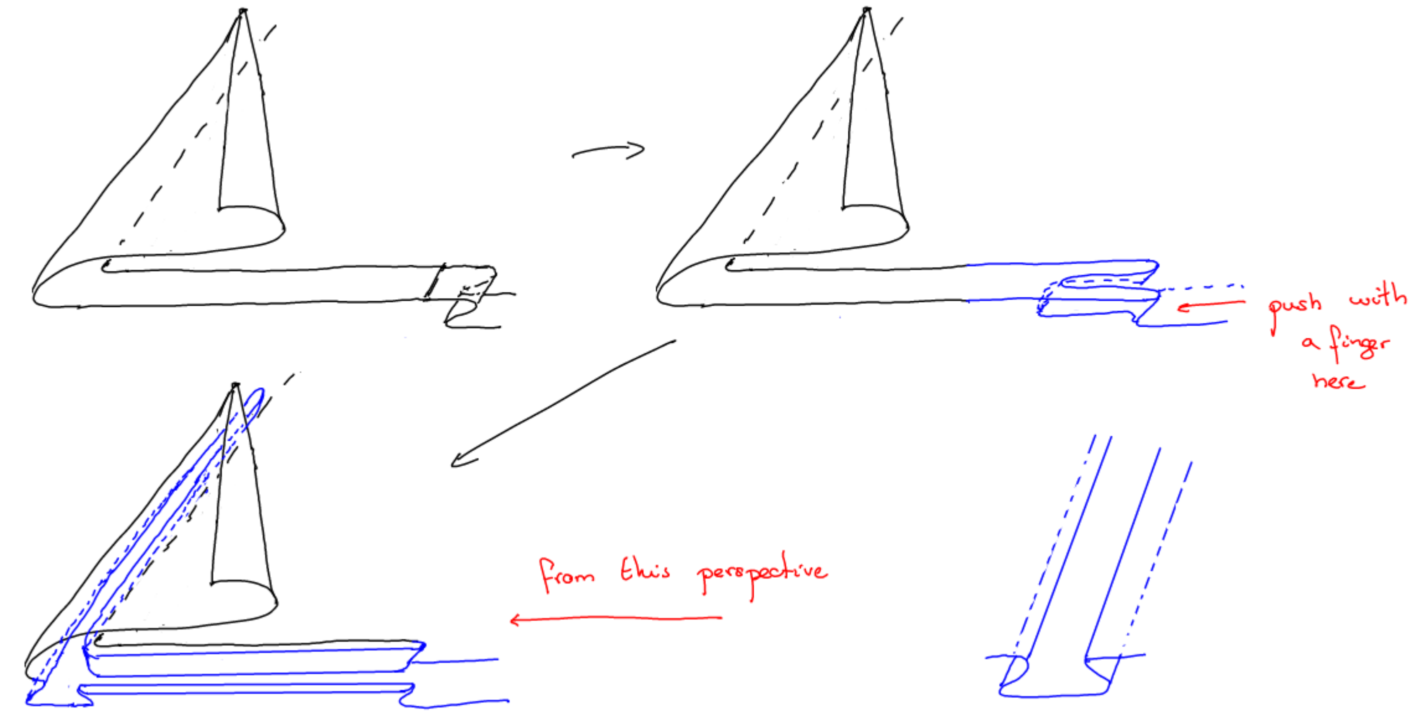


Because then:



the zigzag is gone but the wrinkle too.

So how do we get to this model situation? We make the zig-zag grow a tentacle along $\gamma|_{[1,2]}$ (similar to Lemma 2):

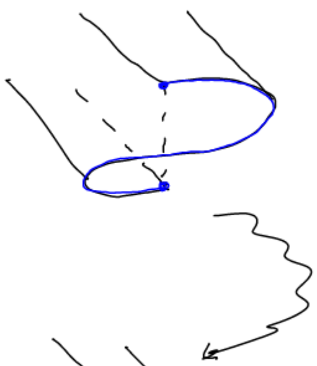


in this area we have the beginning of the standard model $\square$

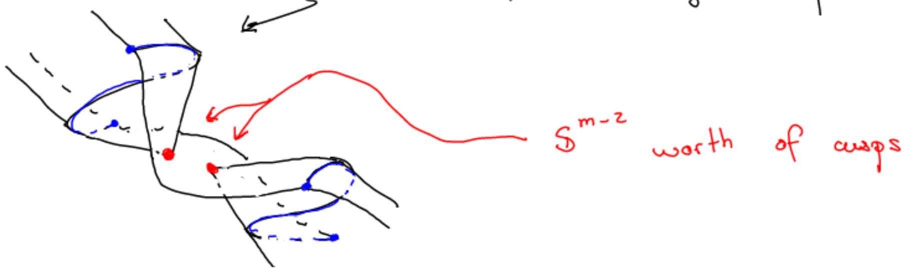
Observation: the proof we have just given is not parametric. We will give a few more details about the parametric case now.

