CLASSIFICATION OF GENERIC DISTRIBUTIONS THROUGH CONVEX INTEGRATION

JAVIER MARTÍNEZ-AGUINAGA AND ÁLVARO DEL PINO

Abstract. This paper tackles the classification of tangent distributions up to homotopy. We focus on distributions satisfying some non-degeneracy constraint in terms of the Lie bracket. Such a constraint implies being bracket-generating but is often stronger.

The main technique that we use is Gromov's convex integration. Our main result states that hyperbolic distributions of rank 4 in dimension 6 satisfy the complete h-principle. Whether similar results hold for other families of non-degenerate distributions is left to future work.

Unlike other differential relations, the constraint defining (4,6) hyperbolic distributions fails to be ample in some principal directions. Due to this, it is not immediate to verify that convex integration applies. Our main contribution is a scheme proving that this is the case. The intuitive idea is that, in some cases, it is sufficient for ampleness to hold in most principal directions.

Contents

1. Introduction

.

2. Bracket–generating distributions

In order to set notation, we recall now some of the key definitions to be studied in this article.

We fix a smooth manifold endowed with a distribution $\mathcal{D}$. Then:

Definition 2.1. *The **(fast) Lie flag** associated to $\mathcal{D}$ is defined iteratively by the formula:*

$$\mathcal{D}^1 := \mathcal{D}, \qquad \mathcal{D}^{k+1} := [\mathcal{D}^k, \mathcal{D}^k].$$

These are, in general, C^∞-modules of vector fields but we will henceforth assume that they have constant rank and define thus a flag of distributions:

$$\mathcal{D}^1 \subset \mathcal{D}^2 \subset \cdots \subset \mathcal{D}^i \subset \cdots \subset TM.$$

We additionally assume that $\mathcal{D}$ is **bracket-generating**, i.e. there exists some i_0 such that $\mathcal{D}^{i_0} = TM$.

2010 *Mathematics Subject Classification.* Primary: 53D10. Secondary: 53D15, 57R17.

2.1. **The dual picture.** Given the subbundle $\mathcal{D} \subset TM$, we denote its annihilator by:

$$\mathcal{D}^{\perp} := \{\alpha \in T^*M \mid \alpha(u) = 0 \, \forall u \in \mathcal{D}\}.$$

Furthermore, we denote by $\mathcal{I}_{\mathcal{D}} \subset \Omega^*(M)$ the ideal of differential forms generated by $\mathcal{D}^{\perp}$, namely:

$$\mathcal{I}_{\mathcal{D}} := \{\omega \in \Omega^*(M) \mid \exists \beta \in \mathcal{D}^{\perp} \text{ s.t. } \omega \wedge \beta = 0\}.$$

Consider the composition of maps

$$\pi \circ d : \mathcal{I}_{\mathcal{D}} \xrightarrow{d} \Omega^*(M) \xrightarrow{\pi} \Omega^*(M)/\mathcal{I}_{\mathcal{D}},$$

where d is the exterior differential and π is the quotient mapping.

Definition 2.2. *The subspace* $\mathcal{I}^2_{\mathcal{D}} := \ker(\pi \circ d)$ *is called the* ***derived ideal*** *of* $\mathcal{I}_{\mathcal{D}}$.

Inductively, we set $\mathcal{I}^{k+1}_{\mathcal{D}}$ to be the derived ideal of $\mathcal{I}^k_{\mathcal{D}}$. Using the Cartan formula we deduce:

Lemma 2.3. *The derived ideal* $\mathcal{I}^{k+1}_{\mathcal{D}}$ *is generated by*

$$\ker(\pi \circ d|_{\mathcal{I}^k_{\mathcal{D}} \cap \Omega^1(M)}) = (\mathcal{D}^{k+1})^{\perp}.$$

That is, $\mathcal{D}$ is bracket-generating in i_0 steps if and only if $\mathcal{I}^{i_0}_{\mathcal{D}} = 0$.

Remark 2.4. *Equivalently, a distribution* $\mathcal{D} := \{\alpha_1, \cdots, \alpha_{n-k}\}$ *is bracket–generating if and only if the curvatures* $d\alpha_i$ *are linearly independent as sections of* $\Omega^2(M)/\mathcal{I}_{\mathcal{D}}$.

2.2. **The pfaffian and degenerate differential forms.** For each pair (k, n) of rank and dimension, we can consider subclasses of distributions satisfying some non-degeneracy/genericity condition. We now introduce maps in the space of forms that measure this.

Consider a local coframe $\langle \beta_1, \cdots, \beta_k \rangle$ for $\Omega^1(\mathcal{D})$. Then a differential two-form restricted to the distribution $\omega \in \Omega^2(\mathcal{D})$ can be expressed as $\omega = \sum a_{i,j} \beta_i \wedge \beta_j$, where the coefficients $a_{i,j}$ form a skew symmetric matrix $M_\omega := (a_{i,j})^n_{i,j=1}$.

Definition 2.5. *We define the dual curvature map as*

$$\begin{array}{rccc} \omega\colon & \mathcal{D}^{\perp} & \longrightarrow & \Omega^2(\mathcal{D}) \\ & \alpha & \longmapsto & d\alpha|_{\mathcal{D}} \end{array}$$

Remark 2.6. *Note that* $d\alpha|_{\mathcal{D}}(X, Y) = -\alpha([X, Y])$ *for any two horizontal vector fields* X, Y.

We now consider the map from the space of two-forms along $\mathcal{D}$ to the space of $\lfloor \frac{k}{2} \rfloor$-forms along $\mathcal{D}$, defined by:

$$\begin{array}{rccc} p\colon & \Omega^2(\mathcal{D}) & \longrightarrow & \Omega^{2\lfloor \frac{k}{2} \rfloor}\mathcal{D} \\ & \beta & \longmapsto & \beta^{\lfloor \frac{k}{2} \rfloor}. \end{array}$$

Definition 2.7. *We define the* ***Pfaffian*** $Pf : \mathcal{D}^* \to \Omega^{2\lfloor \frac{k}{2} \rfloor}(\mathcal{D})$ *as the composition* $Pf = p \circ \omega$*; i.e.*

$$\begin{array}{rccccc} Pf\colon & \mathcal{D}^* & \longrightarrow & \Omega^2(\mathcal{D}) & \longrightarrow & \Omega^{2\lfloor \frac{k}{2} \rfloor}(\mathcal{D}) \\ & \alpha & \longmapsto & d\alpha|_{\mathcal{D}} & \longmapsto & (d\alpha|_{\mathcal{D}})^{\lfloor \frac{k}{2} \rfloor}. \end{array}$$

Note that a two-form gets mapped to a top form in $\mathcal{D}$ via p when the rank of the distribution is even, and to a codimension-1 form if k is odd.

Definition 2.8. *A differential* $2-$*form* β *in* $\mathcal{D}$ *is said to be* ***degenerate*** *if* $p(\beta) = 0$.

Remark 2.9. *The space of degenerate forms is defined as the level set* $\mathcal{C} := p^{-1}(0)$*. As such, it has codimension* 1 *inside the space* $\Omega^2(\mathcal{D})$ *if* k *is even, and codimension* k *if* k *is the odd.*

2.3. $4-$**distributions.** In this Subsection we focus on 4-distributions. Since $k = 4$ is even, the target of p is a $1-$dimensional line bundle. Assuming orientability (which we can do locally) we fix a volume form in $\mathcal{D}$. Then, we can regard the map p as a real quadratic form on $\Omega^4(\mathcal{D})$ and study its signature, which does not depend on the choice of volume form [17]:

Proposition 2.10. *The real quadratic form p has signature $(3, 3)$.*

Proof. Take a local frame $\mathcal{D}^* = \langle \beta_1, \beta_2, \beta_3, \beta_4 \rangle$ compatible with the chosen orientation. Define now the space of self-dual forms $\bigwedge^+(\mathcal{D})$ and the space of anti self-dual forms $\bigwedge^+(\mathcal{D})$ as follows:

$$\overset{+}{\bigwedge}(\mathcal{D}) = \langle a_1 = \beta_1 \wedge \beta_2 + \beta_3 \wedge \beta_4, a_2 = \beta_1 \wedge \beta_3 + \beta_4 \wedge \beta_2, a_3 = \beta_1 \wedge \beta_4 + \beta_2 \wedge \beta_3 \rangle \subset \Omega^2(\mathcal{D})$$

$$\overset{-}{\bigwedge}(\mathcal{D}) = \langle b_1 = \beta_1 \wedge \beta_2 - \beta_3 \wedge \beta_4, b_2 = \beta_1 \wedge \beta_3 - \beta_4 \wedge \beta_2, b_3 = \beta_1 \wedge \beta_4 - \beta_2 \wedge \beta_3 \rangle \subset \Omega^2(\mathcal{D}).$$

A straightforward computation shows that the matrix associated to the bilinear form p for the basis $\langle a_1, a_2, a_3, b_1, b_2, b_3 \rangle$ of $\Omega^2(\mathcal{D})$ consists of an upper-left $Id_{3\times 3}$ identity block and another $-Id_{3\times 3}$ in the right-down corner. This proves the claim. ☐

Since $\mathcal{D}$ is bracket–generating, the dual curvature ω maps $\mathcal{D}^*$ injectively into $\Omega^2(\mathcal{D})$ and we can talk about its signature:

Definition 2.11. *The signature of a distribution $\mathcal{D}$ is the signature of the quadratic form Pf.*

Two remarks are in order. First: the signature is well-defined only once a volume form on $\mathcal{D}$ has been chosem. Otherwise, we cannot distinguish the signatures (i, j, k) and (j, i, k). Furthermore, the signature of $\mathcal{D}$ may vary from point to point.

