WRINKLING h-PRINCIPLES FOR INTEGRAL SUBMANIFOLDS OF JET SPACES

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Abstract. Y. Eliashberg and N. Mishachev introduced the notion of *wrinkled embedding* to show that any tangential homotopy can be approximated by a homotopy of topological embeddings with mild singularities. This concept plays an important role in Contact Topology: The loose legendrian h-principle of E. Murphy relies on wrinkled embeddings to manipulate the legendrian front. Similarly, the simplification of legendrian front singularities was proven by D. Álvarez-Gavela by defining the notion of *wrinkled legendrian*.

This paper generalises these ideas to general jet spaces. Our main theorem proves the analogue of the result by Eliashberg and Mishachev: Any homotopy of the r–order differential information of an embedding can be approximated by a homotopy of embeddings with wrinkle-type singularities (of order r).

Our first application deals with submanifolds of jet spaces tangent to the canonical distribution. Outside of the contact case, we show that there is a complete h-principle as long as the submanifolds have singularities of tangency with respect to the vertical of corank at most 1 (which we dub Whitney singularities of order r). The motto is that, for spaces of jets other than contact, global topological questions can be tackled with h-principle methods, but the local geometry of the singularities with the vertical is non-trivial.

Our second application considers once again tangent submanifolds in jet spaces, but with prescribed corank-1 singularities. We then prove that a complete h-principle holds as long as the submanifold has a concrete local model that we call the *loose chart*, following Murphy. In the front projection, the model is indeed a stabilised zig-zag (of order r) contained in a sufficiently big neighbourhood.

As a corollary of the previous result we obtain an h-principle for singular legendrians with prescribed singularities (modelled on the Whitney singularities of order r). This follows by projecting down from r-jet space.

Contents

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PAPER I:

1. Introduction

1.1. **Wrinkled embeddings.** Let $M \subset N$ be manifolds of dimensions m and n, respectively. In many geometrically meaningful situations, we are interested in producing isotopies of M that simplify its position with respect to some geometric structure in N. For instance, N may be endowed with a foliation $\mathcal{F}$ and we want to isotope M so that it becomes transverse.

In general, such a process is obstructed. The tangent space of M defines a section $\mathrm{Gr}(M) : M \to \mathrm{Gr}(TN, m)|_M$ into the Grassmannian of m-planes of N. If we want to make M transverse to $\mathcal{F}$ by an isotopy, it is certainly necessary that $\mathrm{Gr}(M)$ can be homotoped to be transverse to $\mathcal{F}$. This obstruction is purely algebraic topological in nature and can be analysed using obstruction theory. One may then ponder whether the vanishing of these obstructions (that we call *formal*) is sufficient for the existence of the desired isotopy. The answer is, in general, no. For instance, if N is a fibration over $\mathbb{R}$ and $\mathcal{F}$ is the foliation by fibres, making M transverse to $\mathcal{F}$ would produce a function on M with no critical points, which is impossible if M is closed. This obstruction is geometric and not algebraic.

In situations where singularities may be unavoidable due to geometric reasons, we may attempt to make them as mild as possible instead. This was proven, in the aforementioned setting, by Eliashberg and Mishachev [21]: They showed that, if no formal obstruction exists, M can be isotoped to have simple singularities of tangency (i.e double folds, see Subsection **??**) with respect to $\mathcal{F}$. The idea of the proof is as follows: Since no formal obstruction exists, we are given a homotopy Gr_s starting at $G_0 = \mathrm{Gr}(M)$ and finishing at a bundle G_1 transverse to $\mathcal{F}$. Then, instead of isotoping M, we produce a homotopy M_s of topological submanifolds that may have cuspidal singularities (or so-called *wrinkles*). Despite being singular, the M_s admit well-defined Grassmannian maps $\mathrm{Gr}(M_s)$ and the heart of the argument is that it is possible to choose M_s so that $\mathrm{Gr}(M_s)$ approximates Gr_s; here the flexibility provided by the cuspidal singularities/wrinkles is key. The proof concludes by smoothing out M_s; when we do so, M_1 becomes a smooth submanifold and the cusps/wrinkles are traded for (simple!) singularities of tangency with $\mathcal{F}$.

1.2. **Wrinkled embeddings of higher order.** The starting point of the present article is that the result of Eliashberg-Mishachev is a first order statement. That is: $\mathrm{Gr}(M)$ is the first derivative of M, and their theorem states that any homotopy Gr_s of this first derivative can be approximated by a homotopy M_s of M, as long as we allow for simple singularities in the process. Our main result says that this can be done for higher order data as well:

Theorem 1.1. *Fix an integer r. Any homotopy of the r–order information of an embedding can be approximated by a homotopy of topological embeddings with A_{2r}-zig-zags. Similar statements hold parametrically and relatively by allowing zig-zags to appear and disappear.*

Generalising the first order case requires us to replace the standard cusps by singularities of higher order; an A_{2r}-zig-zag is precisely a pair of r-order cusps. This statement will be stated more formally and proven, as Theorem 9.1, in Section 9.

1.3. **Holonomic approximation.** One can succinctly state Theorem 1.1 by saying that *holonomic approximation holds for submanifolds with zig-zags.* To put this into perspective, let us recall the standard setup for h-principle and geometric PDEs.

Given a smooth bundle $Y \to X$, we can define the *bundle of r-jets* $J^r(Y \to X) \to X$. Its fibres consist of r-order Taylor polynomials of sections of Y. Given any section $f : X \to Y$, we can consider its r-order differential data $j^r f : X \to J^r(Y \to X)$; such a section of jet space is said to be *holonomic*. Most sections $F : X \to J^r(Y \to X)$ are not holonomic and, to emphasise this, we call them *formal sections*.

This provides a very convenient setup to discuss partial differential relations (PDRs). Indeed, we can define a PDR of order r to be a subset $\mathcal{R} \subset J^r(Y \to X)$. It readily follows that a *solution* of $\mathcal{R}$ is a section $f : X \to Y$ whose r-order Taylor polynomial $j^r f$ takes values in $\mathcal{R}$. More generally, we can define *formal solutions* of $\mathcal{R}$ to be sections $F : X \to \mathcal{R}$. The existence of formal solutions is thus a necessary condition for the existence of solutions. One can then compare the spaces of solutions and formal solutions and ask, in particular, whether the two are weakly homotopy equivalent. If this is the case, $\mathcal{R}$ is said to *satisfy the h-principle*.

If the relation $\mathcal{R}$ we consider is open (as is sometimes the case with relations of geometric origin like those describing contact or symplectic structures), we could attempt to find solutions of $\mathcal{R}$ using the following idea: We start with a formal solution F and we find $f : X \to Y$ such that $j^r f$ approximates F. If this approximation is good enough, $j^r f$ will land in $\mathcal{R}$ and f will be a solution. This idea is called, quite descriptively, *holonomic approximation*.

It turns out that this does not work and, indeed, many open relations do not satisfy the h-principle (for instance, symplectic structures). However, using his *method of flexible sheaves*, M. Gromov [] proved that holonomic approximation does hold if we try to approximate only over a subset $K \subset X$ of positive codimension (in fact, one needs to approximate not quite over the given K but over a C^0-close copy of K that is more "wiggly"). Due to the fact that open manifolds can be retracted to the skeleton (which is a positive codimension CW-complex), this can then be used to prove that the h-principle applies for any $\mathcal{R}$ open and Diff-invariant (i.e. invariant under the action of the diffeomorphism group of X), as long as M is open. This result applies to "generic"/"non-degenerate" geometric structures (like contact or symplectic) and generalises prior results about immersions (due to Hirsch-Smale []) and submersions (due to Phillips []).

One can then pose the question: "what can we do for closed manifolds?" We henceforth assume that X is indeed closed.

1.4. **Holonomic approximation for multiply-valued sections.** At each point in $J^r(Y \to X)$, we are given a collection of tautological equations encoding the fact that certain fibre directions should be derivatives of some others. These equations are pointwise linear and define the so-called *Cartan distribution* ξ_{can} in jet space. From its construction, it follows that holonomic sections can be characterised as those sections tangent to ξ_{can}. This led R. Thom [**?**] to define *generalised solutions* of $\mathcal{R}$ as maps $X \to J^r(Y \to X)$ (not necessarily sections!) tangent to ξ_{can} and taking values in $\mathcal{R}$. To emphasise the fact that these are not sections, one may note that it also makes sense to consider general tangent maps $L \to (J^r(Y \to X), \xi_{\text{can}})$, where L is some other manifold of the same dimension as X.

Recall that holonomic sections $j^r f : X \to J^r(Y \to X)$ are in correspondence with their underlying sections $f : X \to Y$. This is (almost) true for generalised solutions as well: $\phi : L \to (J^r(Y \to X), \xi_{\text{can}})$

can be uniquely recovered from its *front projection* $\pi_f \circ \phi : L \to Y$ whenever the *base projection* $\pi_b \circ \phi : L \to X$ is a immersion. It follows that, as long as ϕ is graphical over X in a dense set, it will be uniquely recovered from $\pi_f \circ \phi$. However, the fibre over a point $x \in X$ will often intersect the image of $\pi_f \circ \phi$ in a number of points different than one. Due to this, we will sometimes refer to *generalised solutions* as *multi-sections*.

We can then ask whether it is possible to approximate any formal section by a generalised section. We phrase this as "holonomic approximation holds, even over closed manifolds, if we allow for generalised sections". Remarkably, Thom, preceding Gromov's work, proved this in [**?**], using ideas that seem to be a precursor of the pleating/wrinkling approaches to h-principles [**?**]. However, Thom's argument is somewhat incomplete (particularly regarding higher jets) and, more importantly for us, his method produces generalised solutions with uncontrolled singularities. Later, Gromov provided an argument [], based on microflexibility/pleating, to construct generalised solutions whose only singularities are folds. His approach applies to more general manifolds endowed with bracket-generating distributions, as long as certain dimensional constraints hold[1].

Our result produces generalised solutions that are smooth and embedded. To do so, we use pleating in the front projection, instead of in jet space itself. In particular, our generalised solutions will have topologically embedded front projections with A_{2r}-zig-zags (i.e. cusps of order r in the front, that project to the base manifold M as double folds, and are smooth when lifted to r-jet space). Our second result reads:

Theorem 1.2. *Let $F : X \to J^r(X)$ be an arbitrary section. Then, for any $\varepsilon > 0$, there exists a map $f : X \to J^r(X)$ satisfying:*

- *f is a section with zig-zags;*
- *$|j^r f - F|_{C^0} < \varepsilon$.*

I.e. holonomic approximation holds when we consider multi-valued sections instead of just sections. It then follows immediately:

Corollary 1.3. *Let $\mathcal{R} \subset J^r(X)$ be an open differential relation admitting a formal solution F. Then, $\mathcal{R}$ admits a generalised solution f with $j^r f$ C^0-close to F.*

That is, even though the h-principle does not hold for arbitrary open differential relations, it does hold when we allow generalised solutions. As before, both these statements can be extended to the parametric (in which F varies in a family F_s) and relative settings (where all the F_s are already holonomic in some part of the domain, and some of them are everywhere holonomic). These versions are stated and proven in Section 8.

The reader may wonder what is the relation between Theorems 1.1 and 1.2. The answer is that the latter is the local version of the former. That is: when trying to isotope a submanifold $M \subset N$, we work in its tubular neighbourhood $\mathcal{O}p(M)$, which we identify with the normal bundle $\nu(M) \to M$. Then, sufficiently small homotopies of M as a submanifold with zig-zags correspond to homotopies of the zero section as a multi-section with zig-zags. I.e. Theorem 1.2 implies Theorem 1.1; this is carried out in detail in Section 9.

1.5. **Method of proof.** The proof of Theorem 1.2 has three parts: First we triangulate M very finely and choose holonomic approximations over each top-cell. Secondly, we apply standard holonomic approximation along the codimension-1 skeleton, reducing the statement to the top-dimensional cells. In the last step we are thus working on a ball and we must introduce zig-zags to interpolate between the two holonomic approximations (one coming from the cell and the other one from its boundary).

[1] A distribution ξ can be endowed with a fibrewise inner product g_ξ. Under the bracket-generating assumption, infimising the length of all tangent paths (with respect to g_ξ) produces a metric structure on its ambient manifold Y, known as the Carnot-Carathéodory metric. Lipschitz maps from a Riemannian manifold (N, g_N) into (Y, ξ, g_ξ) are then Lipschitz generalisations of maps tangent to ξ. A result of Gromov [], also in the setting of general distributions, explains how to construct Lipschitz maps using pleating (also under suitable dimensional assumptions). This result was made more explicit, in the setting of jet spaces, by .... in [].

The zig-zags will appear forming concentric spheres close to the boundary. The reader is invited to visualise this as an "accordion" appearing in each top-cell; see Figure **??**.

The overall structure of the proof is quite common in h-principle. The first two steps are often called *reduction* (and they rely on standard techniques) and the last one is called *extension* (which requires more work). This last step boils down to a careful application of holonomic approximation, similar to an idea that appeared first in the contact setting [9]. It differs somewhat significantly from the strategy in the original wrinkled embeddings paper [21], which strongly relies on the fact that one works with first order data. We believe that the parallels between *wrinkling* and *wiggling* are more transparent using our viewpoint.

1.6. **Submanifolds tangent to distributions.** As we explained earlier, $J^r(Y \to X)$ is endowed with the Cartan distribution ξ_{can}, which is *bracket-generating* (i.e. iterated Lie brackets of vector fields tangent to ξ_{can} span the tangent space of $J^r(Y \to X)$). The main motivation behind this article was to understand better the space of embeddings $L \to (J^r(Y \to X), \xi_{\text{can}})$ tangent to ξ_{can}. This is extremely natural: Recall that the first jet space of functions $(J^1(X \times \mathbb{R} \to X), \xi_{\text{can}})$ is a contact manifold and therefore tangent embeddings are precisely legendrians. The study of legendrians is a driving force in Contact Topology, where h-principle results provide flexibility (i.e. classification results) and generating functions/pseudoholomorphic curves/sheaves provide rigidity (in the form of obstructions/invariants).

There are two results on legendrians that we can highlight as inspiration of the present work: The first is the celebrated classification of loose legendrians due to E. Murphy [37], where it is shown that a certain subfamily of legendrians in higher dimensions satisfy the h-principle. The second is the simplification of legendrian singularities of tangency due to D. Álvarez-Gavela [1]; this generalises the work of Eliashberg and Mishachev on wrinkled submanifolds to the contact setting.

However, similar results are not available for higher order jets (or for bundles with larger fibre). One exception is the case of curves tangent to Engel structures (which are locally modelled on $(J^2(\mathbb{R}, \mathbb{R}), \xi_{\text{can}})$), and which was treated in [13, 12].

To put this article into context, let us provide a brief comparison of the contact setting within the general framework of jet spaces. Along the way we state our results about integral submanifolds.

1.6.1. *The failure of the Legendre transform.* The main theme is that contact jet spaces are quite distinct from all other spaces of jets. The reason behind this is that contact structures have plenty of symmetries. Contact transformations, like the Legendre transform, may interexchange variables and derivatives in $J^1(X \times \mathbb{R} \to X)$.

However, this is not true anymore in any other jet space. This is easy to see by dimension counting: in all other jet spaces, there are more the directions corresponding to derivatives than directions corresponding to base variables. One may actually prove that any transformation of $J^r(Y \to X)$ is a lift of a contact transformation in $J^1(Y \to X)$.

1.6.2. *Most projections are intrinsic.* In a contact manifold, we have thus plenty of freedom in choosing how to parametrise our Darboux balls. In particular, we can fix different local front projections by changing the charts we use. This is useful in manipulating legendrians, since we can pick charts adapted to them (often just making them graphical).

From the previous item, it can be deduced that this is not the case for other jet spaces. Namely, the behaviour of ξ_{can} as a distribution defines intrinsically the fibres of the projection $J^r(Y \to X) \to J^{r-1}(Y \to X)$, $r > 1$. Further, if the fibres of $Y \to X$ have dimension at least 2, the fibres of the front projection are similarly intrinsic.

It follows that, if we want to manipulate a tangent submanifold, we may not be able to find "nice" projections adapted to it.

1.6.3. *The geometry of tangent submanifolds is difficult.* Following with the previous remark: The fibre $J^r(Y \to X) \to J^{r-1}(Y \to X)$ is tangent to ξ_{can} but is often larger in dimension than X. In that case, it cannot be homotoped, as a tangent submanifold, to become graphical almost everywhere (i.e. a multi-section).

This issue is also problematic even if we consider a tangent submanifold $L \to J^r(Y \to X)$ with $\dim(L) = \dim(X)$. Its singularities with respect to the fibre could be quite complicated and, since we are not allowed to change our projection to make them better, it is unclear how to perturb L to make them "generic" from a formal point of view. In particular, even if no formal obstructions exist, it is unclear how to simplify the singularities using h-principle arguments.

We bypass this issue by ignoring it: We focus on the subclass of tangent submanifolds whose singularities are of Whitney type. These we are able to deal with using the front projection. Classification results for general submanifolds (with arbitrary singularities) are left as an intriguing open question.

1.6.4. *The topology of tangent submanifolds is flexible.* Non-contact jet spaces have "more room" than contact ones. Namely, the distribution ξ_{can} has codimension larger than 1. This extra room can be used to prove a complete h-principle for tangent submanifolds with Whitney singularities.

For instance, it is possible to manipulate a given submanifold to introduce (through a homotopy of tangent submanifolds) various local models that one may interpret as incarnations of the usual stabilisation of legendrians. This is the main ingredient behind our h-principles.

This is explained in Section 12. The proof requires some auxiliary results on prescription of singularities for smooth manifolds (not necessarily integral), which we review in Appendix 15.

1.6.5. *Odd vs. even.* Jet spaces behave differently depending on the parity of r. In the odd case, the standard tangent homotopy analogous to the Reidemeister I move, does resemble a Reidemeister I move in the front. In the even case, it instead resembles a stabilisation (i.e. a zig-zag).

In practice, this means that one needs to treat each case separately. Namely, for r even, we can simply introduce many of these zig-zags to obtain flexibility. For r odd (and not contact), we introduce standard-looking Reidemeister Is and we then pass one of the resulting cusps to the other side of the original strand (this can only be done through embeddings because we are not contact). This yields a picture of a zig-zag that has one of its cusps stabilised.

To treat both cases in a more streamlined manner, we actually introduce zig-zags for r both even and odd and then, we apply surgery of singularities in the odd case to obtain instead zig-zags with one stabilised cusp.

1.6.6. *The issue of isotopies.* In the contact setting, a homotopy of embedded legendrians yields an isotopy. This is not true anymore for other jet spaces. This follows from the fact that, since the fibres of $J^r(Y \to X) \to J^{r-1}(Y \to X)$ are intrinsic, the tangencies of generalised solution with respect to them are intrinsic as well. In particular, they will be preserved by isotopies. That is, even though we prove flexibility for tangent submanifolds under homotopies, it is still meaningful to ask for classification statements up to isotopy.

We remark that, assuming that the fibres of $Y \to X$ are 1-dimensional, this is related to the classification of legendrians with singularities. Indeed, tangent submanifolds of $(J^r(Y \to X), \xi_{\text{can}})$ project to tangent submanifolds of $(J^1(Y \to X), \xi_{\text{can}})$ and all isotopies upstairs are uniquely determined by contact transformations downstairs. It may thus be possible to tackle this problem, from the rigid viewpoint, using the machinery of generating functions/sheaves/holomorphic curves. This is an open question beyond the scope of this article.