Let us describe now the proof of Lemma 5 using the language of twist markings. This will also clarify a bit more the role of the zig-zag.

Consider the zigzag:



We can collapse the image towards the dotted line, introducing cusps:



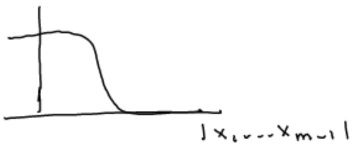
We can call this model the "inside - out" wrinkle. Some remarks:

- Since every zig-zag can be thickened to a I^{m-1} family of zig-zags, we can introduce arbitrarily many inside-out wrinkles.
- In the local model

$$I^m \to I^m$$

$$(x_1, \ldots x_m) \to (x_1, \ldots x_{m-1}, \overbrace{w(x_m)}^{\text{zig-zag}})$$

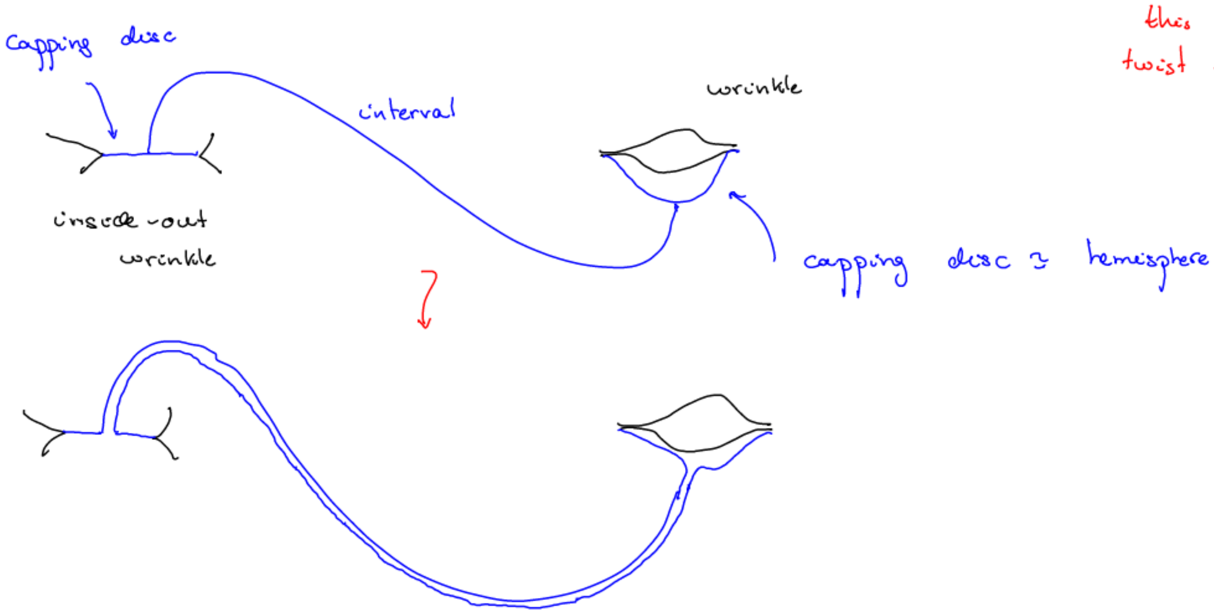
what we do is add $\chi(x_1, \ldots x_{m-1})\, x_m$ to $w(x_m)$, with

χ 

. This cancels the critical points of $w(x_m)$ over some ball $B_\varepsilon(x_1, \ldots x_{m-1})$ whose boundary is the sphere of cusps.

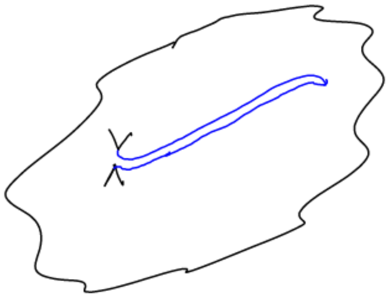
<u>Introducing twist markings</u>: given an inside-out wrinkle and some wrinkle we want to cancel, we find a tube $S^{m-2} \times I$ connecting the corresponding sphere of cusps. This is possible because both spheres have capping disks that can be connected by an interval:

this is the twist marking.