We focus on the case of $(4, 6)-$distributions:

Definition 2.12. *A bracket–generating 4-distribution in a $6-$dimensional manifold M is said to be **non-degenerate** if the following condition is satisfied:*

$$\omega(\mathcal{D}*) \pitchfork p^{-1}(0) \subset \Omega^2(\mathcal{D})$$

This condition says that Pf is non-degenerate on $\mathcal{D}$ and thus the signature of the distribution is independent of the point. Depending on the nature of this intersection we distinguish two different types of $(4, 6)$-distributions:

Definition 2.13. *We say that a non-degenerate $(4, 6)$-distribution is*

i) ***elliptic*** *if $\omega(\mathcal{D}^*) \cap p^{-1}(0) = 0$.*
ii) ***hyperbolic*** *if $\omega(\mathcal{D}^*) \cap p^{-1}(0) \neq 0$.*

Since p has signature $(3, 3)$ and $\mathcal{D}^*$ has dimension 2, there are, up to sign, four possible signatures for $\mathcal{D}$: $(0, 0, 2), (1, 0, 1), (1, 1, 0)$ and $(2, 0, 0)$. Only the cases $(2, 0)$ and $(1, 1)$ are generic, which correspond to the elliptic and hyperbolic cases, respectively.

2.4. **Formal bracket–generating distributions.** We now define formal bracket–generating distributions as we will use them later on. We adopt the viewpoint of 1-forms and thus describe a distribution in terms of its annihilator. This description is valid locally, which is sufficient for our h-principles.

We first consider the following commutative diagram, where $\mathcal{F}$ denotes the fiber of $J^1(T^*M)$ over T^*M, p_1 is the natural projection of $J^1(T^*M) \to T^*M$, and $\pi : T^*M \to M$ is the natural projection onto the manifold. Then, $g = \pi \circ p_1$.

$$\begin{array}{ccc} J^1(T^*M) & \xrightarrow{\phi} & \mathcal{F} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ T^*M & \xrightarrow[\pi]{} & M \end{array}$$

On the other hand, consider the following commutative diagram, where A represents the antisymmetrization morphism; i.e. $A \circ \phi(J^1(\alpha)) = d\alpha$ and q is the natural projection.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{A} & \Omega^2(M) \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle q} & \\ & M & \end{array}$$

We then write:

Definition 2.14. *Fix a smooth n-manifold M. A smooth section φ of the bundle $\oplus^{n-k} J^1(T^*M) \to M$ is a **formal 2-step bracket–generating (k, n)-distribution** if it satisfies:*

- *$p_1 \circ \varphi_1 \wedge \cdots \wedge p_1 \circ \varphi_{n-k}$ is a non-vanishing $n-k$-form. We write Γ for the span $\langle p_1 \circ \varphi_1, \cdots, p_1 \circ \varphi_{n-k} \rangle$.*
- *The 2-forms $\{A \circ \varphi_1, \cdots, A \circ \varphi_{n-k}\}$ are linearly independent in $\Omega^2(M)/\Gamma$.*

We will write $\mathrm{Dist}^f_{(k,n)}(M)$ for the space of such formal distributions. The subspace of holonomic ones is denoted by $\mathrm{Dist}_{(k,n)}(M)$.

For the purposes of studying distributions, we see that whenever we consider a smooth section $\varphi : M \to J^1(T^*M)$, we do not need their full first order data, but only the corresponding 2-form $\omega := A \circ \phi(\varphi)$. Henceforth, for ease of notation and readability and unless explicitly stated otherwise, we write

$$\varphi = (\alpha_i := p_1 \circ \varphi_i, \beta_i := A \circ \varphi_i).$$

Remark 2.15. *The space of formal $(k, n)-$bracket–generating distributions is homotopically equivalent to the space of systems of $n - k$ sections $(\alpha_i, \beta_i) : M \to \Omega^1(M) \oplus \Omega^2(M)$ such that $\alpha_1, \cdots, \alpha_{n-k}$ are linearly independent and $\langle \beta_1 + \mathcal{I}_{\mathcal{D}}, \cdots, \beta_{n-k} + \mathcal{I}_{\mathcal{D}} \rangle$ has rank $n - k$ within the space $\Omega^2(M)/\mathcal{I}_{\mathcal{D}}$.*

We write

$$S \subset \oplus_{n-k} \Omega^1(M) \oplus \Omega^2(M)/\mathcal{I} \subset \oplus_{n-k} \Omega^1(M) \oplus \Omega^2(M)/\mathcal{I}$$

for the condition

$$\beta_1, \cdots, \beta_{n-k} \text{ are pointwise linearly independent .}$$

In Subsection 2.3, we distinguished certain subtypes of generic $4-$distributions based on the way they intersect the cone $\mathcal{C} = p^{-1}(0)$ of degenerate forms. We can also define their associated formal conditions:

Definition 2.16. *A formal $(4, 6)-$bracket–generating distribution (α_i, β_i) that satisfies*

$$\langle \beta_1, \beta_2 \rangle \pitchfork \mathcal{C} \subset \Omega^2(\mathcal{D})$$

is called:

- ***formal elliptic** if $\langle \beta_1, \beta_2 \rangle) \cap \mathcal{C} = 0$.*
- ***formal hyperbolic** if $\langle \beta_1, \beta_2 \rangle \cap \mathcal{C} \neq 0$.*

We denote by $\mathrm{Dist}^f_{ell\,(4,6)}(M)$ the space of formal elliptic $(4, 6)-$bracket–generating distributions in the manifold M and by $\mathrm{Dist}^f_{hyp\,(4,6)}(M)$ the space of formal hyperbolic $(4, 6)-$bracket–generating distributions. Since both spaces are spaces of sections, we equip them with the C^0-topology.

3. $h-$Principle preliminaries

3.1. **Convex integration.** In this Subsection we will recall the main ideas of convex integration, developed by M. Gromov. We essentially reproduce some contents from [13]. We will also refer to some points and remarks in [8], which also treats the topic for the first ordercase; i.e. relations in $X^{(1)}$.

3.1.1. *Main ingredients.* Given a vector bundle $X \to V$, denote by $X^{(r)}$ the space of $r-$jets of sections. We have the projections from the space of $r-$jets to $(r-1)$-jets

$$p^r : X^{(r)} \to X^{(r-1)},$$

which constitute affine fibrations.

Definition 3.1. *We will say that a relation $\mathcal{R} \subset X^{(r)}$ is open/closed if it represents an open/closed subset of $X^{(r)}$.*

Definition 3.2. *Given a hyperplane field $\tau \subset TV$, we denote by $X^{\perp}$ the bundle defined by the following two conditions:*

- *There exist maps*
$$p^r_{\perp} : X^{(r)} \to X^{\perp}$$
and
$$p^{\perp}_{r-1} : X^{\perp} \to X^{(r-1)}$$
such that $p_r = p^{\perp}_{r-1} \circ p^r_{\perp}$,
- *If we denote $J^{\perp}(f) := p^r_{\perp} \circ J^r(f)$, then for any pair f_1, f_2 of germs of smooth sections of $X \to V$:*
$$J^{\perp}(f_1) = J^{\perp}(f_2) \Longleftrightarrow DJ^{r-1}_{f_1}|_{\tau} = DJ^{r-1}_{f_2}|_{\tau}$$

For any given choice of $\tau \subset TV$, the fibers of $X^{(r)} \to X^{\perp}$ are said to be *principal subspaces*. These can be visualised as parallel affine subspaces contained in the fibers of $X^{(r)} \to X^{(r-1)}$.

Given $z \in X^{\perp}$, we denote by $X^{(r)}_z$ the fiber of $X^{(r)} \to X^{\perp}$ over z. For a given relation $\mathcal{R} \subset X^{(r)}$ we denote by $\mathcal{R}_z$ the restriction $\mathcal{R} \cap X^{(r)}_z$.

3.1.2. *Ampleness.* We introduce now the notion of ampleness, which plays a central role in the theory of convex integration. We define ampleness for subsets of affine spaces first, which we then adapt to relations in jet spaces $\mathcal{R} \subset X^{(r)}$.

Definition 3.3. *Given an affine space Y and a path-connected subset $X \subset Y$, we say that X is* ***ample*** *if its convex hull* $\mathrm{Conv}(X)$ *coincides with Y; i.e.* $\mathrm{Conv}(X) = Y$. *A general subset $Z \subset Y$ is* ***ample*** *if each of its path-connected components is ample.*

Example 3.4. *The complement of any stratified subset $\Sigma \subset \mathbb{R}^n$ of codimension greater than* 1 *is ample. Such a set Σ is said to be* thin. *See Figure 1.*

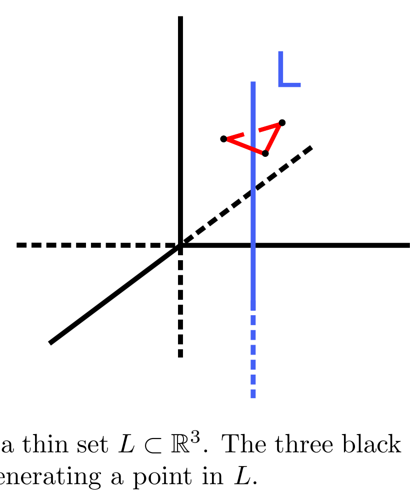


FIGURE 1. Example of a thin set $L \subset \mathbb{R}^3$. The three black points in $\mathbb{R}^3 \setminus L$ represent three points convexly generating a point in L.

Not all ample subsets have thin complements. The following example shows an ample subset whose complement is codimension-1:

Example 3.5. *The subset of $\mathbb{R}^3$ defined by*

$$\mathcal{H}^+ := \{(x, y, z) \in \mathbb{R}^3 : xy - z > 0\}$$

is ample. Indeed: it corresponds to the outer-component of a cone and any point $p \in \mathbb{R}^3 \setminus \mathcal{H}^+$ can be expressed as a convex combination of points in $\mathcal{H}^+$. △

This set will reappear later on in our study of hyperbolic $(4, 6)$ distributions. The other components in the complement of the cone correspond to the elliptic relation.