1.6.7. *Flexibility for isotopies.* Assume that we have a tangent submanifold $L \subset (J^r(Y \to X), \xi_{\text{can}})$, of dimension at least 2, and with Whitney singularities of tangency with respect to the fibre. Under isotopies, we are not allowed to add new singularities of tangency, but we may assume from the get-go that there is a region in L containing a "sufficiently big" zig-zag. This zig-zag can be spread out, through isotopies, to the whole manifold, providing us with flexibility.

That is, as long as there is some special model, we can prove an h-principle for tangent submanifolds up to isotopy. This, of course, resembles the h-principle of Murphy for legendrians with a loose chart. In fact, our result recovers [37] when particularised to the contact setting. Our argument is similar to hers in spirit but differs in its implementation in one key aspect: Since we cannot change the front projection, we need to describe how to pass our "loose zig-zag" across the other singularities of the submanifold.

By projecting to $J^1(Y \to X)$, this can be interpreted as an h-principle for legendrians with singularities of given complexity (namely, singularities arising as projections of tangencies in $J^r(Y \to X)$) and having a loose chart (of the same complexity).

This is explained in Section 14. The surgery arguments needed for the proof appear in Section 13.

Acknowledgments:

2. Distributions

In this Section we review the basics of distributions. They will reappear in the next Section, when we look at the Cartan distribution in jet space. The reader may want to refer to the standard references [34, Chapters 2 and 4], [10], and [23].

2.1. The Lie flag.

Fix a manifold M endowed with a distribution $\xi \subset TM$. The vector fields $\Gamma(\xi)$ tangent to ξ are a C^∞-submodule of the space of all vector fields. It is natural to analyse to what extent this subspace fails to be a Lie subalgebra (with respect to the Lie bracket of vector fields):

Definition 2.1. *The* ***Lie flag*** *associated to ξ is the sequence of C^∞-modules of vector fields defined by the inductive formula:*

$$\Gamma(\xi^{(1)}) := \Gamma(\xi), \qquad \Gamma(\xi^{(i+1)}) := [\Gamma(\xi), \Gamma(\xi^{(i)})],$$

where the rightmost expression denotes taking the C^∞-span of all Lie brackets with entries in $\Gamma(\xi)$ and $\Gamma(\xi^{(i)})$.

In this article we always assume that $\Gamma(\xi^{(i)})$ is the module of sections of a distribution $\xi^{(i)}$. Do note that, in general, this need not be the case. Due to this, we sometimes say that the Lie flag is instead the filtration:

$$\xi^{(1)} := \xi \subset \xi^{(2)} \subset \xi^{(3)} \subset \cdots$$

2.2. Involutive vs. bracket-generating.

By definition, $\Gamma(\xi)$ is a Lie subalgebra if and only if $\Gamma(\xi^{(2)}) = \Gamma(\xi)$. That is, if and only if the associated Lie flag is constant. Such a ξ is said to be **involutive**. Frobenius' theorem states that involutivity of a distribution is equivalent to integrability. The Lie flag is, therefore, a measure of the non-integrability of ξ.

For us, the more interesting case is the complete opposite: ξ is said to be **bracket-generating** if, for some integer r, it holds that $\xi^{(r)} = TM$; i.e. $\Gamma(\xi)$ generates, as an algebra, the space of all vector fields. A well-known theorem of Chow states that any two points in M can be connected by a path tangent to ξ if ξ is bracket-generating. This can be regarded as the first result showing that submanifolds tangent to bracket-generating distributions behave flexibly.

2.3. Curvature and nilpotentisation.

We can define more refined invariants measuring the non-integrability of ξ. By construction, there is a map between sections

$$\Gamma(\xi^{(i)}) \times \Gamma(\xi^{(j)}) \to \Gamma(\xi^{(i+j)}/\xi^{(i+j-1)})$$

induced by Lie bracket. It can be checked that this map is C^∞-linear, allowing us to write:

Definition 2.2. *The **(i,j)-curvature** of ξ is the tensor:*

$$\Omega_{i,j}(\xi) : \xi^{(i)}/\xi^{(i-1)} \times \xi^{(j)}/\xi^{(j-1)} \to \xi^{(i+j)}/\xi^{(i+j-1)}.$$

We can then endow the graded vector bundle

$$L(\xi) := \oplus_{i=1}^{r} L(\xi)_i := \oplus_{i=1}^{r}(\xi^{(i)}/\xi^{(i-1)}) = \xi \oplus (\xi^{(2)}/\xi^{(1)}) \oplus \cdots \oplus (\xi^{(r)}/\xi^{(r-1)})$$

with a fibrewise Lie bracket $\Omega(\xi) = \oplus_{i,j}\Omega_{i,j}(\xi)$ that respects the grading. Then, the pair $(L(\xi), \Omega(\xi))$ is a bundle of graded Lie algebras; we call it the **nilpotentisation** of ξ. Do note that the different fibres may not be isomorphic to one other as Lie algebras (although this will be the case for the Cartan distribution in jet space).

The nilpotentisation should be thought as a linearisation of ξ packaging its infinitesimal behaviour under Lie bracket. From an h-principle viewpoint, once we identify $L(\xi) \cong TM$, the nilpotentisation plays the role of the formal data associated to ξ.

2.4. **Integral elements and submanifolds.** Maps and submanifolds tangent to ξ are said to be **integral**. It is immediate that the first curvature $\Omega_{1,1}(\xi)$ vanishes when restricted to an integral submanifold. This leads us to restrict our attention to those subspaces of ξ that might potentially be tangent to them:

Definition 2.3. *An **integral element** is a linear subspace $W \subset \xi_p \subset T_pM$, $p \in M$, satisfying $\Omega_{1,1}(\xi)_p|_W = 0$.*

The collection of all integral elements $\mathrm{Gr}_{\mathrm{integral}}(\xi, l) \subset \mathrm{Gr}(\xi, l) \subset \mathrm{Gr}(TM, l)$ of a given dimension l is called the **integral Grassmannian bundle**. Its fibres are algebraic subvarieties that may not be smooth nor vary smoothly.

Given an integral element $W \subset \xi_p$ we define its **polar space** as:

$$W^{\xi} := \{v \in \xi_p \mid \Omega_{1,1}(\xi)_p(w, v) = 0, \ \forall w \in W\}.$$

That is, the linear subspace vectors that pair trivially with W using the curvature. Since W is integral, W^{ξ} contains W. Tautologically, extensions of W to an integral element of dimension $\dim(W) + 1$ are in correspondence with lines in the quotient W^{ξ}/W. An element is said to be **maximal** if $W = W^{\xi}$, i.e. if it is not contained in a larger integral element.

Definition 2.4. *A vector $w \in \xi_p$ satisfying $\langle w\rangle^{\xi} = \xi_p$ is called a **Cauchy characteristic**. The linear subspace $\ker(\xi_p)$ spanned by all the Cauchy characteristics is an integral element.*

If the dimension of $\ker(\xi_p)$ does not vary with $p \in M$, then their union is an involutive distribution $\ker(\xi) \subset \xi$ that we call the **characteristic foliation** of ξ. Its leaves are integral submanifolds.

It is immediate that any local diffeomorphism preserving ξ must preserve $\ker(\xi)$. Similarly, its differential can identify two vectors tangent to ξ only if their polar spaces have the same dimension.

Example 2.5. *Let (M, ξ) be a contact manifold. Then the curvature $\Omega_{1,1}(\xi)$ is a nondegenerate 2-form on ξ with values on TM/ξ. Indeed, if $\ker(\alpha) = \xi$, then we have $\alpha \circ \Omega_{0,0}(\xi) = -d\alpha$.*

*A subspace $W \subset \xi_p$ is isotropic if and only if it is integral; maximal integral elements are precisely lagrangians. The polar space W^{ξ} is the usual **symplectic orthogonal**. Integral Grassmannians are thus the same as the Grassmannians of isotropic subspaces.* △

3. Jet spaces

In this Section we recall some elementary notions about jet spaces, putting particular emphasis on their tautological distribution, which is bracket-generating. We go over standard material in order to set up notation.

A standard reference in the Geometry of PDEs literature is [27, Chapter IV], but we also recommend [43, Section 2]. The two standard h-principle references also treat jet spaces, namely [25, Section

1.1] and [20, Chapter 1]. Lastly, the reader may want to look at [26, Section 4.1], whose ideas have certainly inspired parts of this work.

3.1. **Jet spaces of sections.** Let X be an n-dimensional manifold and let $\pi : Y \to X$ be a smooth fibre bundle with k-dimensional fibres. We write $J^r(Y \to X)$ for the space of all r-jets of sections $X \to Y$. When Y is the trivial $\mathbb{R}^k$-bundle over X we often denote it by $J^r(X, \mathbb{R}^k) := J^r(Y \to X)$.

The spaces of r-jets, for varying r, fit in a tower of affine bundles:

$$J^r(Y \to X) \xrightarrow{\pi_{r,r-1}} J^{r-1}(Y \to X) \xrightarrow{\pi_{r-1,r-2}} \cdots \xrightarrow{\pi_{1,0}} J^0(Y \to X) = Y. \tag{3.1.1}$$

that map holonomic sections to holonomic sections. For notational convenience, we single out the **front projection** and the **base projection** which are given, respectively, by the forgetful maps:

$$\pi_f := \pi_{r,0} : J^r(Y \to X) \to Y, \qquad \pi_b : J^r(Y \to X) \to X.$$

3.1.1. *Local coordinates.* By working locally we may assume that X is a n-dimensional vector space, denoted by B, and that the fibre of Y is a k-dimensional vector space, denoted by F. In this local setting the jet space $J^r(Y \to X)$ can be identified with $J^r(B, F)$. To be explicit, we choose coordinates $x := (x_1, \cdots, x_n)$ in B and coordinates $y := (y_1, \cdots, y_k)$ in F. We use (x, y) to endow $J^r(B, F)$ with coordinates, as we now explain.

A point $p \in J^r(B, F)$ is uniquely represented by an r-order Taylor polynomial based at $\pi_b(p) \in X$. Now, the r-order Taylor polynomial of a map $f : B \to F$ at x reads:

$$f(x + h) \cong \sum_{0 \leqslant |I| \leqslant r} (\partial^I f(x)) \frac{dx^{\odot I}}{I!}(h, \ldots, h),$$

where $I = (i_1, \ldots, i_n)$ ranges over all multi-indices of length at most r. Here $\odot$ denotes the symmetric tensor product and we use the notation

$$\mathrm{d}x^{\odot I} := \mathrm{d}x_{i_1} \odot \cdots \odot \mathrm{d}x_{i_n}, \quad I = (i_1, \ldots, i_n).$$

This tells us that $J^r(B, F) \to B$ is a vector bundle and that, formally, we can use the monomials

$$\frac{dx^{\odot I}}{I!} \otimes e_j, \qquad 0 \leqslant |I| \leqslant r' \quad 1 \leqslant j \leqslant k$$

as a framing; here $\{e_j\}_{1 \leqslant j \leqslant k}$ is the standard basis of F in the (y)-coordinates. We can write $z_j^{(I)}$ for the coordinate dual to the vector $\frac{dx^{\odot I}}{I!} \otimes e_j \in \mathrm{Sym}^{|I|}(B^*, F)$. This definition depends only on the choice of coordinates $(x, y) : Y \to B \times F$. We give these coordinates a name:

Definition 3.1. *The coordinates*

$$(x, y, z) := (x, y = z^0, z^1, \ldots, z^r), \qquad z^{r'} := \{z_j^{(I)} \mid |I| = r', 1 \leqslant j \leqslant k\},$$

in $J^r(Y \to X)$ are said to be ***holonomic***.

The monomials above with $|I| = r'$ form a basis of $\mathrm{Sym}^{r'}(B^*, F)$, the space of a symmetric tensors with r' entries in B and values in F. This leads us to write, in more conceptual terms:

Lemma 3.2. $J^r(B, F) = B \times F \times \mathrm{Hom}(B, F) \times \mathrm{Sym}^2(B^*, F) \times \cdots \times \mathrm{Sym}^r(B^*, F)$. *In particular, $\pi_{r,r-1}$ is an affine bundle with fibres modelled on* $\mathrm{Sym}^r(B^*, F)$.

3.2. **The Cartan distribution.** The tautological/Cartan distribution ξ_{can} in $J^r(Y \to X)$ is uniquely defined by the following universal property: a section of $J^r(Y \to X)$ is tangent to ξ_{can} if and only if it is holonomic. The subbundle $V_{\mathrm{can}} := \ker(d\pi_{r,r-1}) \subset \xi_{\mathrm{can}}$ is called the **vertical distribution**.

Images of holonomic sections correspond to integral submanifolds that are everywhere transverse to V_{can}. Integral submanifolds and elements transverse to V_{can} are said to be **horizontal**.

3.2.1. *Local coordinates.* In terms of the holonomic coordinates $(x, y, z) \in J^r(B, F)$ defined above, the holonomic lift of a map $f : B \to F$ reads:

$$\begin{array}{rcl} j^r f : B & \to & J^r(B,F) = \quad B \times F \times \mathrm{Hom}(B,F) \times \mathrm{Sym}^2(B^*,F) \times \cdots \times \mathrm{Sym}^r(B^*,F), \\ x & \to & j^r f(x) \quad = \quad (x, y = f(x), z^1 = (\partial f)(x), z^2 = (\partial^2 f)(x), \ldots, z^r = (\partial^r f)(x)). \end{array}$$

That is, a holonomic section satisfies the relations

$$z_j^{(I)}(x) = (\partial^I y_j)(x), \qquad I = (i_1, \ldots, i_n),\ 0 \leqslant |I| \leqslant r,\ 1 \leqslant j \leqslant k.$$

Equivalently, the tautological distribution ξ_{can} is the simultaneous kernel of the **Cartan 1-forms**:

$$\alpha_j^I = dz_j^{(I)} - \sum_{a=1}^{n} z_j^{(i_1,\cdots,i_a+1,\cdots,i_n)} dx_a, \qquad I = (i_1, \ldots, i_n),\ 0 \leqslant |I| < r,\ 1 \leqslant j \leqslant k. \tag{3.2.1}$$

3.2.2. *Associated distributions.* From this it can be deduced that the Lie flag associated to $(J^r(Y \to X), \xi_{\mathrm{can}})$ is given by the expression:

$$\xi_{\mathrm{can}}^{(i)} = d\pi_{r,r-i}^{-1}(\xi_{\mathrm{can}}),$$

where the right hand side is the preimage of the Cartan distribution on $J^{r-i}(Y \to X)$. In particular, ξ_{can} bracket-generates in $r + 1$ steps. Furthermore:

Lemma 3.3. *The following statements hold:*

- *If $r > 1$, the vertical distribution V_{can} is the characteristic foliation* $\ker(\xi_{\mathrm{can}})$.
- *Inductively,* $\ker(\xi_{\mathrm{can}}^{(i)}) = \ker(d\pi_{r,r-i})$ *for every $0 < i < r$.*
- *Assume $r > 1$ and $k = \dim(Y_x) > 1$. Then, the polar space of a horizontal vector is smaller in dimension than the polar space of a vertical one.*

That is: If we regard $(J^r(Y \to X), \xi_{\mathrm{can}})$ as an abstract manifold endowed with a distribution (i.e. we forget that projections $\pi_{r,r'}$), the Lemma tells us that we can recover the fibres of $\pi_{r,r'}$ intrinsically, as long as we are not in the contact case.

We say that $\ker(\xi_{\mathrm{can}}^{(i)})$ is the i**th characteristic foliation**. If $k = \dim(Y_x) > 1$, we say that the fibres of π_f are the **polar foliation** associated to ξ_{can}.

3.3. **The nilpotentisation.** According to the computations in the previous Subsections, the nilpotentisation of ξ_{can} at any point is isomorphic to the graded Lie algebra:

Definition 3.4. *Let B and F be real vector spaces of dimensions n and k, respectively.*

*The **jet Lie algebra** (depending on n, r, and k) is:*

- *The graded vector space $\mathfrak{g} := \bigoplus_{i=1}^{r+1} \mathfrak{g}_i$ with*
$$\mathfrak{g}_1 := B \oplus \mathrm{Sym}^r(B^*, F), \qquad \mathfrak{g}_i := \mathrm{Sym}^{r-i}(B^*, F).$$
- *Endowed with the Lie bracket defined by the contraction of vectors with tensors*
$$[v, \beta] = \iota_v \beta, \qquad v \in B, \quad \beta \in \mathrm{Sym}^j(B^*, F).$$
All other brackets are either defined by the antisymmetry or zero.

We will often abuse notation and use $\mathfrak{g}$ to denote the graded Lie algebra as a whole.

The degree one part $\mathfrak{g}_0$ is the direct sum $B \oplus \mathrm{Sym}^r(B^*, F)$. When identified with ξ_{can} at a point p, the first part corresponds to a lift of $T_p X$ (in a canonical manner once we choose local coordinates). The second term corresponds to the vertical distribution. We will henceforth say that B is the **horizontal component** and $\mathrm{Sym}^r(B^*, F)$ is the **vertical component**.

Integral elements of ξ_{can} correspond to vector subspaces $W \subset \mathfrak{g}_0$ which are, additionally, Lie subalgebras. Similarly, horizontal elements correspond to Lie subalgebras transverse to the vertical component.

3.4. **Distributions modelled on jet spaces.** Much like contact manifolds look locally like first jet spaces of functions, we can, more generally, consider manifolds with distributions locally modelled on some other jet space.

Definition 3.5. *We say that (M,ξ) is **modelled** on $(J^r(B,F),\xi_{\text{can}})$ if, for each $p \in M$, there are local coordinates (x,y,z) around p, with domain a subset of $J^r(B,F)$, so that $\xi = \xi_{\text{can}}$.*

In particular, the numbers $n = \dim(B)$, $k = \dim(F)$, and r are invariants of ξ. It follows that ξ bracket-generates in r steps and M is endowed with a flag $\{\ker(\xi^{(i)})\}_{i=1,\dots,r-1}$ of **characteristic foliations**. Similarly, when $k > 1$, we also have a well-defined **polar foliation**. In local coordinates these correspond to the fibres of the various projections.

3.4.1. *Automorphisms.*

Definition 3.6. *Let (M,ξ) be a distribution modelled on a jet space. A **(contact) transformation** of (M,ξ) is a ξ-preserving diffeomorphism.*

A more restrictive notion of symmetry, which only makes sense for jet spaces, is the following:

Definition 3.7. *Let $\Psi : Y \to Y$ be a fibre-preserving diffeomorphism lifting a diffeomorphism $\psi : X \to X$. The **point symmetry** lifting Ψ is defined as:*

$$\begin{aligned} j^r\Psi : (J^r(Y \to X),\xi_{\text{can}}) &\to (J^r(Y \to X),\xi_{\text{can}}) \\ j^r f(x) &\to (j^r\Psi)(j^r f(x)) := j^r(\Psi \circ f \circ \psi^{-1})(\psi(x)). \end{aligned}$$

Point symmetries form a subgroup of the group of contact transformations. It is well-known in Contact Geometry that the space of contact transformations of $J^1(X,\mathbb{R})$ is strictly larger than the space of point symmetries. From the existence of the polar and characteristic foliations we deduce:

Lemma 3.8. *Assume $r > 1$ or $k = \dim(Y_x) > 1$. Any contact transformation of $J^r(Y \to X)$ is the lift of a contact transformation of $J^{r-1}(Y \to X)$.*

3.4.2. *Jet spaces of submanifolds.* Let Y be a smooth manifold and fix an integer $n < \dim(Y)$. We say that two n-submanifolds have the same **r-jet** at $p \in Y$ if they are tangent at p with multiplicity r. We denote the space of r-jets of n-submanifolds as $J^r(Y,n)$. We have, just like in the case of sections, a sequence of forgetful projections

$$\pi_{r,r'} : J^r(Y,n) \to J^{r'}(Y,n),$$

with $\pi_f := \pi_{r,0}$ the **front projection**.