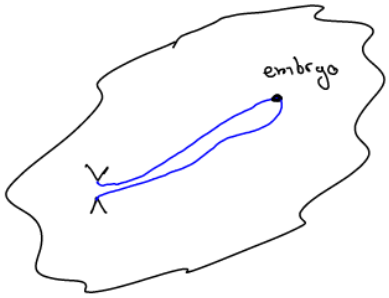


One can just introduce a pair of cusps along the tube $S^{m-1} \times I$ to cancel the cusps, yielding one wrinkle less.

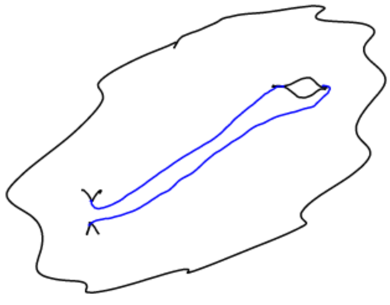
Let us describe pictorially the parametric argument:



At $t = t_0 - \varepsilon$ we spread the capping disc of the inside-out wrinkle

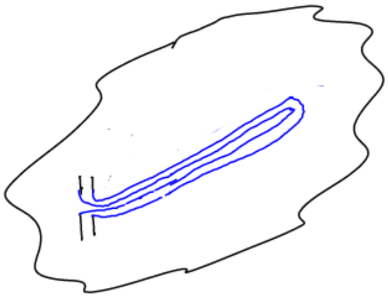


At $t = t_0$ a wrinkle embryo appears. The disc capping of the inside-out wrinkle is there.

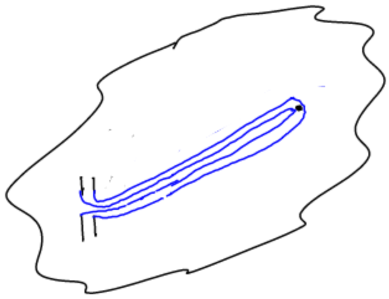


For $t > t_0$ the capping disc has morphed into a cylinder connecting the cusps of the wrinkle and the inside out wrinkle.

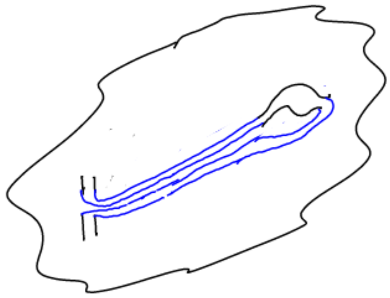
resolve the twist marking



At $t = t_0 - \varepsilon$ a tentacle grows out of the zig-zag



At $t = t_0$ the tentacle plays the role of the embryo



At $t > t_0$ the tentacle evolves as the original wrinkle would have.

One of the subtleties of the argument is how different zig-zags interact. A key part of the argument is that, by chopping the wrinkles, they can be assumed to be short lived in the parameter. We won't describe the chopping again. The point is that the parameter regions where the zig-zags are defined have larger overlaps than the parameter regions where wrinkles are alive:

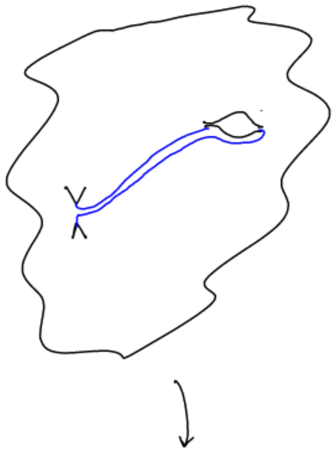
Situation before chopping:

At t_0, a first zig-zag is swallowing a given wrinkle

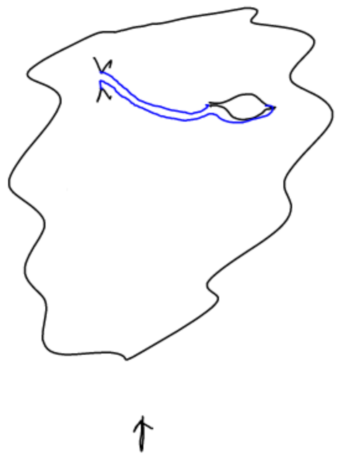
who swallows the wrinkle here?