We now define the strongest notion of ampleness for differential relations:

Definition 3.6. *Given an open relation $\mathcal{R} \subset X^{(r)}$, we say that it is* ***ample*** *(in all principal directions) if, for any principal subspace $X_z^{(r)}$, the restricted relation $\mathcal{R}_z$ is ample.*

In the case of relations in $X^{(1)}$, there exists a less restrictive notion of ampleness [8]:

Definition 3.7. *For a local choice of coordinates, a relation $\mathcal{R} \subset X^{(1)}$ is* ***ample in the coordinate directions*** *if it intersects every principal subspace associated to a coordinate hyperplane in an ample subset.*

Eliashberg and Mishachev pointed out in (p. 171, [8]), referring to relations in $X^{(1)}$, that they "do not know any geometrically interesting examples when the less restrictive notion of ampleness is satisfied but the other one is not". In fact, one can further relax the notion of ampleness (as Gromov does in [13]), but no applications of the convex integration scheme in full generality are known.

We provide the first example (see Section 5) of a "geometrically interesting" relation $\mathcal{R}$ that is ample in coordinate directions but not ample in all principal directions. This relation is defined as follows: We start with the relation defining $(4, 6)$ hyperbolic distributions. This relation is diff-invariant, but not ample in all principal directions. Given a formal solution, we choose coordinates nicely adapted to it (in the sense that the component of the relation containing the formal solution is ample in each coordinate subspace). Furthermore, we refine the relation so that this niceness property is preserved throughout the whole argument (during which the formal data gets replaced). The resulting relation is thus not diff-invariant, but it is now ample in coordinate directions in each step. This will be explained in more detail in Section 5.

Theorem 3.8. *The complete h-principle holds for any open relation $\mathcal{R} \subset X^{(r)}$ satisfying ampleness in coordinate directions.*

We now recall some of the key ideas from convex integration. This overview is our take on the arguments due to Gromov [13] and differs from the later treatment in [8].

3.1.3. *Inductive argument.* In order to explain the inductive process, we first make some remarks. Consider a vector space V, and note that the following morphism

$$\begin{array}{rccc} g\colon & V^* & \longrightarrow & \mathrm{Sym}^r(V) \\ & \tau & \longmapsto & \tau^{\otimes r} \end{array}$$

identifies the space of hyperplane fields with the projectivisation of the subspace of "pure" symmetric homogeneous polynomials of degree r with entries in V. Such a pure element is said to be principal. It is a well known result that every homogeneuos polynomial of degree r can be expressed as a linear combination of pure polynomials.

Now, the fibres of $p : X^{(r)} \to X^{(r-1)}$ can be identified with $\mathrm{Sym}^r(TM, \|_X)$, i.e. the homogeneous polynomials with entries in TM and values in the vertical bundle. Principal elements are thus those of the form $\tau^{\otimes r} \otimes x$, with τ a covector in M and x a vertical vector. The statement above says that a basis of principal directions can be chosen in each fibre.

Definition 3.9. *A* ***principal path*** *$\gamma : [0, 1] \to X^{(r)}$ is a piecewise linear path such that there exists a partition $\{0, \cdots, t_k\} \subset [0, 1]$ for which $\gamma_{[t_i, t_{i+1}]}$ is contained in a principal subspace. The* ***principal length*** *between two points in the same fibre $X^{(r)} \to X^{(r-1)}$ is the minimal k such that a principal path consisting of k segments exists between them.*

It follows that, given a formal solution (f, F), we can always find a principal path of principal length k joining $J^r f$ with F, where k depends only on r (see Figure 2).

Now we present the inductive scheme. We focus on a single solution, but a similar reasoning works parametrically and relatively (see the next Subsection). We start with a principal path $(J^r f, F)$ between $J^r f$ (which is holonomic) and F (which is a formal solution). The inductive parameter will be the principal length, which will be reduced by one in each step (keeping the holonomic/formal dichotomy). Once the induction is complete, the two entries will agree, and we will thus have an actual solution.

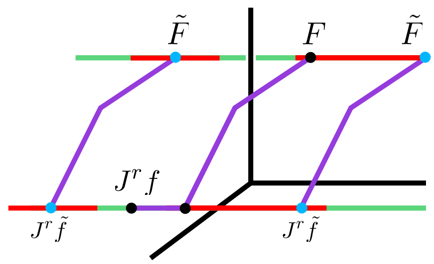


FIGURE 2. Inductive process on the length of principal paths.

We now explain the induction step. We consider the principal subspace $\tilde{P}$ corresponding to the first piece $\gamma|_{[0,t_1]}$ of γ; it contains $J^r f$. Write P for the principal subspace parallel to $\tilde{P}$ and passing trough F. We consider the restriction $\mathcal{R}_P$ of the differential relation $\mathcal{R}$ to P. We can use $\gamma|_{[t_1,1]}$ to shift $\mathcal{R}_P$ affinely to $\tilde{P}$; this transported relation is called $\tilde{\mathcal{R}_P}$. Do note that this is not the same as $\mathcal{R}_{\tilde{P}}$

Assuming $\mathcal{R}_P$ to be ample, we can invoke the $C^\perp$ convex integration lemma. It says that there exists a section g, C^0-close to f in $X^\perp$ (with respect to the principal subspace $\tilde{P}$), such that $J^r g$ lies in a C^0 neighbourhood of $\tilde{R_P}$. We construct g by integrating r-times a "flower" in $\tilde{R_P}$ that convexly surrounds $J^r f$.

If we now shift $J^r g$ back to P we get a new formal data G; this is a formal solution with values in $\mathcal{R}_P$. The outcome of this construction is that the principal length of $(J^r g, G)$ is $k - 1$, proving the induction step.

3.1.4. *From local to global.* Convex integration works relative to domain and parameter. Therefore, the process can be extended from local to global, just by working in one small chart at a time, as follows.

Start with a fine enough cubical cover $\bigcup B_i$ of $M \times K$ and consider slightly smaller open cubes $B_i' \subset B_i$ by slightly shrinking. We require that the union of B_i' still covers $M \times K$. The charts in the smaller cubes are inherited from the bigger ones.

If we start with a given formal solution, by the iterative process described above, we can get a new holonomic solution in B_i' (blue area in the left side of Figure 3) while the formal solution outside B_i remains unaltered. In the zone comprised between B_i and B_i' we get an interpolating formal solution which is not necessarily holonomic (light red area in the left side of Figure 3).

We carry on the process for all cubes in the covering and since $\bigcup_i B_i'$ covers $M \times K$, we will eventually get a globally holonomic solution at the end of the iteration.

3.2. **Linear algebra.** We will now recall that the spaces of non-singular matrices constitute ample sets within the space of matrices. This will be key in subsequent sections for proving flexibility results for spaces of distributions.

Proposition 3.10. *The subspace* $\mathrm{GL}(n)$ *of non-singular matrices is an ample subset of the space of* $(n \times n)$*-matrices* $\mathcal{M}_{n\times n}$ *if and only if* $n \geq 2$.

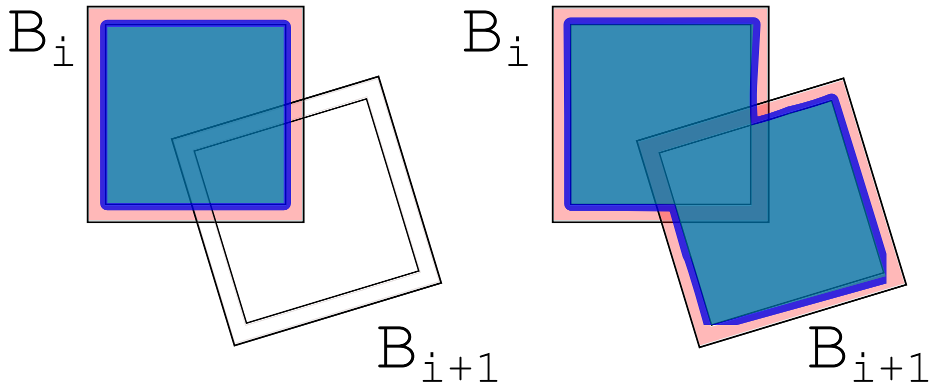


FIGURE 3. Extending the holonomic solution from cube B_i to cube B_{i+1} in the globalization process.

Proof. The two components $\mathrm{GL}^+(n)$ and $\mathrm{GL}^-(n)$ are connected. We have to show that each is individually ample.

First note that every $n \times n$ matrix can be expressed as the convex combination of two non-singular matrices, since

$$M = \frac{1}{2}(2M - 2\lambda \cdot Id) + \frac{1}{2}\lambda \cdot 2Id$$

and the right hand-side is the sum of two non-singular matrices for any choice of $\lambda \notin \mathrm{Spec}(M) \setminus 0$. Therefore, it is enough to show that any matrix $M \in \mathrm{GL}^+(n)$ can be expressed as a convex combination of matrices in $\mathrm{GL}^-(n)$ (and viceversa). This readily follows by writing $M = (v_1, v_2, v_3, \cdots, v_n)$ (expressed in column vectors) as $M = \frac{1}{2}M_1 + \frac{1}{2}M_2$, where

$$M_1 = (-v_1, 3v_2, v_3, \cdots, v_n)$$
$$M_2 = (3v_1, -v_2, v_3, \cdots, v_n)\,.$$

Note that M_1 and M_2 do not belong to the same connected component of $\mathrm{GL}(n)$ as M and, thus, the claim follows. □

Alternate proof. Observe that $\mathcal{M}_{n\times n}$ is convexely spanned by those matrices with a single non-zero entry.