The **holonomic lift** of an n-submanifold $X \subset Y$ is the submanifold $j^rX \subset J^r(Y,n)$ consisting of all the r-jets of N at each of its points. The **Cartan distribution** ξ_{can} in $J^r(Y,n)$ is the smallest distribution which is tangent to every holonomic lift.

Given $X \subset Y$, we can restrict our attention to its tubular neighbourhood and to those submanifolds graphical over X. It follows that $(J^r(Y,n),\xi_{\text{can}})$ is locally modelled on a jet space.

Remark 3.9. *If $n = \dim(Y) - 1$ and $r = 1$, the structure we just constructed is precisely the **space of contact elements**. In general, if $r = 1$, the space $J^1(Y,n)$ is the Grassmannian of n-planes $\mathrm{Gr}(TY,n)$.* △

3.5. **The foliated setting.** Due to the parametric nature of the statements we want to prove, we will need to phrase our constructions in a foliated setting. An alternate (seemingly weaker but ultimately equivalent way) would be to use the fibered setting [20, 6.2.E].

Let $Y \to (M,\mathcal{F})$ be a smooth fiber bundle over a foliated manifold. We write k for the dimension of the fibres and n for the dimension of the leaves. We define the bundle of **foliated r-jets** $J^r(Y \to (M,\mathcal{F}))$ to be the space of equivalence classes of leafwise sections that are r-tangent to one another. The fibres of $J^r(Y \to (M,\mathcal{F})) \to M$ are again modelled on r-order Taylor polynomials of k functions in n variables. Given a global section $f : M \to Y$, we can consider its corresponding leafwise r-jet

$j^r_{\mathcal{F}} f : M \to J^r(Y \to (M, \mathcal{F}))$. Such a section of the space of foliated jets is said to be **holonomic**. Note that $j^r_{\mathcal{F}} f$ encodes no information about the derivatives of f along the normal bundle of $\mathcal{F}$.

Given manifolds X and K, where the latter is thought of as a parameter space, we may consider the foliated manifold

$$(M = X \times K, \mathcal{F} = \coprod_{a \in K} X \times \{a\}).$$

If $Y \to X$ is a fibre bundle, we can pull it back to $X \times K$ using the obvious projection. The corresponding space of foliated r-jets $J^r(Y \to (M, \mathcal{F}))$ is the natural place to carry out parametric arguments for K-families of sections of $Y \to X$.

4. Singularities

The central theme of the wrinkling philosophy is that, sometimes, it is enough to consider maps whose only singularities are simple. We will review some results in this direction in the next Section. For now, we set the stage by introducing the "mild singularities" that we need.

4.1. **The Thom-Boardman stratification theorem.** Let N be endowed with a foliation $\mathcal{F}$ of rank k, and let $f : L \to N$ be an immersion. A point $p \in L$ is a **singularity of tangency** with respect to $\mathcal{F}$ if $d_p f(TL)$ and $\mathcal{F}_{f(p)}$ are not transverse to one another. In our concrete case, N will be a jet space, $\mathcal{F}$ will be $V_{\mathrm{can}} \subset \xi_{\mathrm{can}}$ and L will be an integral submanifold.

We define the **locus of singularities of corank** j

$$\Sigma^j(f, \mathcal{F}) := \{p \in N \mid \dim(df(T_pL) \cap \mathcal{F}_p) - \max(k + n - m, 0) = j\}$$

as the set of points where the dimension of the intersection $df(T_pL) \cap \mathcal{F}_p$ surpasses the transverse case by j.

Assuming that $\Sigma^j(f, \mathcal{F})$ is a submanifold, one can recursively define higher tangency loci of corank $J = j_0, \ldots, j_l$ by setting

$$\Sigma^J(f, \mathcal{F}) := \Sigma^{j_l}(f|_{\Sigma^{j_0 j_1 \cdots j_{l-1}}(f,\mathcal{F})}, \mathcal{F}).$$

Thom [39] and Boardman [8] proved that one may perturb f so that all the $\Sigma^J(f, \mathcal{F})$ are smooth submanifolds forming a stratification. One should think of it as the pullback of the universal stratification of $\mathrm{Gr}(TN, n) \to N$ defined by $\mathcal{F}$.

Given an arbitrary smooth map $g : L \to M$, one can similarly consider the locus of singularities of mapping

$$\Sigma^j(f) := \{p \in N \mid \mathrm{corank}(d_p g) = j\},$$

as well as higher singularities. It is immediate to see that singularities of tangency of an immersion $f : L \to (N, \mathcal{F})$ correspond (in a foliation chart) to singularities of mapping of the quotient map $g : L \to N/\mathcal{F}$.

4.2. **Morin-Whitney singularities.** We now focus on singularities of mapping between equidimensional manifolds. One can provide similar definitions when the source is larger than the target, but this is unnecessary for our purposes.

Write $(x) = (x_1, \cdots, x_{n-1})$ for the coordinates in $\mathbb{R}^{n-1}$ and (x, q) for the coordinates in $\mathbb{R}^n$. Consider the following family of corank-1 singularities of mapping:

Definition 4.1. *The n-**th Whitney singularity** is the germ at the origin of the map:*

$$\begin{aligned} \mathrm{Whit}_n &: \mathbb{R}^n \to \mathbb{R}^n \\ (x, q) &\to (x, q^{n+1} + x_1 q^{n-1} + \cdots + x_{n-1} q). \end{aligned} \tag{4.2.1}$$

For $n = 1, 2$, these maps are referred to as the **fold** and the **pleat**, respectively.

We may think of Whit_n as the $\mathbb{R}^{n-1}$-family of maps $\mathbb{R} \to \mathbb{R}$ that unfolds the singularity $q \to q^{n+1}$. This proves that $\Sigma^2(\mathrm{Whit}_n)$ is indeed empty. We can obtain further corank-1 singularities as follows:

Definition 4.2. *The **i-fold stabilisation** of* Whit_n *is the map:*

$$\begin{aligned} \mathbb{R}^{n+i} &\to \mathbb{R}^{n+i} \\ (s,x,q) &\to (s, \mathrm{Whit}_n(x,q)), \end{aligned}$$

where $s \in \mathbb{R}^i$ *and* $(x,q) \in \mathbb{R}^n$.

H. Whitney proved in [44] that Whit_n is a stable map and then Morin proved the converse [36]: The germ at $p \in \Sigma^1(f)$ of a stable map $f : M \to N$, between manifolds of the same dimension, is left-right equivalent to the germ at the origin of a stabilisation of a Whitney map. Here left and right actions on germs need not to fix the origin (otherwise, the orbit of the Whitney map of index l would have codimension l in the space of all germs).

Whenever we encounter a singularity of mapping, we will say that it is Whitney, a fold, or a pleat, if it is equivalent to the model. We will use the same naming convention if we encounter a singularity of tangency $f : L \to (N, \mathcal{F})$ whose quotient is one of these; do note that this implies that $\dim(L) = \mathrm{corank}(\mathcal{F})$.

4.3. **The equidimensional wrinkle.** The fold and its stabilisations are the simplest (non-trivial) singularities of equidimensional maps. Ideally, we would work in the category of folded maps. However, this is not possible when we consider families of maps: we must, at the very least, allow folds to appear and disappear in birth/death events, i.e. pleats. This leads to the definition:

Definition 4.3. *The (equidimensional) **wrinkle** is the map*

$$\mathrm{Wrin}_n : \mathcal{O}p(\mathbb{D}^n) \to \mathbb{R}^n$$

(4.3.1)

$$(x,q) \mapsto \left(x, w(x,q) = \frac{q^3}{3} + (|x|^2 - 1)q \right).$$

The region bounded by the singular locus, i.e. the interior of the disc $\mathbb{D}^n$ *in the domain, is called the* ***membrane*** *of the wrinkle.*

4.3.1. *Singularity locus.* We see that Wrin_n is a map fibered over $\mathbb{R}^{n-1}$. Its singularities (which are of corank 1) correspond to the vanishing of $\frac{\partial w}{\partial q} = q^2 + |x|^2 - 1$, i.e. the unit sphere $\Sigma(\mathrm{Wrin}_n) = \mathbb{S}^{n-1}$ bounding the membrane. If we further restrict Wrin_n to $\Sigma(\mathrm{Wrin}_n)$ we observe that its singularities live in $\{q = 0\}$, i.e. the equator $\Sigma^{11}(\mathrm{Wrin}_n) = \mathbb{S}^{n-2}$. The map $\mathrm{Wrin}_n|_{\Sigma^{11}(\mathrm{Wrin}_n)}$ is non-singular so

$$\Sigma(\mathrm{Wrin}_n) = \Sigma^{10}(\mathrm{Wrin}_n) \cup \Sigma^{11}(\mathrm{Wrin}_n).$$

Thus, the equator is a codimension-2 sphere of pleats and the two open hemispheres consist of folds. Each two points in $\Sigma^{10}(\mathrm{Wrin}_n)$ sharing the same q–coordinate are a local maximum and a local minimum of the corresponding function $x \to \frac{q^3}{3} + (|x|^2 - 1)q$. As we move in q towards the equator $\Sigma^{11}(\mathrm{Wrin}_n)$, these two points collapse in a birth/death event. Hence, the singularities of the wrinkle are seemingly in cancelling position, but not really: the domain of definition of Wrin_n is not the whole of $\mathbb{R}^n$ (in which the cancellation is possible) but a small neighbourhood of the unit ball $\mathbb{D}^n$.

4.3.2. *Formal desingularisation.* Nonetheless, the singularities of the wrinkle are homotopically inessential from the point of view of obstruction theory: consider the homotopy of functions

$$W_s(x,q) = (q^2 + |x|^2 - 1) + s\rho(x,q), \qquad s \in [0,1],$$

where $\rho : \mathcal{O}p(\mathbb{D}^n) \to [0,\infty)$ is a non-negative function which is greater than 1 over $\mathbb{D}^n$ and identically zero in a neighbourhood of the boundary of its domain. It provides a compactly-supported homotopy between $W_0 = \frac{\partial w}{\partial q}$ and a strictly positive function. We can use W_s to construct a compactly-supported homotopy between the differential $T\mathrm{Wrin}_n$ and a bundle monomorphism. Indeed, we keep the formal derivatives of Wrin_n with respect to the x-coordinates fixed, and we homotope the formal derivative with respect to q using W_s. We call this the **formal desingularisation**. Its existence implies that the wrinkle, as a singularity, represents a trivial class (relative to the boundary of the model).

4.3.3. *The fibered nature of a wrinkle.* Let us regard the wrinkle Wrin_{n+k} as a fibered over $\mathbb{R}^k$ map. Explicitly, we write (s) for the coordinates in $\mathbb{R}^k$ and (x,q) for the coordinates in $\mathbb{R}^n$. The restriction of Wrin_{n+k} to the fibre over a fixed s is left-right equivalent to Wrin_n if $|s|<1$, non-singular if $|s|>1$, and left-right equivalent to:

$$\begin{aligned} \mathcal{O}p(\{0\}) &\to \mathbb{R}^n \\ (x,q) &\mapsto (x, \frac{q^3}{3}+|x|^2 q), \end{aligned} \tag{4.3.2}$$

if $|s|=1$. This singularity is called the **embryo**. It is precisely the event in which a wrinkle Wrin_n is born. It follows similarly that the embryo can be formally desingularised in a unique manner up to homotopy.

4.4. **Double folds, wrinkles, and surgery.** A wrinkle has non-empty Σ^{11}-locus. Sometimes, it is useful to work with maps whose singularity locus is just Σ^{10}; we call such maps, *folded.* A key idea in wrinkling is that one may produce a folded map out of a wrinkled map using *surgery of singularities* [15, 21]. Conversely, one can pass from a map having *double folds*, defined below, to a wrinkled map by a procedure called *wrinkle chopping.* Hence, wrinkles and double folds are essentially equivalent.

4.4.1. *The definition.*

Definition 4.4. *We define the* ***double fold*** *to be the map:*

$$\begin{aligned} f:\mathcal{O}p(\mathbb{S}^{n-1}\times[-1,1]) &\to \mathbb{R}^n \\ (x,q) &\mapsto (x, \frac{q^3}{3}-q). \end{aligned} \tag{4.4.1}$$

The region bounded by the singular locus, i.e. the open annulus $\mathbb{S}^{m-1}\times(-1,1)$ in the domain, is called the ***membrane*** *of f.*

The singularity locus $\Sigma(f)=\Sigma^{10}(f)$ is the union of the spheres bounding the membrane

$$\left\{\frac{\partial f}{\partial q}=q^2-1\right\}=(\mathbb{S}^{n-1}\times\{-1\})\cup(\mathbb{S}^{n-1}\times\{+1\}).$$

At each sphere the singularity is modelled on (a stabilisation of) the usual fold. Like the wrinkle, the two fold points sharing the same q-coordinate seem to be in cancelling position, but they are not due to the size of the domain.

We often speak of the spheres $\mathbb{S}^{n-1}\times\{\pm1\}$ as being the double fold, leaving the existence of the membrane bounding them implicit. We could also define the folds to take place along hypersurfaces other than spheres, but for our purposes this is unnecessary.

4.4.2. *Embryos.* Just like wrinkles are born in an embryo event, we may define the analogous birth/death singularity for double folds. It is given by the following expression:

$$\begin{aligned} f:\mathcal{O}p(\mathbb{S}^{n-1}\times\{0\}) &\to \mathbb{R}^n \\ (x,q) &\mapsto (x, q^3), \end{aligned} \tag{4.4.2}$$

which we call the (double fold) **embryo**. It is simply a parametric version, along a codimension-1 sphere, of the 1-dimensional birth/death critical point.

4.5. **The (first order) wrinkle in positive codimension.** The central idea behind the wrinkled embeddings [21] of Eliashberg and Mishachev is that it is sometimes preferable to replace singularities of tangency by singularities of mapping. Namely, we could eliminate a tangency fold by introducing a cusp along the tangency locus. Motivated by this, we introduce:

Definition 4.5. *The* ***wrinkle*** *(of dimension m into $n>m$, and of order 1) is the map*

$$\begin{aligned} \mathrm{Wrin}_{m,n}:\mathcal{O}p(\mathbb{D}^m) &\to \mathbb{R}^n \\ (x,q) &\to (x, q^3+3(|x|^2-1)q, \int_0^q (t^2+|x|^2-1)^2 dt, 0,\dots,0). \end{aligned}$$

Its projection to $\mathbb{R}^m$ is precisely the wrinkle Wrin_m.

4.5.1. *Singularity locus.* The $(m+1)$th coordinate of $\mathrm{Wrin}_{m,n}$ is a function that has exactly the same singularity locus as Wrin_m. Therefore, the singularity locus $\Sigma(\mathrm{Wrin}_{m,n})$ is the unit sphere $\Sigma^1(\mathrm{Wrin}_{m,n}) = \mathbb{S}^{m-1}$. It is the union of the equator $\Sigma^{11}(\mathrm{Wrin}_{m,n}) = \mathbb{S}^{m-2}$ and its complement $\Sigma^{10}(\mathrm{Wrin}_{m,n})$. The singularity along $\Sigma^{10}(\mathrm{Wrin}_{m,n})$ is a stabilisation of the usual planar semicubic cusp. The families of cusps in each hemisphere approach each other at the equator $\Sigma^{11}(\mathrm{Wrin}_{m,n})$, cancelling in a sphere of open semicubic swallowtails.

4.5.2. *Regularisation.* Unlike Wrin_n, the wrinkle $\mathrm{Wrin}_{m,n}$ is not stable as soon as $m < n$. Indeed, the small perturbation:

$$(x,q) \to (x, q^3 + 3(|x|^2-1)q, \varepsilon q + \int_0^q (t^2+|x|^2-1)^2 dt, 0, \dots, 0)$$

is a smooth embedding. A cut-off may be applied to make this perturbation compactly supported. This smoothing process is unique up to isotopy (which may also be assumed to be compactly supported); we call it the **regularisation**.

4.5.3. *The Gauss map.* Despite being singular, $\mathrm{Wrin}_{m,n}$ has a well-defined lift $\mathrm{Gr}(\mathrm{Wrin}_{m,n})$ to the space of 1-jets of submanifolds; see Subsection 3.4.2. This is clear along the cusp locus $\Sigma^{10}(\mathrm{Wrin}_{m,n})$, because the planar cusp has a well-defined tangent line at every point. We claim that the same is true along the swallowtail region $\Sigma^{11}(\mathrm{Wrin}_{m,n})$. This is a simple computation, but we will justify it, in the setting of integral submanifolds of general jet spaces, in Subsection 7.1.

4.5.4. *Embryos.* Just as in the equidimensional setting, we may think of the wrinkle $\mathrm{Wrin}_{k+m,k+n}$ as a fibered over $\mathbb{R}^k$ map. We write (s) for the coordinates in $\mathbb{R}^k$ and (x,q) for those in $\mathbb{R}^m$. For $|s| < 1$ given, the restriction of $\mathrm{Wrin}_{k+m,k+n}$ to the fibre over s is left-right equivalent to $\mathrm{Wrin}_{m,n}$. For $|s| > 1$, it has no singularities. Lastly, for $|s| = 1$ the map is equivalent to:

$$(x,q) \to (x, q^3 + 3|x|^2 q, \int_0^q (t^2+|x|^2)^2 dt, 0, \dots, 0).$$

whose only singularity is the origin. This is exactly the birth/death phenomenon for $\mathrm{Wrin}_{m,n}$, which we also call **embryo**. It can be regularised as above.

5. The h-principle

The h-principle is a collection of techniques and heuristic approaches whose purpose is to describe the spaces of solutions of partial differential relations. This Section provides a quick overview, and readers familiar with h-principles are invited to skip ahead.

We first review some of the necessary language (Subsections 3.1 and 5.1). Then we go over some classic techniques: *holonomic approximation* in Subsection 5.3, *triangulations in general position* in Subsection 5.4, and *wrinkling* in Subsection 5.5.

For a panoramic view of h-principles we refer the reader to the two standard texts [20] and [25] (which we suggest to check in that order). Wrinkling techniques were introduced first in the *wrinkling saga* [17, 19, 18, 22, 21].

5.1. **Differential relations.** Let $Y \to X$ be a smooth fibre bundle. A **partial differential relation** (PDR) of order r is a subset $\mathcal{R} \subset J^r(Y \to X)$. This provides a framework for PDRs of sections, but one can define PDRs of n-submanifolds of Y as subsets of $J^r(Y,n)$ as well.

Endow $\Gamma(J^r(Y \to X))$ with the C^0-topology. We may use the inclusion

$$j^r : \Gamma(Y \to X) \quad \to \quad \Gamma(J^r(Y \to X)),$$

to pull it back and endow the domain with its usual Whitney C^r-topology. This makes j^r a continuous map. We write $\mathrm{Sol}^f(\mathcal{R})$ for the subspace of sections in $\Gamma(J^r(Y \to X))$ whose image lies in $\mathcal{R}$, i.e. the space of formal solutions. Similarly, we write $\mathrm{Sol}(\mathcal{R})$ for the space of solutions, which is a subspace of $\Gamma(Y)$.