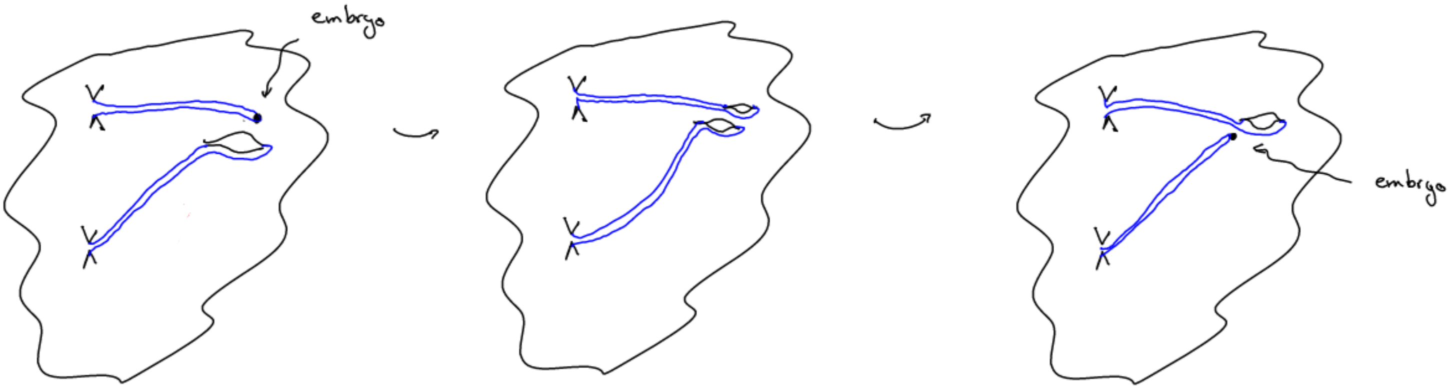
At t_1, the first zig-zag is gone and the second swallows the same wrinkle.

chopping



Now the wrinkle has been chopped into two in time. Each zig zag deals with one.





III. Taut S-immersions

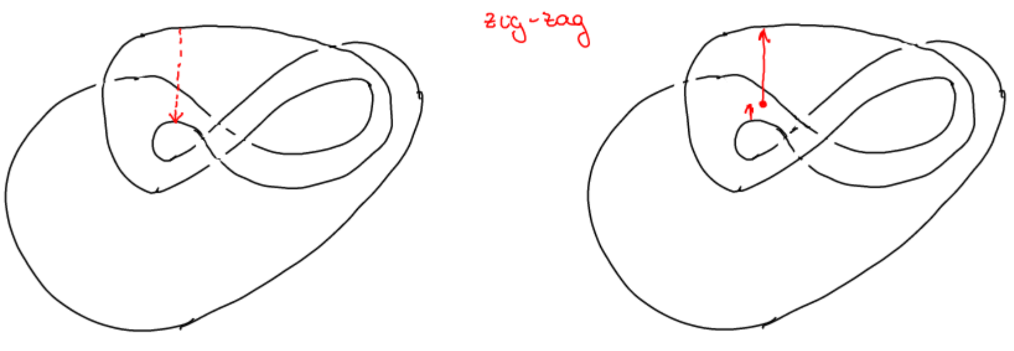
Since a taut S-immersion f has an involution, it is sufficient to understand $f: M/_\alpha \to Q$. $M/_\alpha \cong M^+$ is an open manifold so we can apply hol. approx.

Then: $SImm^{taut}(M, Q; S) \simeq Imm(M^+, Q) \times Involutions(M, S)$ $\boxed{//}$

IV S-immersions $S^2 \to \mathbb{R}^2$

Final example: There are two components in $SImm(S^2, \mathbb{R}^2; S^1)$

- $SImm^{taut}$ contains the projection
- $SImm^{soft}$ contains



immersion of $S^2 \cap \{z \geq 0\}$

immersion of $S^2 \cap \{z \leq 0\}$

Both components have $\pi_1 \simeq \mathbb{Z}$ as their only non-zero homotopy group:

$$SImm^{soft} \simeq \text{Bundle Iso}_{S^2}(\underline{\mathbb{R}^2}, \underline{\mathbb{R}^2}) = \{S^2 \to Gl(2)\} \simeq Gl(2)$$

$$SImm^{taut} \simeq \text{Bundle Iso}_{D^2}(\underline{\mathbb{R}^2}, \underline{\mathbb{R}^2}) \times Diff(D^2, \partial D^2) \simeq Gl(2)$$

$\boxed{//}$