Then: Given a matrix M and a sufficiently large constant C, it holds that M is in the interior of the convex hull of the matrices $e^{\pm}_{i,j}$ whose single non-zero entry is (i,j) with value $\pm C$. The matrix $e^{\pm}_{i,j}$ has zero determinant so it may be perturbed to yield a matrix $\tilde{e}^{\pm}_{i,j}$ with positive (resp. negative) determinant. In doing so, the convex hull is perturbed as well. However, M will remain in the interior if the perturbations are small enough, by continuity. This proves the ampleness of $\mathrm{GL}^+(n)$ (resp. $\mathrm{GL}^-(n)$). □

Corollary 3.11. *The subspace of non-singular matrices in $\mathcal{M}_{n\times m}$ is ample unless $n = m = 1$.*

Proof. We may assume $n \neq m$. Then, the space of non-singular matrices has a single component and is dense. □

3.3. **First order linear differential operators.** In this subsection, we introduce certain notation and definitions about differential operators that will be useful throughout the paper. We reproduce more or less the contents of [8, Chapter 20].

Let X and Z be two vector bundles over the same base V and consider a first order linear operator

$$D : \Gamma(X) \to \Gamma(Z)$$

This means that D factors through $\Gamma(X) \to \Gamma(X^1) \to \Gamma(Z)$, where the first arrow is the $1-$jet extension map and the second one is a fiberwise homomorphism. This morphism is called the symbol of the operator D and we denote it by $\sigma(D)$.

Definition 3.12. *A section $f \in \Gamma(Z)$ is called a $D-$section if there exists $g \in \Gamma(X)$ such that $D \circ g = f$.*

Definition 3.13. *We say that a linear differential operator D has rank r if $\operatorname{rank}(\sigma(D) \circ P) = r$ for any principal subspace $P \subset X^1$.*

Corollary 3.14. *Assume $\operatorname{rank}(D) \geq 2$ and $\sigma(D)$ is an epimorphism. Then any system of linearly independent sections $\{f_1, \cdots, f_n\} \subset \Gamma(Z)$ can be homotoped through systems of linearly independent sections to a system of linearly independent $D-$sections.*

Proof. The assumptions imply that the relation defining linear independence is ample, according to Proposition 3.10. □

3.4. **The symbol of the exterior differential.** The exterior differential d is a first order linear differential operator with surjective symbol $\sigma(d)$ which we will now describe.

Consider local coordinates $(q_1, \cdots, q_n)$ around $q \in M$ and write $(p_1, \cdots, p_n)$ for the corresponding local fibre coordinates in T^*M,

$$(p_1, \cdots, p_n) \mapsto \sum_{i=1}^{n} p_j dq_j.$$

Given a $0-$jet (q, p_j), we take coordinates $(a_{j,k})$, $1 \leq j, k \leq n$ to represent the $1-$jet $\left(q, p_j, \sum a_{jk} \frac{\partial p_k}{\partial q_j}\right)$ over it . In these terms, the symbol $\sigma(d)$ can be described as

$$\sigma(d) \circ (a_{j,k}) = \sum (a_{jk} - a_{kj})\, du_j \wedge du_k$$

In order to check if the method of convex integration applies for certain certain conditions involving bracket–generating distributions, we will provide an explicit description for the **principal subspace** associated to a principal direction u_i. This will be key in subsequent sections.

Lemma 3.15. *Let $\left(p, \sum a_{jk}(p) \frac{\partial y_k}{\partial u_j}\right)$ be a point in $J^1(\Omega^1(\mathcal{O}p(q)))$. Then the principal subspace associated to a principal direction u_i can be expressed as*

$$P_{u_i} = \{\sigma(d) \circ (a_{j,k})(p) + du_i \wedge \alpha : \alpha \in T^*(\mathcal{O}p(p))\}. \tag{1}$$

Proof. Without loss of generality, we will prove it for the first coordinate direction. The coordinates a_{jk} are constant if $j > 1$. We can decompose $\sigma(d) \circ (a_{j,k})$ in a component ω_0 not depending on the coordinate direction and a component corresponding to the u_1-variable part as follows:

$$\sigma(d) \circ (a_{j,k}) = \omega_0 + \sum a_{1k} du_1 \wedge du_k.$$

Since the following equality holds

$$\{\omega_0 + \sum a_{1k} du_1 \wedge du_k : a_{1k} \in \mathbb{R}\} = \{w_0 + du_1 \wedge \alpha : \alpha \in \Omega^1(\mathcal{O}p(p))\}$$

we have that the subspace P_1 over a certain point $\left(p, \sum a_{jk}(p) \frac{\partial y_k}{\partial u_j}\right)$ can be computed as taking the constant shift part ω_0 and adding the family of terms consisting on the wedge of the covector dual to the principal direction with all possible $1-$forms $\alpha \in \Omega^1(\mathcal{O}p(p))$. This proves the claim. □

We denote by $\tilde{d}$ the exterior differential operator when consider into the quotient $\Omega^2(M)/\mathcal{I}$

$$\begin{array}{rcl} \tilde{d} : \Omega^1(M) & \longrightarrow & \Omega^1(M) \oplus \Omega^2(M)/\mathcal{I} \\ \alpha & \longmapsto & (\alpha, d\alpha + \mathcal{I}) \end{array}$$

Since the usual differential exterior first-order linear operator $d : \Omega^1(M) \to \Omega^2(M)$ has fiberwise surjective symbol, so does $\tilde{d} : \Omega^1(M) \to \Omega^1(M) \oplus \Omega^2(M)/\mathcal{I}$.

3.5. **Jiggling for coordinate charts.** This subsection contains a *jiggling* type result for coordinate subspaces. It guarantees the existence of a cubical cover in which any two coordinate subspaces are transverse. Moreover, these charts can be chosen so that the coordinate directions avoid an open set with fibrewise dense complement. This theorem will play an important role in the proof of our main theorem in section 5, but we state it in more generality, since we expect it to be applicable in other situations.

Let us introduce some preliminary notation. Let $\{e_1, \cdots, e_n\}$ be the standard basis of $\mathbb{R}^n$.

Definition 3.16. *A subspace $\Pi \subset \mathbb{R}^n$ is called a coordinate subspace if $\Pi = span\langle e_{i_1}, \cdots, e_{i_j}\rangle$, for some collection of subindices $\{i_1, \cdots, i_j\} \subset \{1, \cdots, n\}$.*

Theorem 3.17. *Let M be a differentiable $n-$manifold. Let $\Sigma \subset TM$ be an open set whose fibrewise complement is dense.*

Then, there exists a cover by cubical charts $\{(K_i, \phi_i : [0,1]^n \subset \mathbb{R}^n \to K_i)\}$ such that, for any indices i, j and any point $p \in M$:

i) *For any coordinate direction ∂_{e_l} it holds:*
$$d_p\phi_i(\partial_{e_l}) \notin \Sigma.$$
ii) *For any two coordinate subspaces Π_1 and Π_2, and any $i \neq j$, it holds:*
$$d_p\phi_i(\Pi_1) \pitchfork d_p\phi_j(\Pi_2).$$

Proof. Denseness and openness of Σ implies that we can find a cubical cover $\mathcal{U}$ of M such that condition (i) is satisfied. $\mathcal{U}$ may be assumed to be finite invoking the compactness of M. We additionally fix a constant $\delta > 0$ such that the δ-neighbourhoods of the coordinate directions of the cover still satisfy condition (i).

Given any constant $r > 0$, we subdivide $[0,1]^n$ into cubes of side r homothetic to the original. We can thicken these by applying a dilation by a factor of 2. Applying this to each $K_i \in \mathcal{U}$ through the charts ϕ_i, we obtain a refinement of $\mathcal{U}$, which we call $\mathcal{U}(r)$. We will say that two cubes in $\mathcal{U}(r)$ are *siblings* if they arise by subdividing the same $K_i \in \mathcal{U}$, which we call the *parent*. The transition functions between two cubes in the subdivision are inherited from their parents.

The idea of the proof is to make r small enough so that a slight *tilting* of the cubes in $\mathcal{U}(r)$ satisfies conditions (i) and (ii). By tilting, we mean that we identify each cube $K \in \mathcal{U}(r)$ with $[-1,1]^n$ (using the chart ϕ_i inherited from its parent $K_i \in \mathcal{U}$ and then scaling and translating) and then we apply a linear transformation to it. When the transformation is given by a linear map ε-close to the identity, we speak of a ε-tilt. The rest of the argument consists of computing rough estimates, depending only on r, that will allow us to apply tilting and prove the result.

First: Given $K \in \mathcal{U}(r)$ we write $\mathcal{U}_K(r)$ for the cubes in the subdivision that intersect K non-trivially. We claim that there is an upper bound for its cardinality that is independent of r. This is trivially true for intersections between siblings. For unrelated cubes we reason as follows: Write $K_i \in \mathcal{U}$ for the parent of K and consider the image $\phi_{ij}(K) \subset K_j$ in some other cube $K_j \in \mathcal{U}$. Since ϕ_{ij} is C^1-bounded by compactness, the diameter of $\phi_{ij}(K)$ behaves as $O(r)$. In particular, it can only intersect $O(1)$ cubes in the image.

Second: We shrink our constant $\delta > 0$ so that all δ-tiltings of cubes in $\mathcal{U}(r)$ still yield a cover. Using the C^1-boundedness of the transition functions, we deduce that this choice does not depend on r. We enlarge $\mathcal{U}_K(r)$ to contain all cubes intersecting any δ-tilt of K; its cardinality is still bounded independently of r.