Definition 5.1. *We say that the* ***(complete)*** h***-principle*** *holds for* $\mathcal{R}$ *if the inclusion*

$$\begin{aligned} \iota_{\mathcal{R}} : \mathrm{Sol}(\mathcal{R}) &\rightarrow \mathrm{Sol}^f(\mathcal{R}) \\ f &\rightarrow \iota_{\mathcal{R}}(f) := j^r f \end{aligned}$$

is a weak homotopy equivalence.

5.2. **Flavours of h-principle.** Sometimes, $\iota_{\mathcal{R}}$ fails to be a weak homotopy equivalence, but partial results hold. For instance, if $\iota_{\mathcal{R}}$ is surjective at the level of connected components, we say that the **existence h-principle** holds. Similarly, if $\iota_{\mathcal{R}}$ is a bijection of connected components, we may say that the h-principle holds in π_0; analogous statements hold for higher homotopy groups.

Furthermore, we may ask whether the h-principle holds over each open set $U \subset X$ in a way that is coherent with respect to the sheaf structure of $\mathrm{Sol}(\mathcal{R})$. This can be phrased as follows. The h-principle is **relative in the domain** when: any family of formal solutions of $\mathcal{R}|_U$, which are already honest solutions in a neighbourhood of a closed set $A \subset U$, can be homotoped to become solutions over the whole of U while remaining unchanged over $\mathcal{O}p(A)$.

Similarly, the h-principle is **relative in the parameter** when: any family of formal solutions $\{F_k\}_{k\in K}$, parametrised by a closed manifold K, and with $F_{k'}$ holonomic for every k' in an open neighbourhood of a fixed closed subset $K' \subset K$, can be homotoped to be holonomic relative to $\mathcal{O}p(K')$.

5.3. **Holonomic approximation.** One of the cornerstones of the classical theory of h-principles is the holonomic approximation theorem. It states that any formal section of a jet bundle can be approximated by a holonomic one in a neighbourhood of a perturbed CW-complex of codimension at least 1. The precise statement reads as follows:

Theorem 5.2 ([20])**.** *Let $Y \rightarrow X$ be a fiber bundle, K a compact manifold, $A \subset M$ a polyhedron of positive codimension, and $(F_{k,0})_{k\in K} : X \rightarrow J^r(Y \rightarrow X)$ a family of formal sections. Then, for any $\varepsilon > 0$ there exists*

- *a family of isotopies $(\phi_{k,t})_{t\in[0,1]} : X \rightarrow X$,*
- *a homotopy of formal sections $(F_{k,t})_{k\in K,t\in[0,1]} : X \rightarrow Y$,*

satisfying:

- *$F_{k,1}$ is holonomic in $\mathcal{O}p(\phi_{k,1}(A))$,*
- *$|\phi_{k,t} - \mathrm{id}|_{C^0} < \varepsilon$ and is supported in a ε-neighbourhood of A,*
- *$|F_{k,t} - F_{k,0}|_{C^0} < \varepsilon$.*

Moreover the following hold:

- *If $V \in \mathfrak{X}(\mathcal{O}p(A))$ is a vector field transverse to A, then we can arrange that $\phi_{k,t}(A)$ is transverse to V for all t and k.*
- *If the $F_{k,t}$ are already holonomic in a neighborhood of a subcomplex $B \subset A$, then we can take $F_{k,t} = F_{k,0}$ and $\phi_{k,t} = \mathrm{id}$ on $\mathcal{O}p(B)$, for all k.*
- *If $F_{k,t}$ is everywhere holonomic for every k in a neighbourhood of a CW-complex $K' \subset K$, then we can take $F_{k,t} = F_{k,0}$ and $\phi_{k,t} = \mathrm{id}$ for $k \in \mathcal{O}p(K')$.*

Remark 5.3. *Note that in the above statement, the inequalities*

$$|\phi_{k,t} - \mathrm{id}|_{C^0} < \varepsilon, \quad |F_{k,t} - F_{k,0}|_{C^0} < \varepsilon,$$

depend on a choice of Riemannian metric on X and Y. △

For the proof and a much longer account of its history, we refer the reader to [20]. Essentially, this theorem recasts the method of flexible sheaves due to M. Gromov (itself a generalisation of the methods used by S. Smale in his proof of the sphere eversion and the general h-principle for immersions) in a different light. Let us go over the statement.

The starting point is the family of formal sections $F_{k,0}$, which we want to homotope until they become holonomic. This is not possible, but the theorem tells us that at least we can achieve holonomicity in a neighbourhood of a set of positive codimension. We are not allowed to fix this set. Instead, we begin with a polyhedron A, which we deform in a C^0 small way to yield an isotopic polyhedron $\phi_{k,1}(A)$. This isotopy occurs in the normal directions of A (which we may prefix by taking a transverse vector field V), and essentially produces a copy $\phi_{k,1}(A)$ of A of greater length. This process is called, descriptively, **wiggling**. The room we gain by wiggling is what allows us to achieve holonomicity: the main idea is that, at each point $p \in A$, we approximate $F_{k,0}$ by the corresponding Taylor polynomial $F_{k,0}(p)$ and then we use the directions normal to A to interpolate between these polynomials keeping control of the derivatives. Hence, we can take the $F_{k,t}$ to be arbitrarily close to our initial data, and the wiggling to be C^0-small. However, if we desire better C^0-bounds, we will be forced to wiggle more aggressively, i.e. the isotopies $\phi_{k,t}$ will become C^1-large.

5.4. **Thurston's triangulations.** An important step in the application of many h-principles (including ours), is the reduction of the global statement (global in the manifold M), to a local statement taking place in a small ball. These reductions allow us not to worry about (global) topological considerations, making the geometric nature of the arguments involved more transparent. Working on small balls (i.e. "zooming-in") usually has the added advantage of making the geometric structures we consider seem "almost constant"; this will play a role later on.

A possible approach to achieve this is to triangulate the ambient manifold M and then work locally simplex by simplex. A small neighbourhood of a simplex is a smooth ball which can be assumed to be arbitrarily small if the subdivision is sufficiently fine; thus, this achieves our intended goal. When we deal with parametric results (phrased using the foliated setup, see subsection 3.5), we want to zoom-in in the parameter space too. This requires us to triangulate in parameter directions as well. For us, this means that we must triangulate a foliated manifold in a manner that is nicely adapted to the foliation.

Let $(M, \mathcal{F})$ be a manifold of dimension n endowed with a foliation of rank k . Given a triangulation $\mathcal{T}$, we write $\mathcal{T}^{(i)}$ for the collection of i-simplices, where $i = 0, \ldots, \dim(M) = n$. We think of each i-simplex $\sigma \in \mathcal{T}^{(i)}$ as being parametrised $\sigma : \Delta^i \to M$, where the domain is the standard simplex in $\mathbb{R}^i$. The parametrisation σ allows us to pull-back data from M to Δ^i. In particular, if σ is a top-dimensional simplex, it is a diffeomorphism with its image and we may assume that σ extends to an embedding $\mathcal{O}p(\Delta^n) \to M$ of a ball.

If the image of σ is sufficiently small, we would expect that the parametrisation σ can be chosen to be reasonable enough so that $\sigma^*\mathcal{F}$ is almost constant. This can be phrased as follows:

Definition 5.4. *A top-dimensional simplex σ is in **general position** with respect to the foliation $\mathcal{F}$ if the linear projection (identifying $T_p\mathbb{R}^n = \mathbb{R}^n$)*

$$\Delta^n/(\sigma^*\mathcal{F})_p \to \mathbb{R}^{n-k}$$

restricts to a map of maximal rank over each subsimplex of σ. In particular, $\sigma^*\mathcal{F}$ is transverse to each subsimplex.

*The triangulation $\mathcal{T}$ is in **general position** with respect to $\mathcal{F}$ if all of its top-simplices are in general position.*

Theorem 5.5. *Let $(M, \mathcal{F})$ be a foliated manifold. Then, there exists a triangulation $\mathcal{T}$ of M which is in general position with respect to $\mathcal{F}$.*

This statement was first stated and proven by W. Thurston in [40, 41], playing a central role in his h-principles for foliations. A sketch of the argument goes roughly as follows: we start with a triangulation $\mathcal{T}'$. We then subdivide it (in a controlled fashion called *crystalline subdivision*, which ensures that angles remain controlled and that the cardinality of the star of a vertex is uniformly bounded). As we subdivide, the foliation seems progressively flatter from the perspective of each simplex. In particular, the measure of the set of planes that intersect the foliation non-transversally goes to zero. This allows us to apply *Thurston's jiggling*: we tilt slightly the vertices, yielding simplices that are transverse to $\mathcal{F}$.

5.5. **Wrinkling.** Wrinkling is an h-principle method whose goal is constructing mildly singular solutions of partial differential relations. It was used by Y. Eliashberg and M. Mishachev to prove flexibility results for submersions [17], equidimensional immersions with prescribed folds [22], foliations [18], and fibrations [16]. It entered the world of Contact Topology with [21], which would then lead to the works of E. Murphy on loose legendrians [37] and D. Álvarez-Gavela on the simplification of front singularities of legendrians [2, 1]. It is also one of the central ingredients in the construction and classification of overtwisted contact structures in all dimensions [9] due to M.S. Borman, Y. Eliashberg, and E. Murphy. More recently, it has been used in Engel Geometry to classify overtwisted Engel structures [14] and integral knots in Engel manifolds [12].

For the reader to have a somewhat complete picture, let us provide a list of sample theorems on wrinkling.

5.5.1. *Wrinkled submersions.* Let M and N be n-dimensional manifolds (we assume equidimensionality for simplicity). It is well-known that the space of submersions $M \to N$ may not be homotopy equivalent to the space of formal submersions if M is closed. The first wrinkling result of Y. Eliashberg and M. Mishachev [17] says that one may salvage the h-principle by relaxing the submersion condition:

Definition 5.6. *A* ***wrinkled submersion*** *is:*

- *a map $f : N \to M$ between n-dimensional manifolds,*
- *a finite collection of disjoint open balls $\{B_i\}$,*

such that:

- *f is a submersion in the complement of the B_i.*
- *$f|_{B_i}$ is left-right equivalent to Wrin_n (Definition 4.3).*

A ***wrinkled submersion with embryos*** *has an additional collection of balls in which f is modelled by the embryo (Equation 4.3.2).*

Using the formal desingularisation of wrinkles and embryos we dedyce that there is a map, well-defined up to homotopy, from the space of wrinkled submersions with embryos to the space of formal submersions. Then:

Theorem 5.7 (Eliashberg and Mishachev [17])**.** *The space of wrinkled submersions with embryos is homotopy equivalent to the space of formal submersions. This h-principle is, additionally, C^0-close.*

We can similarly define **submersions with double folds** to be maps which are submersions in the complement of a finite collection of disjoint annuli in which they are modelled by a double fold. They may additionally have finitely many spheres in which they are modelled by a double fold embryo. Using surgery of singularities one can deduce the equivalent statement:

Corollary 5.8. *The space of submersions with double folds and embryos is homotopy equivalent to the space of formal submersions. This h-principle is, additionally, C^0-close.*

5.5.2. *Wrinkled embeddings.* Let $M \subset N$ be smooth manifolds with $\dim(M) < \dim(N)$. In [21], Y. Eliashberg and M. Mishachev study the problem of isotoping M, as an embedded submanifold of N, to approximate a given tangential homotopy in a holonomic manner. This problem is solvable if we relax the embedding condition:

Definition 5.9. *A smooth map $f : M \to N$ is a* ***wrinkled embedding*** *if:*

- *it is a topological embedding,*

- *it is a smooth embedding away from a collection of disjoint embedded codimension-*1 *spheres* S_i,
- $f|_{\mathcal{O}p(S_i)}$ *is left-right equivalent to* $\mathrm{Wrin}_{\dim(M),\dim(N)}$.

A map $f : M \to N$ *is a* ***wrinkled embedding with embryos*** *if it is a wrinkled embedding in the complement of a finite collection* $\{p_i\}$ *of points and it is left-right equivalent to an embryo in each neighbourhood* $\mathcal{O}p(p_i)$.

Theorem 5.10 (Eliashberg and Mishachev [21])**.** *Let* N *and* K *be smooth manifolds. Let* $(M_k)_{k\in K} \subset N$ *be a* K*-family of submanifolds of* N*. Assume that there is a family of tangential homotopies* $(\nu_{k,s})_{k\in K,s\in[0,1]}$ *starting at* $\nu_{k,0} = TM_k$.

Then, there is a $K \times [0, 1]$*-family of wrinkled submanifolds with embryos* $(M_{k,s})_{k\in K,s\in[0,1]}$*, starting at* $M_{k,0} = M_k$*, such that* $TM_{k,s}$ *is* C^0*-close to* $\nu_{k,s}$.

Furthermore:

- *Assume there is a closed submanifold* $K' \subset K$ *such that* $\nu_{k,s} = TM_k$ *for every* $k \in K'$*. Then, we may assume that* $M_{k,s} = M_k$ *for all* $k \in K'$.
- *Assume there are closed submanifolds* $M'_k \subset M_k$ *such that* $\nu_{k,s}(x) = T_xM_k$ *for all* $x \in M'_k$*. Then we may assume that* $M_{k,s}$ *agrees with* M_k *in* $\mathcal{O}p(M'_k)$.

An equivalent result can be proven using double cusps instead of wrinkles.

6. Generating functions and metasymplectic projections

In this Section we introduce local methods to construct and manipulate Σ^2-free integral submanifolds. Before we get there, we will introduce some notation regarding Grassmannians of integral elements.

In Subsection 6.2 we introduce the formalism of generating functions in jet space. We will review constructions due to Lychagin and Givental using the lens of reduction.

In Subsection 6.3, we introduce the *metasymplectic formalism*. Namely, we will be able to deal with Σ^2-free integral submanifolds by projecting them down to so-called metasymplectic space. This generalises the standard lagrangian projection used in Contact Geometry to the setting of jet spaces.

We denote $\dim(X) = n$ and $\dim(Y) = k$, where $J^r(Y \to X)$ is the jet space of interest. We will quickly pass to local coordinates, and we will replace X by a vector space B and the fibres of Y by a vector space F.

6.1. Grassmannian bundles.

Recall the notation $\mathrm{Gr}_{\mathrm{integral}}(\xi_{\mathrm{can}}, l)$ for the Grassmannian bundle of l-dimensional integral elements of ξ_{can}. According to Proposition **??** we can identify each fibre $(T_pJ^r(Y \to X), \xi_{\mathrm{can}})$ with the jet Lie algebra $\mathfrak{g}$ (uniquely up to point symmetries). From this, it follows that the fibres of $\mathrm{Gr}_{\mathrm{integral}}(\xi_{\mathrm{can}}, l)$ correspond to Grassmannians of Lie subalgebras of $\mathfrak{g}$.

We can further denote the **horizontal Grassmannian** by

$$\mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, l) \subset \mathrm{Gr}_{\mathrm{integral}}(\xi_{\mathrm{can}}, l).$$

We will be interested in integral submanifolds that are horizontal over a dense set. From this it follows that their tangent spaces will take values in the **multi-section Grassmannian**

$$\overline{\mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, l)}.$$

We remark (even though this is not the case in the contact setting) that there may be, in general, integral elements not contained in this closure. This can be checked by dimension counting, noting that V_{can}, which is an integral element, is often larger than X in dimension.

More generally, we can write:

$$\mathrm{Gr}_{\Sigma^i}(\xi_{\mathrm{can}}, l) := \{W \in \mathrm{Gr}_{\mathrm{integral}}(\xi_{\mathrm{can}}, l) \mid \dim(W \cap V_{\mathrm{can}}) = i\}$$

and, since we want to restrict our attention to Whitney singularities, we introduce the Σ^2**-free Grassmannian**

$$\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\xi_{\mathrm{can}}, l) := \mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, l) \cup \mathrm{Gr}_{\Sigma^1}(\xi_{\mathrm{can}}, l).$$

We will prove in the sequel [?] that its fibres are smooth manifolds; this will play no role in the present paper. It is unclear to the authors whether the fibres of $\overline{\mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, l)}$ are smooth.

6.2. **Generating functions.** V. Arnold proved in [4, 5] that front singularities of embedded legendrians/lagrangians can always be (locally) described by generating functions. This is not true for arbitrary integral submanifolds of jet spaces [38, p. 14] [45], but it nonetheless holds that front singularities are rather special compared to the singularities of a general map. This was first explored by V. Lychagin [29] for 1-jet spaces in more than one variable, and later by A. Givental [24] for general jet spaces.

Our goal in this Section is to define what a generating function is for a general jet space. We do this using reduction, which we introduce in Subsection 6.2.1. This allows us, in subsection 6.2.4, to provide a recipe for corank-1 front singularities admitting a generating function description. We will see in subsection 7.1.1 that this recipe can be particularised to recover Givental's description of integral submanifolds that have Whitney type front singularities.

6.2.1. *Reduction.* The main idea behind generating functions is that we can follow a two step process when constructing non-horizontal integral submanifolds: first, we produce a horizontal submanifold over a base of greater dimension. Then, we use a "reduction" procedure to go down to the actual jet space we want to work in. It is in this latter step in which the horizontality condition is lost.

The "enlarged base" will be the total space of a fibration $\pi : E \to X$, endowed with the foliation $\mathcal{F}$ by fibres. We pullback Y to E and we denote it by Y_E; tautologically, there is fibrewise flat connection over each fibre of E that identifies the fibres of Y_E.

We denote by $C^\infty(E, Y_E)$ the space of smooth sections $E \to Y_E$. Using the pullback of π, we have a natural inclusion $\pi^* : C^\infty(X, Y) \to C^\infty(E, Y_E)$, whose image we denote by $C^\infty_{\mathcal{F}}(E, Y_E)$. A function in $C^\infty_{\mathcal{F}}(E, Y_E)$ is said to be **basic**. We collect all the r-jets of basic functions to yield:

Definition 6.1. *The space of **basic** r**-jets** is defined as:*

$$J^r_{\mathcal{F}}(E, Y_E) := \{j^r_e f \in J^r(E, Y_E) \mid e \in E, f \in C^\infty_{\mathcal{F}}(E, Y_E)\}.$$

The canonical projection map

$$\begin{array}{rcl} \tilde{\pi} : J^r_{\mathcal{F}}(E, Y_E) & \mapsto & J^r(X, Y) \\ j^r_e(f \circ \pi) & \mapsto & j^r_{\pi(e)} f, \end{array}$$

*is called the **reduction map**.*

In this general setting, the familiar properties of the contact reduction process still hold. We leave the proof to the reader:

Lemma 6.2. *The following statements hold:*

- $J^r_{\mathcal{F}}(E, Y_E)$ *is a smooth submanifold of* $J^r(E, Y_E)$.
- *The restriction*
$$\xi^{\mathcal{F}}_{\mathrm{can}} := \xi_{\mathrm{can}} \cap TJ^r_{\mathcal{F}}(E, Y_E)$$
has a $\mathrm{rank}(\mathcal{F})$*-dimensional characteristic foliation* $\ker(\xi^{\mathcal{F}}_{\mathrm{can}})$. *It is the lift of* $\mathcal{F}$ *to the fibrewise connection of* Y_E.
- *The reduction map* $\tilde{\pi}$ *preserves the Cartan distribution.*
- *Leaves of the characteristic foliation* $\ker(\xi^{\mathcal{F}}_{\mathrm{can}})$ *correspond to fibres of* $\tilde{\pi}$.