Third: Given $K \in \mathcal{U}(r)$, we look at the coordinate subspaces coming from neighbourhouring cubes. We want to show that these form a small set (so that we may tilt the coordinate subspaces of K to be transverse to them). More formally: Let $\mathrm{Gr}(n,a)$ be the Grassmannian of a-dimensional subspaces of $\mathbb{R}^n$, where we identify $TK \cong K \times \mathbb{R}^n$. For each $K \in \mathcal{U}(r)$ we define:
$$\Pi_{K,a} := \bigcup_{K' \in \mathcal{U}_K(r), p \in K \cap K', \dim(\Pi)=a} \{d_p\phi_{K'K}(\Pi)\} \subset \mathrm{Gr}(n,a),$$

where Π ranges over all a-dimensional coordinate subspaces. We claim that the measure of Π_K behaves like $O(r)$.

Note that the number of coordinate subspaces and cubes in $\mathcal{U}_K(r)$ is finite. As such, it is sufficient to control the measure of a single $\cup_{p\in K'\cap K}\{d_p\phi_{K'K}(\Pi)\}$. We prove instead a stronger statement (which we will need later when we tilt our cubes to achieve condition (ii)): There is an upper bound, depending only on r, for the measure $\cup_{p\in K'\cap K}\{d_p\phi_{K'K}(\Pi)\}$, where K' is any affine cube of radius r contained in one of the $K_j \in \mathcal{U}$ (i.e. not necessarily one of the cubes in the subdivision $\mathcal{U}(r)$). Observe that the claim is true if K' and K are siblings, because the transition $\phi_{K'K}$ between the two is the identity. The idea for K' not a sibling follows the same spirit: as we make r smaller, $\phi_{K'K} = \phi_{ij}$ is approximated better by $d\phi_{K'K}$. As such, we have:

$$d_{p+h}\phi_{K'K} = d_p\phi_{K'K} + O(h)$$

and, since the radius of all cubes is $O(r)$, the claim follows.

Failing to be transverse to a given subspace of $\mathbb{R}^n$ defines a subvariety in each Grassmannian $\mathrm{Gr}(n,a)$. We can consider the union $\Sigma_{K,a}$ of all these subvarieties as we range over the subset $\Pi_{K,a}$. Our reasoning above shows that $\Sigma_{K,a}$ has measure $O(r)$. Similarly, we consider $\mathrm{GL}(n)$, which we think of as the space of tilted cubes in $\mathbb{R}^n$, and we note that the subspace of cubes $\mathrm{GL}_K(n)$ whose coordinate subspaces are not transverse to the elements in $\Pi_{K,a}$ has measure $O(r)$. We can then fix r so that $\mathrm{GL}_K(n)$ does not cover a neighbourhood of the identity of size δ (which we picked earlier).

Now we forget about the original cubes, and we fix an ordering $\{K_i\}$ of the cubes in $\mathcal{U}(r)$. We prove the result by induction on this order; at every step l we produce a new cover $\mathcal{U}^l(r)$. The induction hypothesis is that conditions (i) and (ii) hold for the first l cubes in the cover $\mathcal{U}^l(r)$. The induction step follows by replacing K_{l+1} by some tilting. Performing a δ-tilting ensures that condition (i) holds and that the new $\mathcal{U}^{l+1}(r)$ is still a covering. Condition (ii) (with respect to neighbouring cubes appearing earlier in the list) can be achieved for some appropriate δ-tilting due to the measure considerations above. This concludes the proof. □

4. Flexibility for bracket–generating distributions

4.1. **The contact and even-contact cases.** We revisit two well-known cases of bracket–generating distributions; i.e. the contact and even-contact case. We will show that the failure of the former case corresponds to a $1-$codimension foliation, whereas the latter corresponds to a thin singularity.

In order to study the (non-)ampleness of the contact and even-contact conditions, respectively, in an $n-$dimensional manifold M, consider the following family of maps

$$\begin{array}{rccc} \Gamma\colon & T^*M \oplus \Omega^2(M) & \longrightarrow & \Omega^{2\lfloor \frac{n-1}{2} \rfloor} M \\ & (\alpha,\beta) & \longmapsto & \alpha\wedge\beta^{\lfloor \frac{n-1}{2} \rfloor}. \end{array}$$

Definition 4.1. *Define the relation*

$$\mathcal{R}_\Gamma = J^1\left(\Omega^p(M)\right) \setminus D^{-1}\left(\Gamma^{-1}(0)\right),$$

where

$$D : \Omega^p(M) \overset{id\oplus d}{\longrightarrow} \Omega^p(M) \oplus \Omega^{p+1}(M)$$

is the extended exterior differential.

We study whether $\mathcal{R}_\Gamma$ is ample or not. Fix a coordinate direction ν and consider its dual codirection λ. We will study if, for each principal subspace P_λ associated to λ, the equality $\mathrm{Conv}\left(\mathcal{R}_\Gamma \cap P_\lambda\right) = P_\lambda$ holds.

Split $d\alpha$ in two components $d\alpha = \omega_1 + \omega_2$, where $\omega_1 \in \Omega^2(M)/\langle\nu\rangle$ and $\omega_2 \in \langle\nu\rangle$. Since the component ω_2 belongs to the underlying vector space associated to the principal subspace, the principal subspaces P_1 over $d\alpha$ and over ω_1 coincide. For the ease of computations and without loss of generality, we can therefore assume that $\omega_2 = 0$ so that $d\alpha$ has no component in the direction ν.

The condition for a point in the principal subspace reads as:

$$\Gamma(\alpha, d\alpha + \lambda\wedge\tau) \neq 0.$$

For simplicity, denote $k := \lfloor \frac{n-1}{2} \rfloor$. The left hand side above can be written as:

$$\alpha \wedge (d\alpha + \lambda \wedge \tau)^k = \alpha \wedge \big((d\alpha)^k + (d\alpha)^{k-1} \wedge \lambda \wedge \tau\big)$$

We distinguish two cases:

First: $\alpha \wedge (d\alpha)^k \neq 0$. It follows from the assumption $d\alpha \notin \langle \nu \rangle$ that $\mathcal{R}_\Gamma \cap P_\lambda = P_\lambda$, since perturbations in the principal subspace associated to the ν-direction will not change ω_1.

Second: $\alpha \wedge (d\alpha)^k = 0$. Then

$$\alpha \wedge (d\alpha + \lambda \wedge \tau)^k = \alpha \wedge (d\alpha)^{k-1} \wedge \lambda \wedge \tau \tag{2}$$

Since $(d\alpha)^{k-1} \wedge \lambda$ does not depend on the principal coordinate, the expression (2) is not zero if and only if

$$\alpha \wedge \tau \neq 0 \text{ in the quotient space } \Omega^{2\lfloor \frac{k}{2} \rfloor}(M)/\langle (d\alpha)^{k-1} \wedge \lambda \rangle \tag{3}$$

Pointwise, we can identify $\Omega^{2\lfloor \frac{k}{2} \rfloor}(M)/\langle (d\alpha)^{k-1} \wedge \lambda \rangle$ with $\mathbb{R}^2$. There, τ serves as a formal derivative of α. We deduce:

Proposition 4.2 (The failure of the contact condition is a foliation)**.** *The relation defining the contact condition $\mathcal{R}_\Gamma \subset J^1(T^*M)$ is not ample. Its singularity set is, fibrewise, a codimension-1 foliation.*

Proof. By the previous discussion, the failure for the contact condition for an odd-dimensional manifold M^{2n+1} can be described as

$$\Sigma = \{(p, p') \in \mathbb{R}^2 : p \wedge p' = 0\},$$

which describes the radial foliation in $\mathbb{R}^2$. At each non-zero point its tangent space is a codimension-1 linear subspace, contradicting ampleness. □

Proposition 4.3 (The failure of the even-contact condition is a thin singularity. McDuff, [19])**.** *The relation defining the even-contact condition $\mathcal{R}_\Gamma \subset J^1(T^*M)$ is ample since its singularity set is thin.*

Proof. As in the previous proposition, the singularity set can be described as $\Sigma = \{(p, p') \in \mathbb{R}^3 : p \parallel p'\}$. This corresponds to a singularity of codimension 2 and the claim follows. □

4.2. **h-Principle for one-step bracket-generating distributions.** We can now prove that bracket–generating distributions abide by the $h-$principle, as long as full non-degeneracy is not required.

Theorem 4.4 ($h-$principle for two-step bracket–generating distributions)**.** *The inclusion*

$$\mathrm{Dist}_{(k,n)}(M) \to \mathrm{Dist}^f_{(k,n)}(M^n)$$

is a weak homotopy equivalence for any bidimension (k, n), where $k \geq 3$ and $n \leq \binom{k}{2} + k$.

Proof. Take local coordinates $(u_1, \cdots, u_n)$ around $p \in M$. We will check that for each principal coordinate subspace $P_{u_i} \subset J^1(\Omega^1(M))$, either $\mathrm{rank}(\sigma(\tilde{d}) \circ (P_{u_i})) = 0$ or $\mathrm{rank}(\sigma(\tilde{d}) \circ (P_{u_i})) \geq 2$.

i) If $du_i(p) \in \mathcal{D}^{\perp}_{(k,n)}$ for some $1 \leq i \leq n$, then by Lemma 2.3, $\mathrm{rank}(\sigma(\tilde{d}) \circ (P_{u_i})) = 0$.
ii) Otherwise, $du_i(p) \notin \mathcal{D}$ and $\mathrm{rank}(\sigma(\tilde{d}) \circ (P_{u_i})) \geq 2$. This follows from by equation (1) in Lemma 3.15 and the fact that $k \geq 3$.

In both cases, the resulting determinantal condition is ample. Case $i)$ is trivially ample and case $ii)$ follows from Theorem 3.14. Since convex integration works relative to domain and parameter the result follows. □

Remark 4.5. *This Theorem answers positively the open question (Problem 6.2) raised during the Engel Structures workshop held in April 2017 (American Institute of Mathematics, San Jose, California):*

Problem 6.2. Are there examples of pairs (n, k) with $n > k \geq 2$ such that for any parallelizable $n-$manifold, there exists a $k-$plane field $D \subset TM$ with maximal growth vector?