We say that $J^r(X, Y)$ is the **reduction** of $J^r(E, Y_E)$ with respect to $\ker(\xi^{\mathcal{F}}_{\mathrm{can}})$. We may study next how integral submanifolds interact with the reduction process:

Definition 6.3. *Let $L \subset J^r(E, Y_E)$ be an integral submanifold. Its* ***reduction*** *is the set*

$$L/\mathcal{F} := \tilde{\pi}(L \cap J^r_{\mathcal{F}}(E, Y_E)) \subset J^r(X, Y).$$

We say that $f : E \to Y_E$ is the ***generating function*** *for*

$$L_f := \mathrm{Image}(j^r f)/\mathcal{F}.$$

As suggested by the definition, even if the intersection $L \cap J^r_{\mathcal{F}}(E, Y_E)$ is a smooth submanifold, it may have singularities of tangency with $\ker(\xi^{\mathcal{F}}_{\mathrm{can}})$. Therefore, the reduction $L/\mathcal{F}$ is often not smooth. However, it is integral (in the sense that it is the image of an integral map).

6.2.2. *Reduction in concrete terms.* We now describe the local situation. Fix vector spaces B and A and endow their product $B \times A$ with coordinates (x, q). Similarly, take the fibre of Y to be a vector space F. In this manner, the reduced space is $J^r(B, F)$.

Lemma 6.4. *A function $G : B \times A \to F$ generates the subset:*

$$L_G = \{(x, G(x,q), \partial_x G(x,q), \cdots, \partial^r_x G(x,q)) \mid \forall (x,q) \text{ s.t. } \partial^t_x \partial^s_q G(x,q) = 0 \quad \forall s \neq 0, t\}. \tag{6.2.1}$$

Proof. The lift of G is given by the expression:

$$j^r G(x,q) = (x, q, G(x,q), \partial_x G(x,q), \partial_q G(x,q), \partial^2_x G(x,q), \partial_x \partial_q G(x,q), \ldots, \partial^r_q G(x,q)).$$

The intersection of $j^r G$ with the space of basic r-jets is the subset of $j^r G$ in which all derivatives of G involving q at least once are zero. Do note that this set is contained in the locus of fibrewise critical points of G and the two agree if $r = 1$. □

6.2.3. *Remark: dimension counting.* In the contact case the collection of leafwise critical points on a given fibre $\{x\} \times A$ is, generically, a finite collection of points and, for most fibres, the points are of Morse type. In particular, the reduction L_G is a legendrian, that can be regarded as the 1-jet of a multiply-valued function $B \to \mathbb{R}$.

For $mr > 1$, having derivative purely in the q-directions is an overdetermined condition. The expected dimension of L_G may be computed to be:

$$(n+m) - k \sum_{l=1}^{r} \left(\binom{n+m+l-1}{l} - \binom{n+l-1}{l} \right).$$

The expected dimension is n only in the contact setting, and it is non-negative only if $r = 1$ and $n \geqslant (k-1)m$. Otherwise, and in particular for all higher jet spaces, the expected dimension is negative.

This tells us that any generating function theory for higher jet spaces would not rely on generic functions, but rather on a subclass of functions (of positive codimension given by the formula above) with prescribed singularities. We will look at one particularly manageable example next. Developing a general theory is left as an open question.

6.2.4. *Integral expressions.* Consider now the situation where $E = X \times \mathbb{R}$. Being a rank-1 bundle over M, any integral manifold we produce by reduction will be Σ^2-free. We work in the product case and assume the target is the vector space F.

Given a submersion $H : E \to \mathbb{R}$, we define:

$$\begin{aligned} G : X \times \mathbb{R} \quad &\to \quad F \\ (x,q) \quad &\to \quad \left(G_1(x,q) := \int_0^q H(x,t)^r dt, 0, \ldots, 0 \right). \end{aligned}$$

The only relevant entry is G_1, since the other $(k-1)$ entries are zero and therefore singular everywhere. We see that $\partial_q G_1(x,q) = H(x,q)^r$. Furthermore, using induction one may show that:

Lemma 6.5. *Let $t \geqslant 0$ and $s > 0$ be integers. Then, there are functions $\Psi_l : X \times \mathbb{R} \to \mathbb{R}$ such that*
$$\partial_x^t \partial_q^s G_1(x,q) = \sum_{l=0}^{s+t-1} H^{r-l}(x,q)\Psi_l(x,q).$$

That is, all the derivatives (up to order r) involving q at least once vanish at the fibrewise critical points of G. Therefore:

Lemma 6.6. *The reduction L_G is parametrised by the locus of zeroes of H:*
$$L_G = \{(x, G(x,q), \partial_x G(x,q), \cdots, \partial_q^r G(x,q)) \mid \forall (x,q) \text{ s.t. } H(x,q) = 0\}.$$
In particular:

- *L_G is an Σ^2-free integral submanifold in $J^r(X,F)$.*
- *Its singularities of tangency with the vertical distribution correspond to the singularities of tangency of $H^{-1}(0)$ with $\mathcal{F}$.*

Proof. The concrete expression for L_G follows from the previous Lemma. The integrality condition is automatic since we are using generating functions. The Σ^2-free condition follows because the fibres of E are 1-dimensional. All we have to do is check that L_G is smooth and then describe its singularities.

Due to the submersion condition, the locus of zeroes $H^{-1}(0)$ is a smooth hypersurface in E, which we can use to parametrise L_G. In each branch of $H^{-1}(0)$ graphical over the x-coordinates, the variable q can be regarded as a function of x. Hence, branches of $H^{-1}(0)$ are mapped to branches of L_G simply by taking the graph $j^r(G(x, q(x))$, which is thus smooth.

We focus then on the tangencies of H^{-1} with $\mathcal{F}$. Fix $(\tilde{x}, \tilde{q}) \in \Sigma(H^{-1}(0), \mathcal{F})$. Since H is a submersion, we have that $\partial_{x_i} H(\tilde{x}, \tilde{q}) \neq 0$, for some i. We may then compute:
$$\partial_q \partial_{x_i}^r G_1(\tilde{x}, \tilde{q}) = r![\partial_{x_i} H(\tilde{x}, \tilde{q})]^r \neq 0$$
because all other terms involve H and are zero. Therefore, the map $q \to \partial_{x_i}^r G_1(x,q)$ is a local diffeomorphism of $\mathbb{R}$. This implies that $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, q)$ locally parametrises L_G as a smooth embedded manifold, concluding the proof. □

We can define additional Σ^2-free integral subvarieties, for every $0 \leqslant l < r$, as follows:
$$\pi_{r,l}(L_G) = \{(x, G(x,q), \partial_x G(x,q), \cdots, \partial_x^l G(x,q)) \mid \forall (x,q) \text{ s.t. } \partial_q G(x,q) = 0\} \subset J^l(X,F),$$
which are none other than the usual projections of L_G to lower jet spaces. All of them are generated by G and have a well-defined Gauss map into the horizontal Grassmannian. They have singularities of mapping corresponding to the front tangencies of L_G.

6.3. **Metasymplectic projections and lifts.** In Contact Topology it is fruitful to manipulate legendrian knots using their lagrangian projection. In this Subsection we describe the analogue of this process for general jet spaces. We work locally in $J^r(B,F)$, with B and F vector spaces. We fix holonomic coordinates (x,y,z).

We will project $J^r(B,F)$ to the so-called *standard metasympletic space*. Morally speaking, this amounts to projecting to ξ_{can} endowed with its curvature (seen as a vector-valued 2-form). In this manner, integral submanifolds will project to isotropics. This is explained in Subsection 6.3.1.

In Subsection 6.3.2 we prove Theorem 6.14: isotropic submanifolds in standard metasympletic space can be lifted to $J^r(B,F)$. This is sufficient to manipulate 1-dimensional integral submanifolds; see Subsection 6.3.3.

For higher-dimensional integral submanifolds the story is more complicated, because it is non-trivial to manipulate their metasymplectic projections directly. To address this, we work "*one direction at a time*", effectively thinking about them as parametric families of curves. This is done in Subsection 6.3.4.

6.3.1. *Standard metasymplectic space.* Recall the Cartan 1-forms defining ξ_{can}, as introduced in Subsection 3.2.1:

$$\alpha_j^I = dz_j^{(I)} - \sum_{a=1}^{n} z_j^{(i_1,\cdots,i_a+1,\cdots,i_n)} dx_a, \qquad I = (i_1,\dots,i_n),\ |I| = r-1,\ 1 \leqslant j \leqslant k,$$

which only depend on the coordinates z^r. Their differentials are the 2-forms:

$$\Omega_j^I = \sum_{a=1}^{n} dx_a \wedge z_j^{(i_1,\cdots,i_a+1,\cdots,i_n)}, \qquad I = (i_1,\dots,i_n),\ |I| = r-1,\ 1 \leqslant j \leqslant k,$$

which, by construction, are pullbacks of forms in the product $B \oplus \operatorname{Sym}^r(B^*, F)$ (which have the same coordinate expression, so we abuse notation and denote them the same). We can package them in the following intrinsic manner:

Definition 6.7. *The* ***canonical metasymplectic structure*** *in $B \oplus \operatorname{Sym}^r(B^*, F)$ is the 2-form:*

$$\Omega_{\text{can}} := (\Omega_j^I)_{|I|=r-1,\,1\leqslant j\leqslant k} : \quad \wedge^2(B \oplus \operatorname{Sym}^r(B^*, F)) \quad \to \quad \operatorname{Sym}^{r-1}(B^*, F).$$

The pair $(B \oplus \operatorname{Sym}^r(B^*, F), \Omega_{\text{can}})$ is called ***standard metasymplectic space****.*

We remark that we can regard standard metasymplectic space as a vector space endowed with a (vector-valued) linear 2-form, or as a manifold endowed with a smooth 2-form. The tangent fibres of the latter are, of course, isomorphic to the former. We can readily check:

Lemma 6.8. *Given a point $p \in B \oplus \operatorname{Sym}^r(B^*, F)$ and vectors $v_i + A_i \in T_p(B \oplus \operatorname{Sym}^r(B^*, F)) \cong B \oplus \operatorname{Sym}^r(B^*, F)$:*

$$\Omega_{\text{can}}(v_0 + A_0, v_1 + A_1) = \iota_{v_0} A_1 - \iota_{v_1} A_0.$$

I.e. the canonical metasymplectic structure is precisely the contraction map of tensors with vectors. When $r = k = 1$, the standard metasymplectic space $(B \oplus B^*, \Omega_{\text{can}})$ is simply $\mathbb{R}^{2n}$ endowed with its linear symplectic form. We then generalise the lagrangian projection:

Definition 6.9. *The* ***metasymplectic projection*** *is the map*

$$\begin{aligned} \pi_L : J^r(B,F) \quad &\to \quad B \oplus \operatorname{Sym}^r(B^*, F) \\ (x,y,z) \quad &\to \quad \pi_L(x,y,z) := (x, z^r). \end{aligned}$$

By construction, the differential at each point

$$d_p\pi_L : T_pJ^r(B,F) \quad \to \quad T_{\pi_L(p)}(B \oplus \operatorname{Sym}^r(B^*, F))$$

is an epimorphism that restricts to an isomorphism $(\xi_{\text{can}})_p \to T_{\pi_L(p)}(B \oplus \operatorname{Sym}^r(B^*, F))$. Furthermore, using the duality between distributions and their annihilators, it readily follows that:

Lemma 6.10. *The differential is an isomorphism of metasymplectic linear spaces:*

$$d_p\pi_L : ((\xi_{\text{can}})_p, \Omega(\xi_{\text{can}})) \to (T_{\pi_L(p)}(B \oplus \operatorname{Sym}^r(B^*, F)), \Omega_{\text{can}}),$$

where $\Omega(\xi_{\text{can}})$ is the curvature of ξ_{can}.

It is convenient to define: A vector subspace V of the standard metasymplectic (linear) space is said to be an **isotropic element** if $(\Omega_{\text{can}})|_V = 0$. An isotropic element is maximal if it is not contained in a larger isotropic subspace. Similarly, a submanifold of standard metasymplectic space is **isotropic** if all its tangent subspaces are isotropic elements. Then, it readily follows:

Corollary 6.11. *Let $f : N \to J^r(B,F)$ be a map. Then:*

- *f is integral if and only if $\pi_L \circ f$ is isotropic.*
- *If f is integral then f is an immersion if and only if $\pi_L \circ f$ is an immersion.*

6.3.2. *Integral lift of an isotropic.* Our next goal is proving the converse: every isotropic submanifold can be lifted to an integral one. First we need an auxiliary concept:

Definition 6.12. *The **standard Liouville form***

$$\lambda_{\text{can}} \in \Omega^1(B \oplus \text{Sym}^r(B^*, F); \text{Sym}^{r-1}(B^*, F))$$

is defined, at a point (v, A) in standard metasymplectic space, by the following tautological expression:

$$\lambda_{\text{can}}(v, A)(w, B) := -\iota_w A.$$

The computations in Subsection 6.3.1 imply that:

Lemma 6.13. *Then following statements hold:*

- *The Liouville form can be explicitly written as:*
$$\lambda_{\text{can}}(x, z^r) = \left(-\sum_{a=1}^{n} z_j^{(i_1,\cdots,i_a+1,\cdots,i_n)} dx_a\right)_{|(i_1,\cdots,i_a,\cdots,i_n)|=r-1}.$$
- *The Cartan 1-forms $\alpha^r \in \Omega^1(J^r(B, F); \text{Sym}^{r-1}(B^*, F))$ are given by the expression*
$$\alpha_r(x, y, z) = dz_{r-1} + \lambda_{\text{can}}(x, z^r).$$
- *In particular, $d\lambda_{\text{can}} = \Omega_{\text{can}}$.*

That is, the familiar properties for the Liouville form in the symplectic/contact setting hold as well in more general jet spaces. Then:

Theorem 6.14. *Let N be a disc. Given an isotropic map*

$$g : N \to (B \oplus \text{Sym}^r(B^*, F), \Omega_{\text{can}})$$

there exists an integral map

$$\text{Lift}(g) : N \to J^r(B, F)$$

satisfying $\pi_L \circ \text{Lift}(g) = g$. The lift $\text{Lift}(g)$ is unique once we fix $\text{Lift}(g)(x)$ for some $x \in N$.

Proof. Write $g(p) = (x(p), z^r(p))$. By construction, $g^*\Omega_{\text{can}} = 0$. We deduce that each component of $g^*\lambda_{\text{can}}$ is closed and thus exact. We choose primitives, which we denote suggestively by $z^{r-1} : N \to \text{Sym}^{r-1}(B^*, F)$. These functions are unique up to a shift by an element of $\text{Sym}^{r-1}(B^*, F)$.

We put together g with the chosen primitives to produce a map

$$h := (x, z^r, z^{r-1}) : N \to B \oplus \text{Sym}^r(B^*, F) \oplus \text{Sym}^{r-1}(B^*, F).$$

We can readily check, using Lemma 6.13, that

$$h^*\alpha^r = dz^{r-1} + g^*\lambda_{\text{can}} = 0.$$

Furthermore, consider the 2-form with values in $\text{Sym}^{r-2}(B^*, F)$:

$$\Omega_{\text{can}}^{r-1} = \left(\sum_{a=1}^{n} dx_a \wedge dz_j^{(i_1,\cdots,i_a+1,\cdots,i_n)}\right)_{|(i_1,\cdots,i_a,\cdots,i_n)|=r-2}.$$

It corresponds to the curvature of $\xi_{\text{can}}^{(1)}$, which depends only on the coordinates (x, z^{r-1}). We can compute:

$$h^*\Omega_{\text{can}}^{r-1} = h^*\left(-\sum_{a,b=1}^{n} z_j^{(i_1,\cdots,i_a+1,\cdots,i_b+1,\cdots,i_n)} dx_a \wedge dx_b\right) = (0).$$

In the last step we get zero because cross derivatives agree. This computation tells us that the map

$$(x, z^{r-1}) : N \to B \oplus \text{Sym}^{r-1}(B^*, F)$$

is isotropic. Therefore, the argument can be iterated for decreasing r to produce a lift. □

The contractibility assumption on N is used in the proof to ensure that the restriction of the Liouville form at each step is exact.

6.3.3. *Lifting curves.* Let us particularise now to the case $\dim(B) = 1$. Then, in holonomic coordinates $(x, y = z^0, z)$ the Cartan 1-forms read

$$\alpha^l = dz^l - z^{l+1}dx, \qquad l = 0, \dots, r-1.$$

The particular flexibility of curves (compared to higher dimensional integral submanifolds) stems from the fact that any

$$g(t) = (x(t), z_r(t)) : [0,1] \to B \oplus \mathrm{Sym}^r(B^*, F)$$

is automatically isotropic. Then, following the recipe outlined in the proof of Theorem 6.14, we solve for the z^{r-1} coordinates using α^r:

$$g^*\alpha^r = z_{r-1}(t)dt - z_r(t)x'(t)dt$$

leading to the integral expression

$$z_{r-1}(t) = z_{r-1}(0) + \int_0^t z_r(s)x'(s)ds$$

which uniquely recovers z_{r-1} up to the choice of lift $z_{r-1}(0)$. Proceeding decreasingly in l we can solve for all the $z^l(t)$, effectively lifting g to an integral curve $\mathrm{Lift}(g) : [0,1] \to J^r(B, F)$.

According to Lemma 6.11, the lift $\mathrm{Lift}(g)$ is immersed if and only if g was immersed. Assuming g is immersed, the front tangencies $\Sigma(\mathrm{Lift}(g), \pi_f)$ correspond precisely to the singularities of tangency $\Sigma(g, \pi_b)$. This implies that to control the singularities of an integral curve it is sufficient to control the singularities of its metasymplectic projection, which is a smooth curve with no constraints.

6.3.4. *Restricted metasymplectic projection.* Unlike curves, higher-dimensional isotropic/integral submanifolds cannot be deformed freely. To get rid of differential constraints we consider instead:

Definition 6.15. *The **principal metasymplectic projection** with respect to the principal direction determined by the coordinate x_n is the map:*

$$\begin{aligned} \pi_L^n : J^r(B,F) &\to B \oplus \mathrm{Sym}^r(\mathbb{R}^*, F) \\ (x,y,z) &\to (x, z^{(0,\dots,0,r)}). \end{aligned}$$

That is, we only remember the pure r-order derivatives associated to x_n. We then work with Σ^2-free maps whose rank drops along the x_n-directions. We think of them as $(n-1)$-families of curves, allowing us to prove:

Lemma 6.16. *Given a smooth map:*

$$\begin{aligned} g : B &\to B \oplus \mathrm{Sym}^r(\mathbb{R}^*, F) \\ (t) = (\tilde{t}, t_n) = (t_1, \dots, t_n) &\to (\tilde{t}, x_n(t), z^{(0,\dots,0,r)}(t)), \end{aligned}$$

there exists an integral map $\mathrm{Lift}(g) : B \to J^r(B,F)$ satisfying $\pi_L^n \circ \mathrm{Lift}(g) = g$.

The map $\mathrm{Lift}(g)$ is unique up to the choice of $\mathrm{Lift}(g)|_{\{t_n=0\}}$.

Proof. The integral lift $\mathrm{Lift}(g)$ is given by the formula:

$$\begin{aligned} (t) \quad \mapsto \quad (&\tilde{t}, x_n; \\ &y = z^{(0,\dots,0,0)}; \\ &\partial_{\tilde{t}} y, z^{(0,\dots,0,1)}; \\ &\partial_{\tilde{t}}^2 y, \partial_{\tilde{t}} z^{(0,\dots,0,1)}, z^{(0,\dots,0,2)}; \\ &\dots; \\ &\partial_{\tilde{t}}^r y, \dots, \partial_{\tilde{t}} z^{(0,\dots,0,r-1)}, z^{(0,\dots,0,r)}). \end{aligned}$$

All the terms on the right hand side depend only on t. Let us explain how the other functions are obtained from t, x_n and $z^{(0,\dots,0,r)}$.

The term $z^{(0,\dots,0,l)}$ is the (formal) pure derivative of order l in the direction of x_n and it is defined (for decreasing l) by the integral expression:

$$z^{(0,\dots,0,l)}(t) := z^{(0,\dots,0,l)}(\tilde{t},0) + \int_0^{t_n} z^{(0,\dots,0,l+1)}(\tilde{t},s)x_n'(\tilde{y},s)ds,$$

following what we did in the previous Subsection for curves. In particular, the coordinate $y = z^{(0,\dots,0,0)}$ is recovered by integrating r times. At every step we can choose the value of $z^{(0,\dots,0,l)}(\tilde{t},0)$.

All other functions are derivatives of the form $\partial_{\tilde{t}}^i z^{(0,\dots,0,j)}$, for some integers i and j. Hence, we obtain them, uniquely, by differentiation. □

Note that the polar space of a $(n-1)$-dimensional horizontal element is $(n+k)$-dimensional and it consists of the $(n-1)$ original directions, the additional missing direction from the base, and the corresponding k pure derivative directions along the fibre. This indicates that any Σ^2-free integral map can be reconstructed by the lifting method we just used.

Most of the key properties of the lift can be read from the projected map:

Corollary 6.17. *Let g be a map into a principal metasymplectic projection. Then:*

- *The map $\mathrm{Lift}(g)$ is well-defined, smooth, integral and Σ^2-free.*
- *The singularities of mapping $\Sigma(\mathrm{Lift}(g))$ are in correspondence with $\Sigma(g)$.*
- *The singularities of tangency $\Sigma(\mathrm{Lift}(g), V_{\mathrm{can}})$ wtih respect to the vertical are in correspondence with $\Sigma(g, \mathrm{Sym}^r(B^*, F))$.*

7. Singularities of integral submanifolds

In this Section we introduce the models of integral Σ^2-free singularities needed for our h-principles.

Remark 7.1. *Our naming conventions for singularities reflect the behaviour of the integral maps themselves, not their front projections. In particular, the names we use often refer to their singularities of tangency with the vertical distribution. When singularities of mapping are present, we point it out explicitly.* △

In Section 7.1 we describe singularities of tangency of Whitney type. In Subsection 7.2 we use these to define models of singularities of tangency along submanifolds (as opposed to germs at points). Lastly, in Subsection 7.3 we look at singularities of mapping.

Recall our notation: We work on $J^r(Y \to X)$, where X is n-dimensional and k is the dimension of the fibres of Y. Sometimes we pass to local coordinates, in which case we write B for the base and F for the fibre.

7.1. Whitney singularities in jet spaces.

We introduced smooth Whitney singularities in Definition 4.1. We now discuss their integral analogues in jet space:

Definition 7.2. *Let $f : N \to (J^r(Y \to X), \xi_{\mathrm{can}})$ be a Σ^2-free integral mapping. The germ of f at a point p is a* ***Whitney singularity*** *(of tangency with respect to the vertical V_{can}) if:*

- *f is an immersion at p, and*
- *the base map $\pi_b \circ f$ has a Whitney singularity (Definition 4.1) at p.*

In particular, a germ of integral immersion f is said to be a fold/pleat if $\pi_b \circ f$ is a fold/pleat.

The stability of Whitney singularities due to Morin generalises to jet spaces through Givental's stability theorem [24]:

Theorem 7.3 (A. Givental)**.** *Assume $k = 1$. Then, up to point symmetries, integral Whitney singularities have no moduli.*

The statement for general k is not addressed in [32, 24] and, to our knowledge, it remains open.

This potential lack of uniqueness will not play a role in our arguments. We will always rely on concrete models produced using either generating functions or metasymplectic lifts. Every time a singularity appears, we will always mean one of these concrete local models, and not a general singularity given by Definition 7.2. By construction, these models will only make use of "one direction" in F and, as such, we will be able to pass between them invoking Givental's result.

7.1.1. *Generating functions.* Recall the notation from Subsections 4.2 and 6.2.4: Endow $B\times\mathbb{R}$ with coordinates $(x_1,\dots,x_n,q)$ and denote $x=(\tilde{x},x_n)=(x_1,\dots,x_n)$ and $\widehat{x}_l:=(x_1,\dots,x_{l-1},x_{l+1},\dots,x_n)$. Consider the fibration $\pi:B\times\mathbb{R}\to B$ defined by $(x,q)\mapsto x$. We set

$$\begin{array}{rcl} H_l(x,q):B\times\mathbb{R} & \to & \mathbb{R}, \\ (x,q) & \mapsto & q^{l+1}+x_1q^{l-1}+\cdots+x_l, \end{array}$$

and we let $\Gamma_l:=H_l^{-1}(0)$ be the locus of roots of $q\to H_l(x,q)$. The coordinates $(\widehat{x}_l,q)$ parametrise Γ_l, and we denote the parametrisation by

$$\begin{array}{rcl} s_l:\mathbb{R}^n & \to & B\times\mathbb{R}, \\ (\widehat{x}_l,q) & \mapsto & (x_1,\dots,X_l(\widehat{x}_l,x):=-q^{l+1}-x_1q^{l-1}-\cdots-x_{l-1}q,\dots,x_n,q). \end{array}$$

Define the generating functions:

$$\begin{gathered} G_{r,l}:B\times\mathbb{R}\to F \\ (x,q)\mapsto\left(\int_0^q (H_l(x,t))^r dt,0,\dots,0\right), \end{gathered} \tag{7.1.1}$$

where r is the order of the jet space and $l\leqslant n$.

The function H_l a submersion, as required in Subsection 6.2.4. Therefore, the loci $L_{G_{r,l}}$ are smooth integral manifolds which are parametrised by the locus of roots $\Gamma_l\cong\mathbb{R}^n(\widehat{x}_l,q)$. This is shown as the dashed diagonal arrow in the following diagram:

$$\begin{array}{ccccc} \mathbb{R}^n(\widehat{x}_l,q) & \xrightarrow{s_l} & \Gamma_l\subset B\times\mathbb{R} & \xrightarrow{j^rG_{r,l}} & J^r(B\times\mathbb{R},F) \\ & & \downarrow\pi & \searrow & \downarrow \\ & & B & \xleftarrow[\pi_b]{} & L_{G_{r,l}}\subset J^r(B,F) \end{array}$$

where $L_{G_{r,l}}$, is the reduction of $J^rG_{r,l}$ along the fibers of π, as in Definition 6.3. The diagram provides a parametrisation $\mathbb{R}^n(\widehat{x}_l,x)\to L_{G_{r,l}}$. The composition of the parametrisation with the base projection $\pi_b:J^r(B,F)\to B$ is precisely the $(n-l)$-fold stabilisation of the l-th Whitney map.

We now provide further details on folds and pleats.

7.1.2. *Folds.* Fix holonomic coordinates (x,y,z) in $J^r(B,F)$. We use the restricted metasymplectic projection $(x,z^{(0,\dots,r)})$ defined by the first $(n-1)$-coordinates. The following map

$$\begin{array}{rcl} g_r:B & \to & B\times\mathrm{Sym}^r(\mathbb{R}^*,F) \\ (x_1,\cdots,x_n) & \to & \left(x_1,\cdots,x_{n-1},X_n(x)=\dfrac{x_n^2}{2},z_1^{(0,\cdots,0,r)}(x)=x_n,0,\dots,0\right) \end{array}$$

has a fold singularity with respect to the vertical $\mathrm{Sym}^r(\mathbb{R}^*,F)$ along the hyperplane

$$\Sigma(A_{2r})=\Sigma^{10}(A_{2r})=\{x_n=0\}.$$

We can integrate $z_1^{(0,\cdots,0,r)}$ with respect to X_n r times, yielding the integral lift:

$$(x_1,\cdots,x_n) \to \Big(x_1,\cdots,x_{n-1}, X_n(x) = \frac{x_n^2}{2},\, y_1(x) = \frac{x_n^{2r+1}}{(2r+1)(2r-1)\dots 1}, 0, \dots$$
$$z_1^{(0,\cdots,0,i)}(x) = \frac{x_n^{2r-2i+1}}{(2r-2i+1)(2r-2i-1)\dots 1}, 0, \dots$$
$$z_1^{(0,\cdots,0,r)}(x) = x_n, 0, \dots, 0\Big).$$

This integral mapping can be recovered, using differentiation, from its front projection:

Definition 7.4. *The A_{2r}-**cusp** is the germ at the origin of the map:*

$$\begin{aligned} A_{2r} : B \quad &\to \quad B \times F \\ (x_1,\cdots,x_n) \quad &\to \quad \left(x_1,\cdots,x_{n-1}, X_n(x) = \frac{x_n^2}{2}, y_1(x) = \frac{x_n^{2r+1}}{(2r+1)(2r-1)\cdots}, 0, \dots, 0\right). \end{aligned}$$

That is, the front projection is described by multivalued function whose only non-zero entry is y_1. Invoking Theorem 7.3] we deduce that that the integral map we started with is equivalent to $L_{G_r,1}$, the fold we obtained using generating functions.

7.1.3. *Pleats.* We continue using the same setup. The following singularity of tangency with respect to the vertical is a pleat:

$$\begin{aligned} B \quad &\to \quad B \times \mathrm{Sym}^r(\mathbb{R}^*, F) \\ (x_1,\cdots,x_n) \quad &\to \quad \left(x_1,\cdots,x_{n-1}, X_n(x) = x_n^3/3 - x_1x_n, z_1^{(0,\cdots,0,r)}(x) = x_n, 0, \dots, 0\right). \end{aligned}$$

This restricted metasymplectic projection can then be lifted to produce a formula for the pleat in $J^r(B,F)$. To avoid cluttering the text, we write only the formula for the front:

Definition 7.5. *The A_{2r}-**swallowtail** is the germ at the origin of the mapping:*

$$\begin{aligned} \mathrm{Sw}_{2r} : B \quad &\to \quad B \times F \\ (x) \quad &\to \quad \left(x_1,\cdots,x_{n-1}, X_n(x) = x_n^3/3 - x_1x_n, y_1(x) = \int_0^{x_n}\int_0^{s_1}\dots\int_0^{s_{r-1}} s_r \prod_j (s_j^2 - x_1) ds_r \dots ds_1, \dots\right). \end{aligned}$$

Its singularity locus reads:

$$\Sigma^1(\mathrm{Sw}_{2r}) = \{x_n^2 - x_1 = 0\}, \qquad \Sigma^{11}(\mathrm{Sw}_{2r}) = \{x_n, x_1 = 0\}.$$

Givental's theorem tells us once more that this is equivalent to $L_{G_r,2}$. We leave it to the reader to produce similar formulas for the higher Whitney singularities using lifting.

7.1.4. *The Reidemeister I move.* The A_{2r}-swallowtail has a fibered nature. Consider a vector space K, serving as parameter space. Endow $K \times B$ with coordinates (s, x) and consider the fronts $B \to B \times F$ obtained from

$$\mathrm{Sw}_{2r}(k, x) : K \times B \to K \times B \times F$$

by freezing the coordinates (s).

If $s > 0$, the map has no singularities and is graphical over the base. If $s < 0$, the map has a pair of A_{2r}-cusps. At $|s| = 0$, the following birth/death phenomenon takes place:

$$(x_{l+1}, \dots, x_{n-1}, x) \mapsto \left(x_{l+1}, \dots, x_{n-1}, \frac{x_n^3}{3}, \frac{x_n^{3r+1}}{(3r+1)(3r-2)\dots 1}, 0, \dots, 0\right),$$

its lift to r-jet space is an embedded integral manifold whose tangency with respect to the vertical is a cubic singularity. In particular, it is not of Whitney type.

Definition 7.6. *The family of integral maps, obtained from $\mathrm{Sw}_{2r}(s,x)$ by lifting to $J^r(B,F)$, parametrically in s, is called the **first Reidemeister move**.*

One can define similar homotopies for integral mappings by lifting the higher Whitney singularities (as fibered maps over some of their coordinates).

7.2. **Semi-local singularities of tangency.** We go on describing singularities of tangency for Σ^2-free integral embeddings. The singularities we present are semi-local in the sense that they are not germs at points but germs along higher dimensional submanifolds.

The singularities we go through are: the double fold (Subsection 7.2.1), the regularised wrinkle (Subsection 7.2.3) and the stabilisation (Subsection 7.2.5). We also discuss their birth/death phenomena.

7.2.1. *The double fold.* Regard $D = \mathbb{S}^{n-1} \times \mathbb{R}$ as a fibration over $\mathbb{S}^{n-1}$ and endow it with coordinates $(\tilde{x}, x_n)$. We make use of the restricted metasymplectic projection in $J^r(D, F)$ associated to the foliation by spheres.

The following map has two consecutive folds along the x_n direction:

$$\begin{aligned} \sigma : D \quad &\rightarrow \quad D \oplus \mathrm{Sym}^r(\mathbb{R}^*, F) \\ (\tilde{x}, x_n) \quad &\rightarrow \quad (\tilde{x}, X_n(x) = x_n^3/3 - x_n; z_1^{(0,\dots,r)}(x) = x_n, 0, \dots 0). \end{aligned}$$

WHhen $k = 1$, the two folds have opposite orientations. The reader should think of this as having opposite Maslov coorientations once we lift. This will play no role in the present paper and we refer to the sequel [**?**] for further details.

Definition 7.7. *We say that an integral embedding*

$$f : N \rightarrow (J^r(Y \rightarrow X), \xi_{\mathrm{can}})$$

*has a **double fold** in an annulus $A \subset N$ if*

$$f|_{\mathcal{O}p(\partial A)} \quad \cong \quad \mathrm{Lift}^r(\sigma)|_{\mathcal{O}p(\mathbb{S}^{n-1} \times [-1,1])}$$

*up to point symmetry. The interior of A is often called the **membrane**.*

Do note that we do not require for the identification with the model to extend to the membrane, but the membrane is still part of the data of a double fold. The reason behind this is that, in our arguments, we will allow for double folds to appear nested inside one another, but we still want to remember how they are paired up.

The front of a double fold is equivalent to the map:

$$(x) \rightarrow \left(\tilde{x}, x_n^3/3 - x_n; \int_0^{x_n} \int_0^{s_1} \dots \int_0^{s_{r-1}} s_r \prod_j (s_j^2 - 1) ds_r \dots ds_1, 0 \dots, 0 \right), \tag{7.2.1}$$

whose singularity locus is comprised of two spheres $\{|x_n| = 1\}$ of A_{2r}-cusps.

7.2.2. *Fibered double folds.* We now describe how a double fold may appear in a family. Denote still $D := \mathbb{S}^{n-1} \times \mathbb{R}$ and fix additionally coordinates s in the parameter space $K := \mathbb{R}^m$. Then, consider the family of maps:

$$\begin{aligned} \sigma : K \times D \quad &\rightarrow \quad K \times D \oplus \mathrm{Sym}^r(\mathbb{R}^*, F) \\ (s, \tilde{x}, x_n) \quad &\rightarrow \quad (\tilde{x}, X_n(x) = x_n^3/3 + (|s|^2 - 1)x_n; z_1^{(0,\dots,r)}(x) = x_n, 0, \dots 0). \end{aligned}$$

For $|s| < 1$ fixed, the singularities are double folds of tangency with the vertical. When $|s| = 1$, the double folds merge into a $\mathbb{S}^{n-1}$ of cubic singularities. If $|s| > 1$, there are no singularities. Looking at σ as a single function, its singularities appear along:

$$\Sigma = \{x_n^2 + |s|^2 = 1\} = \mathbb{D}^m \times (\mathbb{S}^{n-1} \times \{\pm(1 - |s|^2)\} \cong \mathbb{S}^m \times \mathbb{S}^{n-1}.$$

and the cubical ones correspond to $\mathbb{S}^{m-1} \times \mathbb{S}^{n-1} \subset K \times X$.

Lifting σ, parametrically in s, we obtain the **fibered double fold**. Using Givental's theorem we readily see that, for $|s| < 1$, the resulting lift is indeed a double fold in r-jet space (in the sense of Definition 7.7).

Definition 7.8. *A family of integral maps fibered over* K

$$f : K \times N \to K \times (J^r(Y \to X), \xi_{\text{can}})$$

has a ***fibered double fold*** *if there is a fibered over* K *embedding of* $\Sigma \subset K \times N$ *such that*

$$f|_{\mathcal{O}p(\Sigma)} \quad \cong \quad \text{Lift}^r(\sigma)|_{\mathcal{O}p(\Sigma)}$$

up to fibrewise point symmetry.

7.2.3. *Wrinkles of tangency.* We now define singularities of tangency of wrinkle type. The reader should compare with the smooth Definition 4.3.

Consider now the restricted metasymplectic projection associated to the first $(n-1)$-coordinates of B. The following map has wrinkle-type singularities of tangency with respect to the vertical:

$$\begin{aligned} \sigma : B \quad &\to \quad X \oplus \text{Sym}^r(\mathbb{R}^*, F) \\ (\tilde{x}, x_n) \quad &\to \quad (\tilde{x}, x_n^3/3 + (|\tilde{x}|^2 - 1)x_n; z_1^{(0,\dots,r)} = x_n, 0, \dots 0). \end{aligned} \tag{7.2.2}$$

Definition 7.9. *An integral embedding of a ball* D *is a* ***tangency wrinkle*** *if its germ along* ∂D *is equivalent to* $\text{Lift}^r(\sigma)|_{\mathcal{O}p(\mathbb{S}^{n-1})}$.

The interior of D is called the membrane. A wrinkle may have further singularities in its membrane.

The front projection of a wrinkle reads:

$$(\tilde{x}, x_n) \mapsto \left(\tilde{x}, x_n^3/3 + (|\tilde{x}|^2 - 1)x_n; \int_0^{x_n} \int_0^{s_1} \dots \int_0^{s_{r-1}} s_r \prod_j (s_j^2 + |\tilde{x}|^2 + 1) ds_r \dots s_1, 0, \dots, 0\right).$$

7.2.4. *Fibered tangency wrinkles.* Usual smooth wrinkles are fibered, as explained in Subsection 4.3.3. The same is true for the tangency wrinkle in r-jet space. We let $D = \mathbb{R}^{m+n-1}$, where the first m-coordinates (q) are regarded as parameters and the last $(n-1)$-coordinates $(\tilde{x})$ are domain coordinates. We fix $X = \mathbb{R}^n$, with coordinates $(x) = (\tilde{x}, x_n)$.

A particular incarnation of the embryo is given by lifting the map:

$$(x) \mapsto (\tilde{x}, x_n^3/3 + |\tilde{x}|^2 x_n, ; z_1^{(0,\dots,r)} = x_n, 0, \dots 0).$$

7.2.5. *The stabilisation.*

Definition 7.10. *Set* $D = \mathbb{S}^{n-1}$. *A fibered over* D *integral embedding*

$$f : D \times \mathcal{O}p([0,1]) \quad \to \quad J^r(X, F)$$

is a ***stabilisation*** *if*

$$\Sigma(f, V_{\text{can}}) = D \times \{0\} \cup D \times \{1\}$$

and these are folds with the same Maslov coorientation. The image $f(D \times (0,1))$ *is called the* ***membrane*** *of the stabilisation.*

For a model we may consider the lift $\text{Lift}^r(\sigma)$ of the map:

$$\begin{aligned} \sigma : D \times \mathcal{O}p([0,1]) \quad &\to \quad X \oplus \text{Sym}^r(\mathbb{R}^*, F) \\ (\tilde{x}, x_n) \quad &\to \quad (\tilde{x}, x_n^3/3 - x_n; z_1^{(0,\dots,r)} = x_n^2, 0, \dots 0). \end{aligned} \tag{7.2.3}$$

7.2.6. *Zig-zags.* In the proof of Lemma **??** we see one of the incarnations of a phenomenon we call *open/closed switching*. It was first observed by A. Givental in [24]. Let us explain what it is.