Theorem 4.4 proves that bracket–generating distributions are flexible as long as we do not require any further non-degeneracy constraints. Nonetheless, there are two cases where the $h-$principle for non-degenerate distributions readily follows from Theorem 4.4:

Corollary 4.6. *Non-degenerate* $(3,5)$ *and* $(3,6)$ *distributions satisfy a full* $h-$*principle.*

Proof. This readily follows from Theorem 4.4 and the fact that the spaces of such non-degenerate distributions are connected. □

5. h-Principle for hyperbolic $(4,6)$-distributions

In this Section we tackle the proof of the following result:

Theorem 5.1. *The inclusion*

$$\mathrm{Dist}_{hyp\,(4,6)}(M) \to \mathrm{Dist}^f_{hyp\,(4,6)}(M)$$

is a weak homotopy equivalence.

Identically: Let K be a closed manifold, serving as parameter space, and let

$$\phi : K \times M \to \mathrm{Dist}^f_{hyp\,(4,6)}(M)$$

be a smooth section $\phi \equiv (\beta_1^k, w_1^k, \beta_2^k, w_2^k)$. We will construct a homotopy of sections $\tilde{\phi}_t : M \times K \to \mathrm{Dist}^f_{hyp\,(4,6)}(M)$ such that:

- $\tilde{\phi}_0 = \phi$
- $\tilde{\phi}_1 : K \times M \to \mathrm{Dist}_{hyp\,(4,6)}(M)$ is holonomic.
- The homotopy $\tilde{\phi}_t$ is relative in the domain and parameter.

We will proceed iteratively homotoping our formal solution ϕ in each cube of a covering of $M \times K$ to a holonomic solution $\tilde{\phi}_1$ relative to domain and parameter. We will achieve this through convex integration. In order to apply this method, we must ensure that the considered relation is ample.

It turns out that the hyperbolic relation $\mathrm{Dist}^f_{hyp\,(4,6)}(M)$ does not intersect all principal directions in ample sets. Nonetheless, ampleness does hold for a dense set of principal directions, as expalined in Lemma 5.2.

Take a principal direction u, and denote by λ its dual codirection. Consider a formal solution $\varphi = (\beta_1, \beta_2, w_1, w_2) \in \mathrm{Dist}^f_{hyp\,(4,6)}(M)$ and denote by P_λ the principal subspace over φ associated to the dual direction with respect to u. The following lemma establishes sufficient conditions over a formal solution and principal codirection λ in order the relation to be ample when intersected with the corresponding Principal subspace.

Lemma 5.2. *Let* $\varphi = (\beta_1, \beta_2, w_1, w_2) \in \mathrm{Dist}^f_{hyp\,(4,6)}(\mathbb{R}^6)$*. If* $\lambda \in T^*\mathbb{R}^6$ *satisfies that* $\omega_1 \wedge \lambda$ *and* $\omega_2 \wedge \lambda$ *are linearly independent as forms in the quotient space* $\Omega^3(\mathbb{R}^6)/\mathcal{D}$*, then* $P_\lambda \cap \mathrm{Dist}^f_{hyp\,(4,6)}(\mathbb{R}^6)$ *is ample in* P_λ*.*

Proof. Denote $\mathcal{D} = \ker\beta_1 \cap \ker\beta_2$. The principal subspace P_λ over φ can be described as:

$$P_\lambda = \{(w_1 + \lambda_1 \wedge \alpha_1, w_2 + \lambda_1 \wedge \alpha_2) : \alpha_1, \alpha_2 \in \mathcal{D}^*\}$$

Note that we are working in the quotient space $\Omega^2(M)/\mathcal{D}$, since it suffices to prove ampleness in this space (see Remark 2.15).

Write $g_{11} := (w_1 + \lambda_1 \wedge \alpha_1)^2$, $g_{22} := (w_2 + \lambda_1 \wedge \alpha_2)^2$ and $g_{12} := (w_1 + \lambda_1 \wedge \alpha_1) \wedge (w_2 + \lambda_1 \wedge \alpha_2)$. Then $\mathcal{R}_{hyp} \cap P_\lambda$ reads as

$$\begin{vmatrix} g_{11} & g_{1,2} \\ g_{12} & g_{2,2} \end{vmatrix} < 0 \tag{4}$$

We have the following explicit expressions for each component:

$$g_{11}(\alpha_1,\alpha_2) = w_1^2 + 2\lambda_1 \wedge \alpha_1 \wedge w_1 \tag{5}$$

$$g_{22}(\alpha_1,\alpha_2) = w_2^2 + 2\lambda_1 \wedge \alpha_2 \wedge w_2 \tag{6}$$

$$g_{12}(\alpha_1,\alpha_2) = w_1 \wedge w_2 + w_1 \wedge \lambda_1 \wedge \alpha_2 + \lambda_1 \wedge \alpha_1 \wedge w_2 \tag{7}$$

Identify $\Omega^4(D)$ with $\mathbb{R}$ by fixing a volume form $\omega \in \Omega^4(\mathbb{R}^4)$. Therefore we can identify each term of the expressions above with a real function and consider the following affine map to $\mathbb{R}^3$.

$$\begin{array}{rccc} \psi_{w_1,w_2}\colon & (\mathcal{D}^*)\times(\mathcal{D}^*) & \longrightarrow & \mathbb{R}^3 \\ & (\alpha_1,\alpha_2) & \longmapsto & (g_{11}(\alpha_1,\alpha_2), g_{12}(\alpha_1,\alpha_2), g_{22}(\alpha_1,\alpha_2)) \end{array}$$

Note that $\omega_1 \wedge \lambda$ and $\omega_2 \wedge \lambda$ being linearly independent implies the surjectivity of ψ_{w_1,w_2}. Define the following quadratic form on $\mathbb{R}^3$:

$$\begin{array}{rccc} \Phi\colon & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (x,y,z) & \longmapsto & xy - z^2. \end{array}$$

Its polar form takes the following matrix form:

$$M_\Phi = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

as its matrix representation. Its signature is $(0,1,2)$ and the subspace $X^- \subset \mathbb{R}^3$ corresponding to the negative part is ample in $\mathbb{R}^3$. Since the condition $\mathcal{R}_{hyp} \cap P_\lambda$ described in (4) is tantamount to $\Phi \circ \psi_{w_1,w_2} < 0$, the ampleness of the negative subspace $X^- \subset \mathbb{R}^3$ together with the surjectivity of Φ implies the claim.

□

In order to achieve the condition defining Lemma 5.2 throughout the proof, so as to get ampleness and, thus, apply convex integration, we will need the following Lemmas. They state that the violations of two conditions over 1−jets of forms constitute, respectively, thin singularities. This result will be key in the proof.

Let $\lambda, \lambda_1, \lambda_2 \in T^*M$ such that $\lambda \wedge \lambda_1 \wedge \lambda_2 \neq 0$. Denote by $B_{\lambda_1,\lambda} \subset J^1(T^*M \times T^*M)$ the space of pairs of jets (j_1, j_2) defined by

$$A \circ \phi(j_1) \wedge \lambda_1 \wedge \lambda = 0 \text{ and } A \circ \phi(j_2) \wedge \lambda_1 \wedge \lambda = 0,$$

where A is the antisymmetrization morphism and ϕ the map defined together with A in Section 2.4. A small remark in order to provide some context for the impatient reader: The definition of the set $B_{\lambda_1,\lambda}$ will be justified in the proof of Lemma 5.4. Also, when we apply the following two lemmas later in the proof, λ will play the role of the codirection associated to a Principal subspace, whereas the so called λ_1 and λ_2 will be some other principal codirections. Since we check that the violation of these conditions represent thin singularities inside P_λ, we will be able to apply convex integration along this Principal (co)direction λ.

Lemma 5.3. *Define the set Σ inside the space of $J^1(T^*M \times T^*M \setminus B_{\lambda_1,\lambda})$ as those pairs of jets (j_1, j_2), where if $\omega_i := A \circ \phi(j_i)$, are characterised by:*

$$\omega_1 \wedge \lambda_1 \wedge \lambda_2 = 0 \textit{ and } \omega_2 \wedge \lambda_1 \wedge \lambda_2 = 0. \tag{8}$$

Then Σ represents a thin singularity within $J^1(T^*M \times T^*M \setminus B_{\lambda_1,\lambda}) \cap P_\lambda$.

Proof. Condition (8) is given by the intersection of two sets, each given by the equality $\omega_1 \wedge \lambda_1 \wedge \lambda_2 = 0$ together with $\omega_2 \wedge \lambda_1 \wedge \lambda_2 = 0$. Note that ω_1 and ω_2 are linearly independent and, thus, each equality determines a codimension 1 set within $J^1(T^*M \times T^*M \setminus B_{\lambda_1,\lambda})$. But, since λ_1 and λ_2 are transverse with respect to λ, these two equations do not become trivial when intersecting $J^1(T^*M \times T^*M \setminus B_{\lambda_1,\lambda})$ with P_λ. Therefore, condition (8) represents a codimension 2 subset of $J^1(T^*M \times T^*M \setminus B) \cap P_\lambda$. ☐

Lemma 5.4. *Let $\lambda, \lambda_1, \lambda_2 \in T^*M$ such that $\lambda \wedge \lambda_1 \wedge \lambda_2 \neq 0$. Define the set Σ inside the space of $J^1(T^*M \times T^*M \setminus B_{\lambda_1,\lambda})$ as those pairs of jets (j_1, j_2), where if $\omega_i := A \circ \phi(j_i)$, are characterised by:*

(9) $$\omega_1 \wedge \lambda_1 \text{ and } \omega_2 \wedge \lambda_1 \text{ are linearly dependent.}$$

Then Σ represents a thin singularity within $J^1(T^*M \times T^*M \setminus B_{\lambda_1,\lambda}) \cap P_\lambda$.