Let us recall Equation 7.2.1, which defines the front projection of a double fold:

$$f(\tilde{x},t) = \left(\tilde{x}, x_n = t^3/3 - t; y_1 = \int_0^t \int_0^{s_1} \dots \int_0^{s_{r-1}} s_r \prod_j (s_j^2 - 1) ds_r \dots ds_1, 0 \dots, 0\right).$$

The term y_1 is defined by an iterated integral, as explained in Lemma 6.16. The way in which we obtained it was as follows: let $j^r f(\tilde{x},t)$ be the holonomic lift of f to a multi-section. Consider one of its components, the odd function

$$(z_1^{(0,\dots,0,r)} \circ j^r f)(\tilde{x},t) = t.$$

We then multiply it by $t^2 - 1$, so it remains odd, and then we integrate it once to yield the even function

$$(z_1^{(0,\dots,0,r-1)} \circ j^r f)(\tilde{x},t) = \int_0^t s_r(s_r^2 - 1) ds_r.$$

Inductively we see that:

Lemma 7.11. *The function $z_1^{(0,\dots,0,r-l)} \circ j^r f$ is:*

- *odd if l is even,*
- *even if l is odd.*

This alternation between even and odd is precisely what we call open/closed switching. It can be rephrased using Maslov coorientations in each $(r-l)$-jet space, but we leave this for the reader. We can interpret it geometrically:

Lemma 7.12. *The following statements hold:*

- *If r is even, the function y_1 increases at a fold point if and only if it increases at the other.*
- *If r is odd, the function y_1 increases at a fold point if and only if decreases at the other.*

Proof. Being critical points, when we say increase/decrease we mean as continuous functions, without considerations on the derivative. Note that the model at each fold point tells us that y_1 must be either increasing or decreasing.

If r is even, the function y_1 is odd. This is equivalent to the first statement. Similarly, if r is odd, the function y_1 is even, so the second statement follows. □

We can reason in exactly the same manner for the stabilisation and prove that the situation is exactly the opposite.

Lemma 7.13. *Let g be a stabilisation:*

- *If r is odd, the function $y_1 \circ g$ increases at a fold point if and only if it increases at the other.*
- *If r is even, the function $y_1 \circ g$ increases at a fold point if and only if decreases at the other.*

What this means is that if we want to have two A_{2r}-singularities in the front projection forming a "*zig-zag*" shape, we must use a double fold if r is even and a stabilisation if r is odd. We define:

Definition 7.14. *Set $D = \mathbb{S}^{n-1}$. A fibered over D integral embedding*

$$f : D \times \mathcal{O}p([0,1]) \quad \to \quad J^r(X,F)$$

*is a **zig-zag** if:*

- *r is even and f is a double fold,*
- *r is odd and f is a stabilisation.*

The front of the zig-zag is what we would call an *open* shape, and the other two situations (double fold with r odd, stabilisation with r even) we would call them *closed*. The importance of zig-zags is that they can be stacked on top of each other keeping the front projection embedded. This will be central in our h-principle in Section 8.

7.3. **Singularities of mapping.** The singularities we have presented so far are all of tangency, i.e. the integral maps themselves are non-singular. We will now look at singularities of mapping having well-defined Gauss map taking values in $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\xi_{\mathrm{can}}, n)$.

The main source of examples of singularities of mapping are projections of singularities of tangency (from a higher jet space). We make some remarks in this direction in subsection 7.3.1. We then define several germs: the cusp in its two incarnations (subsections 7.3.2 and **??**) and the swallowtail (subsection 7.3.3). These are the pieces we need to then define some semi-local singularities: the wrinkly stabilisation (subsection 7.3.4), the double cusp (subsection 7.3.6), and the wrinkle (subsection 7.3.7).

We continue using the notation from the previous Subsection 7.2.

7.3.1. *Projecting singularities.* Let $f : N \to J^r(B, F)$ be an integral map. Then the projection $\pi_{r,r-1} \circ f : N \to J^{r-1}(B, F)$ is integral as well. In Lemma **??** we additionally showed that if f is a multi-section then $\pi_{r,r-1} \circ f$ is a multi-section with a well-defined Gauss map $\mathrm{Gr}(\pi_{r,r-1} \circ f) = f$ into the horizontal elements (where we use the identification between horizontal elements and lifts to $J^r(B, F)$). Hence, when we project, singularities of tangency become singularities of mapping.

Some of the singularities we will describe below are obtained by projecting an r-times differentiable Whitney singularity. For instance, in subsections 7.1.2 and 7.1.3 we already saw that the front projection of the fold and the pleat are the A_{2r} cusp and swallowtail, respectively.

One important observation is:

Lemma 7.15. *Assume* $\dim(F) = 1$*. Let* $f : N \to J^r(B, F)$ *be a topologically embedded multi-section of the form* $f = \pi_{r+l,r} \circ g$*, with*
$$g : N \to J^{r+l}(B, F)$$
an embedded multi-section with Whitney singularities.

Then f *is stable among multi-sections lifting to* $J^{r+l}(B, F)$*.*

Proof. Let $(f_s)_{s\in[0,1]}$ be a deformation of $f_0 := f$ and let $(g_s)_{s\in[0,1]}$ be the corresponding deformation of $g_0 := g$ lifting it. Observe that the lifts, when they exist, are uniquely defined (by lifting on each branch).

According to Corollary **??**, the map g is stable up to contact transformation germs. Higher contact transformations are lifts of contact transformations in $J^r(B, F)$ (Lemma 3.8). This implies that the isotopy of contact transformations identifying g_s with g is a lift of an isotopy taking f_s to f, proving the claim. □

Remark 7.16. *We will encounter below singularities of mapping that have a well-defined Gauss map taking values in* $\mathrm{Gr}_{\Sigma^1}(\xi_{\mathrm{can}}, n)$*. Therefore, none of those singularities can admit a lift to* $J^{r+1}(B, F)$*. However, one may instead look the total space of*
$$\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\xi_{\mathrm{can}}, n) \to J^r(B, F)$$
and endow it with its tautological distribution. This partially compactifies $J^{r+1}(B, F)$ *and, by definition, the singularities we describe admit a lift to* $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\xi_{\mathrm{can}}, n)$*.*

For $\dim(B) = \dim(F)$*, iterating this construction yields the* Monster tower*, as introduced by R. Montgomery and M. Zhitomirskii in the treatise [35]. They show that there is a correspondence between points in the tower and singularities of fronts. Their results should partly translate to our context of* Σ^2*-free singularities, but we point out some difficulties in Remark 7.22 below.*

An intriguing question is whether the whole Grassmannian of multi-section elements $\overline{\mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, n)}$ *is smooth. If this were true, the natural next step would be to construct the analogue of the Monster tower.*

7.3.2. *The horizontal cusp.* As we prove below, projecting a fold down one level yields:

Definition 7.17. *An integral map (Definition ??)*

$$f : \mathcal{O}p(\{0\}) \quad \to \quad J^r(X, F)$$

is a ***horizontal cusp*** *if:*

- *The singularities of* $\pi^n_L \circ f$ *form a hypersurface of semicubic cusps.*
- $\mathrm{Gr}(f)$ *takes values in* $\mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, n)$.

A explicit fibered model can be obtained by lifting

$$(\tilde{x}, x_n) \mapsto (\tilde{x}, x_n^2; z_1^{(0,\dots,0,r)} = x_n^3, 0, \dots, 0).$$

Lemma 7.18. *Let* $\dim(F) = 1$. *Then any horizontal cusp is equivalent to the model (using point symmetries in the target, and diffeomorphisms in the domain).*

Proof. By assumption f can be lifted to an integral map $\mathrm{Gr}(f) : N \to J^{r+1}(X, F)$. Since its meta-symplectic projection has semicubic cusps, this lift is an embedding. The singularities of mapping of f correspond to fold singularities of tangency of $\mathrm{Gr}(f)$. The claim follows from Lemma 7.15. □

In particular, a horizontal cusp f is a topological embedding, even if it is not an immersion. Its front singularities are A_{2r+2}-cusps.

7.3.3. *The swallowtail.* In subsection 4.5.1 we defined the smooth the open semicubic swallowtail within the context of the wrinkle in positive codimension (Subsection 4.5). Now we define its jet space analogue:

Definition 7.19. *An integral map (Definition ??)*

$$f : \mathcal{O}p(\{0\}) \quad \to \quad J^r(X, F)$$

is a ***horizontal swallowtail*** *if:*

- $\pi^n_L \circ f$ *has a open semi-cubic swallowtail at the origin.*
- $\mathrm{Gr}(f)$ *takes values in* $\mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, n)$.

It is yet again a topological embedding because that is the case for $\pi^n_L \circ f$.

We can produce a model by lifting the following map into a principal metasymplectic projection:

$$(\tilde{x}, x_n) \mapsto (\tilde{x}, \int_0^{x_n} (s^2 - x_1) ds; z_1^{(0,\dots,0,r)} = \int_0^{x_n} (s^2 - x_1)^2 ds, 0, \dots, 0).$$

Its singularity locus Γ consists of the parabola $\{x_n^2 = x_1\}$, which is tangent to the x_n-lines along the codimension-2 linear subspace $A = \{x_n = x_1 = 0\}$. A is the locus of swallowtails, and its complement in Γ consists of horizontal cusps. Hence, the swallowtail serves as a birth/death of cusps (as is the case in the smooth setting).

Lemma 7.20. *Let* $\dim(F) = 1$. *Then any horizontal swallowtail is equivalent (using point symmetries in the target, and diffeomorphisms in the domain) to the model.*

Proof. We lift f to $\mathrm{Gr}(f) : \mathcal{O}p(\{0\}) \to J^{r+1}(X, F)$, which is smooth, embedded, and has a pleat at the origin. Lemma 7.15 applies. □

One can also consider *vertical* swallowtails or swallowtails with singularity locus becoming vertical over a submanifold. We will not study this.

7.3.4. *The wrinkly stabilisation.* We explained in subsection **??** that there is a correspondence between smooth wrinkles and double folds by performing surgeries. We will not provide a justification of this, but the same is true in jet spaces. For instance, the double fold (subsection 7.2.1) and the regularised wrinkle (subsection 7.2.3) are, up to surgery, equivalent. Similarly, there is a "wrinkle" analogue of the stabilisation, and one can pass between them through surgeries. It is defined as follows:

Definition 7.21. *Set $D = \mathbb{R}^{n-1}$. An integral map (Definition* **??***) fibered over D*

$$f : \mathcal{O}p(\mathbb{S}^{n-1}) \quad \to \quad J^r(X, F)$$

is a ***wrinkly stabilisation*** *if:*

- *$\Sigma^{10}(f) = \mathbb{S}^{n-2}$ is a locus of vertical cusps,*
- *$\Sigma^{10}(f, V_{\text{can}}) = \mathbb{S}^{n-1}$,*
- *The hemispheres $\mathbb{S}^{n-1} \backslash \mathbb{S}^{n-2}$ are folds with the same Maslov coorientation.*
- *It is a topological embeddeding and has no other singularities.*

Note that along $\mathbb{S}^{n-2}$ there is discontinuity in the Gauss map. Hence, the wrinkly stabilisation is not a multi-section in the sense of Definition **??**.

Remark 7.22. *This is a continuation of Remark 7.16 above. The wrinkly stabilisation shows the first difficulty with the Monster tower approach for higher dimensional manifolds: some singularities do not admit a continuous Gauss map.*

If we look at the maps induced by f on each fibre, we see that if $|\tilde{x}| < 1$ then they are curves with two folds, if $|\tilde{x}| > 1$ they are curves graphical over the zero section, and if $|\tilde{x}| = 1$, they are vertical cusps. That is, it corresponds to the standard unfolding of the cusp. Thus, not admitting a continuous Gauss map corresponds to a phenomenon already observed in [35, Section 9.1]*: the lifting procedure to the Monster tower is not continuous in the unfolding parameter. This is something to be explored in future work.*

Lemma 7.23. *The topological embedding condition is implied, in the vicinity of its cusp locus, from the first three items.*

Proof. For $|\tilde{x}|$ smaller than but close to one, the curve $\pi_L^n \circ f(\{\tilde{x}\} \times \mathbb{R})$ is an unfolding of the cusp. It describes a little loop when projected to $(x_n, z_1^{(0,\dots,0,r)})$. In particular, it has a self-intersection point. However, according to the subsection 6.3.3, the two intersection points have different lifts by integration. □

A model we may consider is the lift of

$$(\tilde{x}, x) \mapsto (\tilde{x}, x_n^3/3 + (|\tilde{x}|^2 - 1)x_n; z_1^{(0,\dots,0,r)} = x_n^2, 0, \dots, 0).$$

The principal metasymplectic projection of any wrinkly stabilisation is equivalent, as a smooth map, to this model. However, it is unclear whether the model is unique up to point symmetries.

7.3.5. *The wrinkled zig-zag.*

Definition 7.24. *An integral embedding*

$$f : \mathcal{O}p(\mathbb{S}^{n-1}) \to J^r(X, F)$$

is a ***wrinkled zig-zag*** *if:*

- *r is odd and f is a regularized wrinkle (Definition* **??***),*
- *r is even and f is a wrinkled stabilization (Definition 7.21).*

7.3.6. *The double (horizontal) cusp.* Now we consider two spheres of horizontal cusps bounding an annulus:

Definition 7.25. *Set $D = \mathbb{S}^{n-1}$. A fibered over D integral map (Definition* **??***)*

$$f : D \times \mathcal{O}p([0,1]) \quad \to \quad J^r(X,F)$$

is a ***double cusp*** *if*

- *f is a topological embedding.*
- *$\mathrm{Gr}(f) : D \times \mathcal{O}p([0,1]) \to J^{r+1}(X,F)$ is a stabilisation.*

The image $f(D \times (0,1))$ is called the ***membrane*** *of f.*

In particular, we are requiring that

$$\Sigma(f) = D \times \{0\} \cup D \times \{1\}$$

are horizontal cusps. If that is the case, the lift $\mathrm{Gr}(f)$ exists and is an immersion with two folds. Hence, it may be a double fold or a stabilisation. We require that it is the latter.

The key property here is:

Lemma 7.26. *The front singularities of the double cusp are two A_{2r+2}-cusps in an open configuration (i.e. a zig-zag).*

This follows from the open/closed switching from Lemma 7.13, see subsection 7.2.6.

7.3.7. *The wrinkle.* The "wrinkly" analogue of the double cusp is precisely:

Definition 7.27. *Set $D = \mathbb{R}^{n-1}$. An integral map (Definition* **??***), fibered over D,*

$$f : \mathcal{O}p(\mathbb{S}^{n-1}) \quad \to \quad J^r(X,F)$$

is a ***wrinkle*** *if*

- *$\mathrm{Gr}(f) : D \times \mathcal{O}p([0,1]) \to J^{r+1}(X,F)$ is a wrinkly stabilisation (Definition* **??***).*
- *f is a topological embedding.*

The image $f(D \times (0,1))$ is called the ***membrane****.*

A possible model is the lift of the wrinkled map of positive codimension (see Subsection 4.5):

$$F(\tilde{x}, x_n) = (\tilde{x}, \int_0^{x_n} (s^2 + |\tilde{x}|^2 - 1)ds; z_1^{(0,\dots,0,r)} = \int_0^{x} (s^2 + |\tilde{x}|^2 - 1)^2 ds, 0, \dots, 0).$$

We do not know if $\mathrm{Lift}(F)$ is the only possible model. However, the principal metasymplectic projection of a wrinkle is equivalent to F if we let left equivalences be diffeomorphisms preserving the base projection. From this we deduce:

Lemma 7.28. *Equivalently, a wrinkle is an integral topological embedding*

$$f : \mathcal{O}p(\mathbb{S}^{n-1}) \quad \to \quad J^r(X,F)$$

with singularity locus $\Sigma(f) = \mathbb{S}^{n-1}$ satisfying:

- *The equator $\mathbb{S}^{n-2}$ consists of semicubic swallowtails.*
- *The hemispheres are horizontal cusps.*

Remark 7.29. *The wrinkle is unique for smooth maps (i.e. $r = 0$). Uniqueness for $r > 0$, as we stated, is unknown. In the contact case (i.e. $r = 1$ and $\dim(F) = 1$), wrinkles for legendrians were defined by D. Álvarez-Gavela in [1], providing a explicit model. Although not stated explicitly in his paper, it seems like uniqueness follows from the constructions he provides.*

7.3.8. *Fibered wrinkles.* Let us present the fibered version. We fix coordinates (q) in $\mathbb{R}^m$ and (x) in $X = \mathbb{R}^n$.

Definition 7.30. *A* ***fibered over*** $\mathbb{R}^m$ ***wrinkle*** *is a map*

$$f : \mathcal{O}p(\mathbb{S}^{m+n-1}) \quad \to \quad \mathbb{R}^m \times J^r(X, F),$$

which we regard as a m-parameter family of integral topological embeddings $f_q(x) = f(q,x)$ with singularity locus $\mathbb{S}^{m+n-1}$ satisfying:

- $\Sigma^{110}(\pi^n_L \circ f_q) = \mathbb{S}^{m+n-2}$ *are open semicubic swallowtails,*
- $\Sigma^{10}(\pi^n_L \circ f_q) = \mathbb{S}^{m+n-1} \backslash \mathbb{S}^{m+n-2}$ *are horizontal cusps.*

The maps with $|q| = 1$ are called (wrinkle) ***embryos****.*

A possible model for the principal metasymplectic projection of an embryo reads:

$$(\tilde{x}, x_n) \to (\tilde{x}, \int_0^{x_n} (s^2 + |\tilde{x}|^2) ds; z_1^{(0,\dots,0,r)} = \int_0^x (s^2 + |\tilde{x}|^2)^2 ds, 0, \dots, 0).$$

However, we do not know whether this model is unique.

8. Holonomic approximation by multi-sections

The main result of this Section is an h-principle with PDE flavour. It states that the holonomic approximation Theorem 5.2 applies to closed manifolds as long as we are willing to be flexible and allow for multi-sections. A particular consequence is that any open partial differential relation admits a solution in the class of multi-sections.

The interesting part of the result is that it is sufficient to work with multi-sections with simple singularities. Namely, they will satisfy that:

- Their only singularities are folds in a zig-zag configuration.
- Their front projection is topologically embedded.

In Subsection 8.1 we formulate this formally. In Subsection 8.2 we present the key geometric insight needed for our arguments. Lastly, in Subsection 8.3 we provide the proof.

As in previous Sections, we fix a smooth fibre bundle $Y \to X$, with X compact. We work on the jet space $J^r(Y \to X)$. In order to quantify how close two sections of $J^r(Y \to X)$ are, we fix a metric.