Proof. In order to check that Condition (9) is thin, note first that $\omega_1 \wedge \lambda_1 \neq 0$ or $\omega_2 \wedge \lambda_1 \neq 0$. Since the principal subspace takes the form

$$P_\lambda = \{(w_1 + \lambda \wedge \alpha_1, w_2 + \lambda \wedge \alpha_2) : \alpha_1, \alpha_2 \in \mathcal{D}^*\},$$

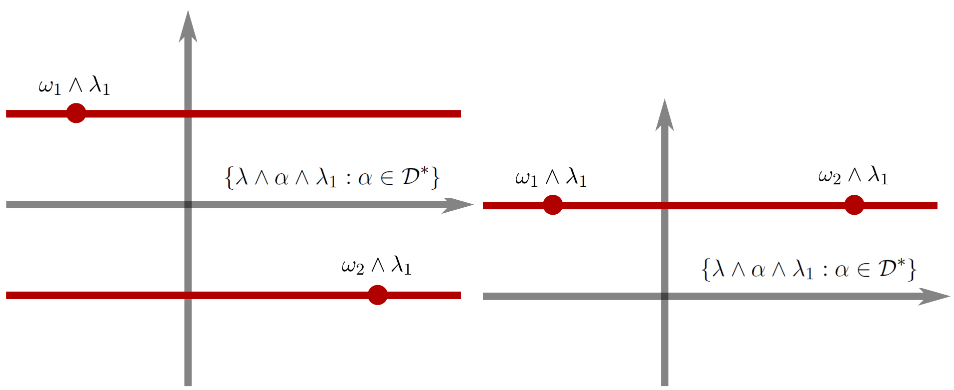
then $P_\lambda \cap \Sigma$ corresponds to pairs of the form

$$(\omega_1 + \lambda \wedge \alpha_1, \omega_2 + \lambda \wedge \alpha_2), \text{ with } \alpha_1, \alpha_2 \in \mathcal{D}^*$$

such that

$$\omega_1 \wedge \lambda_1 + \lambda \wedge \alpha_1 \wedge \lambda_1 \text{ and } \omega_2 \wedge \lambda_1 + \lambda \wedge \alpha_2 \wedge \lambda_1 \text{ are linearly dependent.}$$

We will check that this set has codimension greater than 1. For such purpose, note first that each of the components of P_λ can be regarded as an affine space whose underlying vector space is $\{\lambda \wedge \alpha \wedge \lambda_1 : \alpha \in \mathcal{D}^*\}$, with certain affine shift which is $\omega_1 \wedge \lambda_1$ for the first component and $\omega_2 \wedge \lambda_1$ for the second one. Therefore, we will distinguish the following cases depending on the relative positions of the affine shifts:



(A) Case *i*).$(\omega_1 \wedge \lambda_1 - \omega_2 \wedge \lambda_1) \notin \{\lambda \wedge \alpha \wedge \lambda_1 : \alpha \in \mathcal{D}^*\}$.

(B) Case *ii*).$(\omega_1 \wedge \lambda_1 - \omega_2 \wedge \lambda_1) \in \{\lambda \wedge \alpha \wedge \lambda_1 : \alpha \in \mathcal{D}^*\}$.

FIGURE 4. Case distinction depending on the relative positions of the affine shifts.

i) If $(\omega_1 \wedge \lambda_1 - \omega_2 \wedge \lambda_1) \notin \{\lambda \wedge \alpha \wedge \lambda_1 : \alpha \in \mathcal{D}^*\}$. Note that for each choice of α_1, there is an only (and, thus, 0-dimensional) choice of α_2 among all forms in the 2−dimensional space $\mathcal{D}^*$ such that
$$\omega_1 \wedge \lambda_1 + \lambda \wedge \alpha_1 \wedge \lambda_1 \text{ and } \omega_2 \wedge \lambda_1 + \lambda \wedge \alpha_2 \wedge \lambda_1$$
are linearly dependent (and viceversa). Therefore, the codimension of $\Sigma \subset J^1(T^*M \times T^*M \setminus B) \cap P_\lambda$ is 2.

ii) If $(\omega_1 \wedge \lambda_1 - \omega_2 \wedge \lambda_1) \in \{\lambda \wedge \alpha \wedge \lambda_1 : \alpha \in \mathcal{D}^*\}$. The possibility that both $\omega_1 \wedge \lambda_1$ and $\omega_2 \wedge \lambda_1$ lie in $\{\lambda \wedge \alpha \wedge \lambda_1 : \alpha \in \mathcal{D}^*\}$ is discarded since we excluded the set $B_{\lambda_1,\lambda}$ from the beginning and, thus, $\omega_1 \wedge \lambda_1 \wedge \lambda \neq 0$ or $\omega_2 \wedge \lambda_1 \wedge \lambda \neq 0$. Then, they both lie in a certain affine space
$$\beta + \{\lambda \wedge \alpha \wedge \lambda_1 : \alpha \in \mathcal{D}^*\},$$

with $\beta \neq 0$. So, again, for each choice of α_1, there is an only choice (0−dimensional) of α_2 in the 2−dimensional space $\mathcal{D}^*$ such that

$$\omega_1 \wedge \lambda_1 + \lambda \wedge \alpha_1 \wedge \lambda_1 \text{ and } \omega_2 \wedge \lambda_1 + \lambda \wedge \alpha_2 \wedge \lambda_1$$

are linearly dependent, and viceversa. Therefore, the same conclussion as in the previous case holds.

□

With these two lemmas at our disposal, we will proceed with the proof of Theorem 5.1.

Take a fine enough cubic open cover $\bigcup_i B_i$ of $K \times M$. For each i, there exists a local frame of $TB_i \cap TM$ of principal directions $\langle X_1^i, \cdots, X_6^i \rangle$.

Denote by Λ_i the finite set of indices of cubes intersecting B_i; i.e.

$$\Lambda_i := \{j \in \mathbb{Z} : B_j \cap B_i \neq 0\}.$$

Then, in order to get ampleness, at every B_i, we want to achieve for all $j \in \Lambda_i$ and $1 \leq \ell \leq 6$,

$$w_1 \wedge \lambda_\ell^j \text{ and } w_2 \wedge \lambda_\ell^j \text{ are linearly independent.} \tag{10}$$

where λ_ℓ^i **stands for the dual covector of X_ℓ^i.** The following lemma shows that the set of covectors satisfying condition (10) is dense and thus exists.

Lemma 5.5. *The set of codirections $\lambda_1 \in \mathcal{D}^*$ not satisfying condition (10) has codimension equal or greater than 2 inside $\mathcal{D}^*$.*

Proof. Since ω_1 and ω_2 constitute a formal solution for $\mathcal{R}_{(4,6)}^{hyp}$, then the span $\langle \omega_1, \omega_2 \rangle$ is a hyperbolic plane; i.e. there exists a basis $\{\beta_1, \beta_2\}$ of this plane such that $\text{rank}(\beta_1) = \text{rank}(\beta_2) = 2$. In order to determine the codimension of the set of bad codirections; we take a local model where

$$\beta_1 = dx_1 \wedge dx_2$$
$$\beta_2 = dx_3 \wedge dx_4.$$

Therefore, the set of bad codirections corresponds to those forms $\lambda \in \mathcal{D}^*$ such that either $\beta_1 \wedge \lambda = 0$ or $\beta_2 \wedge \lambda = 0$. But those codirections $\lambda \in \mathcal{D}^*$ that make $\beta_1 \wedge \lambda = 0$, correspond to the span $\langle dx_1, dx_2 \rangle$, and this set has codimension 2 in $\mathcal{D}^*$. The same computation holds for β_2.

□

Another condition that will be needed at the beginning of the proof is the following one:

for all $j \in \Lambda_i$ and $1 \leq \ell \leq 6$,

$$w_1 \wedge \lambda_\ell^i \wedge \lambda_{\ell+1}^j \neq 0 \text{ or } w_2 \wedge \lambda_\ell^i \wedge \lambda_{\ell+1}^j \neq 0. \tag{11}$$

The following lemma states that the violation of condition 11 also represents a thin singularity in the space of covectors. This fact will be needed in the proof of the Main Theorem.

Lemma 5.6. *The set of codirections $\lambda_1, \lambda_2 \in \mathcal{D}^* \times \mathcal{D}^*$ not satisfying condition (11) has codimension equal or greater than 2 inside $\mathcal{D}^* \times \mathcal{D}^*$.*

Proof. Again, since ω_1 and ω_2 constitute a formal solution for $\mathcal{R}_{(4,6)}^{hyp}$, then the span $\langle \omega_1, \omega_2 \rangle$ is a hyperbolic plane and there exists a basis $\{\beta_1, \beta_2\}$ of this plane such that $\text{rank}(\beta_1) = \text{rank}(\beta_2) = 2$. In order to determine the codimension of the set of bad codirections; we take a local model where

$$\beta_1 = dx_1 \wedge dx_2$$
$$\beta_2 = dx_3 \wedge dx_4.$$

In particular, those pair of forms that make

$$w_1 \wedge \lambda_j^i \wedge \lambda_{j+1}^i = 0$$

and

$$w_2 \wedge \lambda_j^i \wedge \lambda_{j+1}^i = 0$$

correspond to the forms lying in the hyperplane of forms $\langle dx_1\rangle \cup \langle dx_2\rangle$.

This equation, together with the analogous one for ω_2, determines the intersection of two hyperplanes within the space of forms. Therefore we get a codimension 2 set. □

In order to apply convex integration iteratively at each cube, we need our formal solution to satisfy Condition (10) at the beginning of each step (for every principal direction $1 \leq \ell \leq 6$). Nonetheless, once our section is perturbed in a certain direction at some cube, the new solution could potentially violate the aforementioned conditions for adjacent cubes or for other principal directions X_ℓ^i in the same cube. So as to avoid that scenario, we will proceed in two steps.

The first step is a preparation where we choose appropriate local charts in order to be able to apply convex integration method. Note that the Hyperbolic relation does not always lead to an ample set when intersecting it with a principal subspace over a given formal solution. This is why we need to choose local charts where the condition described in Lemma 5.2, together with some other technical condition needed later along the proof, is satisfied. Note that these choices of local charts depend on the formal data we start with.

Once we get this preparation, we will carry out an iterative process where we will progressively homotope the formal solution to a new solution that is holonomic with respect to a new direction at a time. This iterative process consists on two parts. First, we explain how to do it within a cube in the cover. This is achieved by homotoping the formal solution to some other formal solution holonomic with respect to each principal direction, one by one. Secondly, we argue that this process can be naturally extended to adjacent cubes in the cover by applying Theorem 3.17. So, once we finish the process within the last directions in a cube, we continue with the first directions of the next cube in the induction. A brief notation reminder before getting into the the proof: recall that λ_ℓ^i stands for the dual covector of X_ℓ^i.

Step 1. Preparation

We choose local coordinates such that for all B_i in the cover, $j \in \Lambda_i$ and $1 \leq \ell < h \leq 6$:

i) $\omega_1 \wedge \lambda_\ell^j$ and $\omega_2 \wedge \lambda_\ell^j$ are linearly independent.
ii) $\omega_1 \wedge \lambda_\ell^j \wedge \lambda_h^j \neq 0$ or $\omega_2 \wedge \lambda_\ell^j \wedge \lambda_h^j \neq 0$.

Condition $ii)$ is a technical condition in order the inductive step to work, as we will see. Note that this choice exists since $i)$ and $ii)$ determine a finite intersection of dense sets of local charts by Lemmas 5.5 and 5.6, and thus it determines a dense set of charts. We will need to perform a more refined preparation in this step but this will be explained later in Step 3.

We will proceed inductively on the principal directions working on a cube at a time and, later on, extending the iteration to adjacent cubes:

Step 2. Iterative process in a cube

Upon the choice of local charts made in the previous step, we define a new relation $\tilde{\mathcal{R}}$ which is a refinement of the hyperbolic relation:

$$\tilde{\mathcal{R}} := \mathcal{R}_{hyp} \setminus \bigcup_{\ell,k} \Sigma_\ell^j,$$

where each Σ_ℓ^j is given by the following two conditions:

i) $\omega_1 \wedge \lambda_\ell^j$ and $\omega_2 \wedge \lambda_\ell^j$ are linearly independent.
ii) $\omega_1 \wedge \lambda_{\ell+1}^j \wedge \lambda_{\ell+2}^j \neq 0$ or $\omega_2 \wedge \lambda_{\ell+1}^j \wedge \lambda_{\ell+2}^j \neq 0$ for $1 \leq \ell \leq 6$.

By Lemmas 5.3 and 5.4, every Σ_ℓ^j is a thin singularity and, thus, so is $\bigcup_{\ell,j} \Sigma_\ell^j$. Moreover, because of the choice of coordinates in the Preparation step, the formal solution we begin with is also a formal solution for the refined relation $\tilde{\mathcal{R}}$. Also, since we are under the conditions of Lemma 5.2 for each cube and coordinate direction within the cube, $\tilde{\mathcal{R}}$ is ample with respect to these choices of principal directions. Therefore, we can apply convex integration in order to homotope our initial formal solution to a new formal solution for $\tilde{\mathcal{R}}$ which is holonomic for each coordinate direction, one at a time.

We get, via this iteration, a homotopy of the initial solution which is holonomic with respect to the direction considered at each step. So, ideally, one would expect to get a solution that is holonomic with respect to all directions when this process ends.

Nonetheless, note that this process, the way it has been described, does not work when we reach the last two steps (5^{th} and 6^{th}) in the iteration within the same cube. Indeed, the iterative process continues by taking into account directions from the $i+1$-th cube. So, we must ensure that the process does work when taking into account adjacent cubes too. We will justify why this can be done in the next step, where we extend the iteration to principal directions of subsequent cubes.

Step 3. Extension of the iteration to subsequent cubes in the cover.

We will now extend the iterative process from the $i-th$ cube to the $i+1-th$ one. One may like to just follow the schema described in Subsection 3.1.4, but this does not work if some adjustments are not done first. In fact, the conditions defining the relation $\tilde{\mathcal{R}}$ strongly depend on the choice of principal directions and, more specifically, it requires the condition of linearly independence of such directions. Since the linear independence of principal directions from one cube with respect to the ones chosen at an adjacent cube is not guaranteed a priori, this is something we have to ask from the beginning of the process.

We must be careful, since a necessary condition in the inductive step to make sense when adding these new directions is ii); i.e. that λ_{j+1} and λ_{j+2} must be linearly independent with respect to the considered principal direction. Otherwise, condition ii) wouldn't be thin. Therefore, the relation would not be ample and the process would not work.

In order to overcome this issue, we will ask in the Preparation step at the beginning that codirections from the $i-th$ cube must be linearly independent with respect to the directions from all other cubes B_j which intersect B_i; i.e. those such that $B_i \cap B_j \neq \emptyset$. But that can be achieved since the beginning in the Preparation step by Theorem 3.17, by asking transversality of the associated dual directions:

$$\lambda^i_\ell \wedge \lambda^i_h \wedge \lambda^j_m \neq 0 \text{ for all } 1 \leq \ell < h < m \leq 6 \text{ and } i,j \text{ such that } B_i \cap B_j \neq \emptyset. \tag{12}$$

Since convex integration works relative to domain and parameter, the process explained in subsection 3.1.4 works and we end up with a holonomic solution.

References

[1] J. Adachi. *Classification of horizontal loops in standard Engel space.* Int. Math. Res. Not. 2007 Vol. 3 (2007).

[2] V.I. Arnol'd, S.P. Novikov. *Dynamical systems VII.* Enc. of Math. Sci., vol. 16, Springer-Verlag, Berlin Heidelberg, 1994.

[3] R.L. Bryant, L. Hsu. *Rigidity of integral curves of rank 2 distributions.* Invent. Math. 114 (1993), no. 2, 435–461.

[4] E. Cartan. *Sur quelques quadratures dont l'élément différentiel contient des fonctions arbitraires.* Bull. Soc. Math. France 29 (1901), 118–130.

[5] R. Casals, J.L. Pérez, Á. del Pino, F. Presas. *Existence h-principle for Engel structures.* Invent. Math. 210 (2017), 417-451

[6] R. Casals, Á. del Pino, F. Presas. *Loose Engel structures.* arXiv:1712.09283

[7] Y. Eliashberg, D. Kotschick, E. Murphy, T. Vogel. *Engel structures.* AIM Workshop report (2017).

[8] Y. Eliashberg, N. Mishachev. *Introduction to the h-principle.* Graduate Studies in Mathematics, 48. American Mathematical Society, Providence, RI, 2002.

[9] Y. Eliashberg, N. Mishachev. *Wrinkled embeddings.* Foliations, geometry, and topology, 207–232, Contemp. Math., 498, Amer. Math. Soc., Providence, RI, 2009.

[10] G. Frobenius, *Uber das Pfaffsche Problem.* J. für Reine und Agnew. Math., 82 (1877) 230–315.

[11] Z. Ge. *Horizontal path spaces and Carnot-Carathéodory metrics.* Pacific J. Math. Volume 161, Number 2 (1993), 255-286.

[12] H. Geiges. *Horizontal loops in Engel space.* Math. Ann. Volume 342, Issue 2, (2008) 291–296.

[13] M. Gromov. *Partial differential relations.* Ergebnisse Math. und ihrer Gren. , Springer–Verlag, Berlin, 1986.

[14] M. Gromov, *Carnot-Carathéodory spaces seen from within.* Sub-Riemannian geometry, Progr. Math., vol. 144, Birkhäuser, Basel, 1996.

[15] M. Mauhart, P. W. Michor. *Commutators of flows and fields.* Archivum Mathematicum (Brno) 28 (1992), 228–236.

[16] R. Montgomery. *Engel deformations and contact structures.* Northern California Symplectic Geometry Seminar, 103–117, Amer. Math. Soc. Transl. Ser. 2, 196, Amer. Math. Soc., Providence, RI, 1999.

[17] R. Montgomery. *A tour of subriemannian geometries, their geodesics and applications.* Vol. 91. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2002.

[18] J. Martínez-Aguinaga, Á. del Pino. *$h-$Principle for regular curves in bracket-generating distributions* In preparation.

[19] D. McDuff. *Applications of convex integration to symplectic and contact geometry.* Ann. Inst. Fourier 37 (1987), 107–133.

[20] E. Murphy. *Loose Legendrian Embeddings in high dimensional contact manifolds*, arXiv:1201.2245.

[21] Á. del Pino, F. Presas. *Flexibility for tangent and transverse immersions in Engel manifolds* arXiv:1609.09306.

[22] Á. del Pino, T. Vogel. *The Engel-Lutz twist and overtwisted Engel structures.* arXiv:1712.09286

[23] T. Vogel. *Existence of Engel structures.* Ann. of Math. (2) 169 (2009), no. 1, 79–137.

Universidad Complutense de Madrid, Facultad de Matemáticas, Plaza de Ciencias 3, 28040 Madrid, Spain

Email address: javier.martinez.aguinaga@icmat.es

Utrecht University, Department of Mathematics, Budapestlaan 6, 3584 Utrecht, The Netherlands

Email address: a.delpinogomez@uu.nl