8.1. **Statement of the result.** Recall the notion of zig-zag from subsection 7.2.6. We are interested in multi-sections of the form:

Definition 8.1. *A* ***section with zig-zags*** *is:*

- *an embedded multi-section $f : X \to J^r(Y \to X)$,*
- *a finite collection of disjoint annuli $\{A_j \subset X\}$,*

satisfying:

- $\pi_f \circ f$ *is a topological embedding,*
- $f|_{X \backslash (\cup_j A_j)}$ *is horizontal,*
- $f|_{A_j}$ *is a zig-zag.*

Our main result is the natural multi-section version of the holonomic approximation Theorem 5.2:

Theorem 8.2. *Let $\sigma : X \to J^r(Y \to X)$ an arbitrary section. Then, for any $\varepsilon > 0$, there exists a map $f : X \to J^r(Y \to X)$ satisfying:*

- *f is a section with zig-zags;*

- $|f - \sigma|_{C^0} < \varepsilon$.

It should be immediate to the reader experienced in h-principles, after inspecting the proof, that a parametric and relative (in the domain and the parameter) version also holds. The parametric version is stated and proven later in this Section.

Furthermore, the Theorem is the graphical case of the analogous result about approximating r-jets of submanifolds through submanifolds with zig-zags (that is, the generalisation to higher jets of the wrinkled embeddings Theorem 5.10). This will be addresed in the next Section.

8.2. **The key ingredient of the proof.** We now present the simple observation that constitutes the basis of our work:

Definition 8.3. *Let $I = [a, b]$ be an interval. An **asymptotically flat sequence of zig-zag bump functions** is a sequence of maps*

$$(\rho_N)_{N\in\mathbb{N}} : [a, b] \to J^0([a, b], \mathbb{R})$$

satisfying

- *their holonomic lifts $j^r\rho_N : [a, b] \to J^r([a, b], \mathbb{R})$ are sections with zig-zags,*
- $\rho_N|_{\mathcal{O}p(a)}(t) = (x = t, y = 0)$,
- $\rho_N|_{\mathcal{O}p(b)}(t) = (x = t, y = 1)$,
- $|z^{(r')} \circ \rho_N| < \frac{1}{N}$ *for all $r' > 0$.*

The name follows from the fact that an element ρ_N, with N sufficiently large, allows us to interpolate between two given sections without introducing big derivatives (unlike a normal bump function).

Proposition 8.4. *An asymptotically flat sequence of zig-zag bump functions exists on any interval.*

Before we provide a proof, let us explain a Corollary that showcases this.

Corollary 8.5. *Let $\varepsilon, \delta > 0$ be given. Consider sections $s_0, s_1 : \mathbb{D}^n \to \mathbb{R}^k$ satisfying $|s_0 - s_1|_{C^r} < \varepsilon$.*

Then, there exists a section with zig-zags $f : \mathbb{D}^n \to J^r(\mathbb{D}^n, \mathbb{R}^k)$ satisfying:

- $(\pi_f \circ f)|_{\mathbb{D}^n_{1-\delta}} = s_0$,
- $(\pi_f \circ f)|_{\mathcal{O}p(\partial\mathbb{D}^n)} = s_1$,
- $|j^r s_0 - f|_{C^0} < 4\varepsilon$.

Proof. We write $(y_1, \ldots, y_k)$ for the coordinates in the fibre $\mathbb{R}^k$ and (x) for the coordinates in the base. We break down the proof into elementary steps.

The pushing trick. Since $|s_0 - s_1|_{C^0} < \varepsilon$, we can shift s_0 by adding a constant in $\mathbb{R}^k$:

$$\tilde{s}_0(x) := s_0(x) + (2\varepsilon, 0, \ldots, 0).$$

Replacing s_0 by $\tilde{s}_0$ guarantees that:

$$\tilde{s}_0(x) \neq s_1(x), \qquad \text{for every } x \in \mathbb{S}^{n-1} \times [1 - \delta, 1],$$

while retaining a bound $|\tilde{s}_0 - s_1|_{C^r} < 3\varepsilon$. We henceforth restrict the domain of $\tilde{s}_0$ and s_1 to the region of interest $\mathbb{S}^{n-1} \times [1 - \delta, 1]$.

First simplification. We can simplify the setup by applying the fibrewise translation:

$$\begin{array}{rcl} J^0(\mathbb{S}^{n-1} \times [1-\delta, 1], \mathbb{R}^k) & \to & J^0(\mathbb{S}^{n-1} \times [1-\delta, 1], \mathbb{R}^k) \\ p & \to & p - \tilde{s}_0(\pi_b(p)), \end{array}$$

It preserves the C^r–distance and maps $\tilde{s}_0$ to the zero section. The section s_1 is mapped to $s := s_1 - \tilde{s}_0$. Consequently, we just need to explain how to interpolate between the zero section and some arbitrary section s satisfying $|s|_{C^r} < 3\varepsilon$ and $s(x) \neq 0$ for all x.

<u>Second simplification.</u> A second symmetry allows us to put s in normal form. Due to the nature of the shift we performed, we have that

$$\varepsilon < |y_1 \circ s(x)| < 3\varepsilon$$

for all x. This allows us to define a framing

$$\begin{aligned} A : \mathbb{S}^{n-1} \times [1-\delta, 1] &\to \mathrm{GL}(\mathbb{R}^k) \\ A(x) &= (s, e_2, e_3, \dots, e_k), \end{aligned}$$

where $\{e_j\}_{j=1,\dots,k}$ is the framing dual to the coordinates y_i in $\mathbb{R}^k$. The framing A defines a fibre-preserving transformation of the $\mathbb{R}^k$-bundle by left multiplication. By construction $Ae_1 = s$.

<u>Main construction.</u> Apply Proposition 8.4 to produce an asymptotically flat sequence of zig-zag bump functions

$$(\rho_N)_{N\in\mathbb{N}} : [1-\delta, 1] \quad \to \quad J^0([1-\delta, 1], \mathbb{R}).$$

We use it to define a sequence of front projections:

$$\begin{aligned} Z_N : \mathbb{S}^{n-1} \times [1-\delta, 1] &\to J^0(\mathbb{S}^{n-1} \times [1-\delta, 1], \mathbb{R}^k) \\ (\tilde{x}, t) &\to A[\rho_N(t)e_1]. \end{aligned}$$

We claim that, for N large enough, the holonomic lift $f_N := j^r Z_N$ satisfies the properties prescribed.

<u>Checking the claimed properties.</u> We first observe that f_N is a section with zigzags. This follows from the fact that $j^r(\rho_N e_1)$ is a section with zigzags and f_N is obtained from it by applying the point symmetry $j^r A$. In particular, the singularities of f_N are codimension-1 spheres of folds, corresponding to the values of t in which ρ_N has an A_{2r}-singularity.

The second and final claim is that $|f_N|_{C^0} < 4\varepsilon$ if N is large enough. Equivalently, we have to bound the C^r-size of:

$$A(\rho_N e_1) = \rho_N s.$$

Note that we can pretend that ρ_N is an actual function, because this is true over a dense set. Therefore, for each multi-index I with $|I| \leqslant r$ we compute:

$$|\partial^I(\rho_N s)|^2 = \left| \sum_{I'+I''=I} (\partial^{I'}\rho_N)(\partial^{I''}s) \right|^2 \leqslant \sum_{I'+I''=I} |\partial^{I'}\rho_N|^2 |\partial^{I''}s|^2$$

Now, each derivative $|\partial^{I'}\rho_N|$ is smaller than $1/N$, with the exception of $|\rho_N| = 1$. Similarly, $|\partial^{I''}s| < 3\varepsilon$ for all I''.

Let K_1 be the maximum number of decompositions $I' + I'' = I$ that a multi-index $|I| \leqslant r$ in n variables and k outputs may have. Let K_2 be the number of multi-indices $|I| \leqslant r$. Then:

$$\begin{aligned} |\partial^I(\rho_N s)|^2 < &\quad |\partial^I s|^2 + \frac{9K_1}{N^2}\varepsilon^2 \\ |\rho_N s|^2_{C^r} < &\quad \sum_I \left(|\partial^I s|^2 + \frac{9K_1}{N^2}\varepsilon^2 \right) < |s|^2_{C^r} + \frac{9K_1K_2}{N^2}\varepsilon^2. \end{aligned}$$

Therefore, by setting $N^2 > 9K_1K_2$, we conclude:

$$|f_N|_{C^0} = |\rho_N s|_{C^r} < |s|_{C^r} + \varepsilon < 4\varepsilon.$$

□

Remark 8.6. *An interesting feature of the proof is that the sections with zig-zags we construct are obtained from the "standard" sections with zig-zags $j^r(\rho_N e_1)$ by applying a point symmetry. The same argument would work if instead of $j^r\rho_N$ we used a particular model of wrinkle (subsection 7.3.7). Hence, we can bypass the potential uniqueness issues for wrinkles pointed out in Remark 7.29.*

Now we construct the zig-zag bump functions:

*Proof of Proposition 8.4. Observe that i*t is sufficient to prove the claim for $I = [0,1]$, since any two intervals are diffeomorphic by a scaling and a translation. The scaling dilates the fibres of jet space in a homogeneous manner, so any asymptotically flat sequence is mapped to an asymptotically flat sequence.

Fix N. We will construct ρ_N as the holonomic lift $\rho_N = j^r(\pi_f \circ \rho_N)$ of its front projection $\pi_f \circ \rho_N$.

The infinite zig-zag. We first define:

$$\begin{aligned} Z : \mathbb{R} \quad &\to \quad J^0([0,1],\mathbb{R}), \\ (t) \quad &\to \quad \left(x(t) = \frac{1}{2}\int_0^t \sin(s)ds, y(t) = \int_0^t \sin(s)^{2r} ds \right). \end{aligned}$$

We claim that, at each of its critical points $\{t = 0, \pi, 2\pi, \dots\}$, the map Z is modelled on the A_{2r}-singularity. To prove this we compute the Taylor expansion at each of these points:

$$\sin(l\pi + h) = \frac{h}{2} + O(h^3), \qquad \sin(l\pi + h)^{2r} = h^{2r} + O(h^{2r+2}),$$
$$x(l\pi + h) = \frac{h^2}{4} + O(h^4), \qquad y(l\pi + h) = \frac{h^{2r+1}}{2r+1} + O(h^{2r+3}).$$

Which proves the claim because the A_{2r} singularity is stable.

From this computation we deduce that the lift

$$j^r Z : \mathbb{R} \to J^r([0,1],\mathbb{R})$$

is an integral mapping with fold singularities. Since its front is topologically embedded, $j^r Z$ is embedded. Lastly, according to the definition in Subsubection 7.2.6, the germ $j^r Z|_{\mathcal{O}p([(2l-1)\pi, 2l\pi])}$ is a zig-zag. The section with zig-zags $j^r Z$ has infinitely many of them stacked.

A piece of the infinite zig-zag. Next, observe that Z is graphical over $[0,1]$ in the intervals $(2l\pi,(2l+1)\pi)$. In particular, we can flatten Z in $\mathcal{O}p(0)$ so that it is identically 0, without introducing self-intersections of the front. Similarly, for any l, we can flatten Z in the region $\mathcal{O}p((2l+1)\pi)$ so that it is identically $Z((2l+1)\pi)$. Lastly, we can scale this modification of Z, dividing by the constant $Z((2l+1)\pi)$. In this manner we obtain a front that is identically 0 and 1 in $\mathcal{O}p(0)$ and $\mathcal{O}p((2l+1)\pi)$, respectively. We denote it by Z_N.

We claim that, if l is large enough, then $|z^{(a)} \circ j^r Z_N| < \varepsilon$ for all $a > 0$. This follows immediately from the scaling we just did: Z was 2π-periodic, so the quantities $z^{(a)} \circ j^r Z$ were bounded. The quantity $Z((2l+1)\pi)$ goes to infinity as l does, so a sufficiently large choice guarantees that the derivatives of $j^r Z_N$ are smaller than $1/N$.

Lastly, we simply reparametrise

$$\pi_f \circ \rho_N(t) = Z_N \circ \phi(t),$$

where $\phi : [0,1] \to [0,(2l+1)\pi]$ is a suitable diffeomorphism. □

8.3. **The proof.** The proof of Theorem 8.2 follows the standard structure of an h-principle.

In subsection 8.3.2 we prove the *reduction step*. Its output is a holonomic section g, defined along the codimension-1 skeleton of X and approximating the given formal section σ.

In subsection 8.3.3 we provide the *extension argument*: we extend g to the interior of the top dimensional cells. In order to obtain a good approximation of σ, the extension to the interior must be a multi-section, as presented in Corollary 8.5.

8.3.1. *Preliminaries.* We must fix some auxiliary data first. Depending on the constant $\varepsilon > 0$ we fix a finite collection of pairs $\{(U_i, f_i)\}$ such that

- $\{U_i\}$ is a covering of X by balls,
- $f_i : U_i \to J^r(Y|_{U_i} \to U_i)$ is a holonomic section satisfying $|f_i - \sigma|_{U_i}| < \varepsilon$.

The existence of such a collection follows from the standard holonomic approximation Theorem 5.2 applied to each point in X. By compactness of X we get a finite refinement.

We then triangulate X, yielding a triangulation $\mathcal{T}$. We assume that this triangulation is fine enough to guarantee that each simplex is contained in one of the U_i. Given a top-simplex $\Delta \in \mathcal{T}$, we choose a preferred U_i and we denote the corresponding section f_i by f_Δ.

We remark that $Y|_{U_i}$ is trivial, so we can make the identification $J^r(Y|_{U_i} \to U_i) \cong J^r(\mathbb{D}^n, \mathbb{R}^k)$. We can then relate the C^0-norm in the former with the standard C^0-norm in the latter. By finiteness of the cover there is a constant bounding one in terms of the other. We assume this constant is 1 to avoid cluttering the notation.

8.3.2. *Reduction.* The codimension-1 skeleton of X is a CW-complex of positive codimension. Thus, according to Theorem 5.2, there exists:

- a wiggled version $\tilde{\mathcal{T}}$ of $\mathcal{T}$,
- a holonomic section $g : \mathcal{O}p(\tilde{\mathcal{T}}) \to Y$ satisfying $|\sigma - j^r g| < \varepsilon$.

The wiggling can be assumed to be C^0-small, so each top-simplex $\Delta \in \tilde{\mathcal{T}}$ is contained in the same U_i as the original simplex. I.e., we have sections g (defined over $\mathcal{O}p(\partial\Delta)$) and f_Δ (defined over the whole of Δ), both of them approximating σ.

8.3.3. *Extension.* We focus on a single top-simplex $\Delta \in \tilde{\mathcal{T}}$ because the argument is the same for all of them. We simply observe that Corollary 8.5 applies to g and f_Δ over the annulus $\mathcal{O}p(\partial\Delta)$, producing the desired multi-section extension f of $j^r g$ to the interior of Δ. The Corollary guarantees that:

$$|f - \sigma| < |f - j^r f_\Delta| + |j^r f_\Delta - \sigma| < 5\varepsilon.$$

This concludes the proof of Theorem 8.2. □

We close with an extremely biased remark about the proof: the idea presented (zig-zag bump functions together with the pushing trick) seems simpler than the path followed in [21] (reducing to simple tangential homotopies and approximating them with a model zig-zag). Additionally, it has a more transparent connection with holonomic approximation. Therefore, Theorem 8.2 provides a new understanding even in the classic case $r = 1$.

8.4. **Parametric and relative version.** To state the parametric version of Theorem 8.2 we work in the foliated setting of Section 3.5. Let X be an n-dimensional manifold endowed with a codimension-k foliation $\mathcal{F}$. Then, given a fiber bundle $Y \to X$, we consider the leafwise jet bundle $J^r(Y \to (X, \mathcal{F}))$.

In parametric families, double folds (Definition 7.7) can appear and disappear. Thus, we need to extend the notion of a section with zig-zags (Definition 8.1) to include the appropriate birth-death behavior.

Definition 8.7. *A **foliated section with zig-zags** is:*

- *an embedded multi-section $f : (X, \mathcal{F}) \to J^r(Y \to (X, \mathcal{F}))$ (Definition* **??***);*
- *a finite collection of disjoint (embedded) cylinders $\{C_j \subset X\}$ each isomorphic to $\mathbb{S}^{n-k-1} \times \mathbb{D}^{k+1}$.*

This data is assumed to satisfy:

- *$\pi_f \circ f$ is a topological embedding;*
- *$f|_{X \setminus (\cup_j C_j)}$ is transverse to the fibers of $\pi_b : J^r(Y \to (X, \mathcal{F})) \to X$;*
- *$\mathcal{F}|_{C_j}$ is equal to the foliation induced by the projection $\pi : \mathbb{S}^{n-k-1} \times \mathbb{D}^{k+1} \to \mathbb{D}^k$ onto the first k coordinates of $\mathbb{D}^{k+1}$;*
- *$f|_{C_j}$ is equal to the $\mathbb{S}^{n-k-1}$ stabilization of a wrinkled zig-zag (Definition 7.24) fibered over $\mathbb{D}^{k+1} \to \mathbb{D}^k$.*

Thus, the singular locus of a foliated zig-zag equals $\Sigma(f) = \Sigma^{10} \cup \Sigma^{110}$ where

$$\Sigma^{10}(f|_{C_j}) = \mathbb{S}^k \backslash \mathbb{S}^{k-1} \quad \Sigma^{110}(f|_{C_j}) = \mathbb{S}^{k-1},$$

and each cylinder C_j and for leaf $\mathcal{L}$ of $\mathcal{F}$, one of the following holds:

- $C_j \cap \mathcal{L} = \varnothing$,
- $C_j \cap \mathcal{L} = \mathbb{S}^{n-k-1}$, consisting of double fold embryos (Equation 4.4.2),
- $C_j \cap \mathcal{L} = \mathbb{S}^{n-k-1} \times I$, and the restriction of f to this annulus is a zig-zag (Definition 7.14).

The parametric version of Theorem 8.2 is then stated as follows:

Theorem 8.8. *Let $\sigma : X \to J^r(Y \to (X, \mathcal{F}))$ be an arbitrary section. Then, for any $\varepsilon > 0$, there exists a map $f : X \to Y$ satisfying:*

- *f is a foliated section with zig-zags,*
- *$|f - \sigma|_{C^0} < \varepsilon$.*

Moreover, if σ is holonomic in a neighborhood of a polyhedron $A \subset M$, then we can arrange $f = \sigma$ on a neighborhood of A.

8.5. **The key ingredient of the proof.** The proof closely follows that of Theorem 8.2. The additional ingredient is a description of the birth/death of zig-zag bump functions as in Definition 8.3.

Definition 8.9. *Let $[-\delta, \delta] \times [a, b] \subset \mathbb{R}$ be a square and fix coordinates $(s, t) \in [-\delta, \delta] \times [a, b]$. An* ***aymptotically flat sequence of zig-zag bump functions with birth/death of order*** *r is a sequence of maps*

$$(\rho_N)_{N \in \mathbb{N}} : [-\delta, \delta] \times [a, b] \to [-\delta, \delta] \times J^0([a, b], \mathbb{R})$$

satisfying:

(i) *their holonomic lifts $j^r \rho_N : [-\delta, \delta] \times [a, b] \to [-\delta, \delta] \times J^r([a, b], \mathbb{R})$ are multi-sections with zig-zag birth/deaths, should define this describe the singular locus inside the domain?,*
(ii) *$\rho_N(\delta, t)$ is an asymptotically flat sequence of zig-zag bump functions as in Definition 8.3,*
(iii) *$\rho_N(-\delta, t) = (x = t, y = 0)$,*
(iv) *$\partial_t \rho_N(s, t) = 0$ for all $t \in \mathcal{O}p(\partial[a, b])$, and $\rho_N(s, t) = (x = t, y = 0)$ for all $t \in \mathcal{O}p(a)$.*
(v) *$|z^{r'} \circ \rho_N| < \frac{1}{N}$ for all $0 < r' \leqslant r$,*

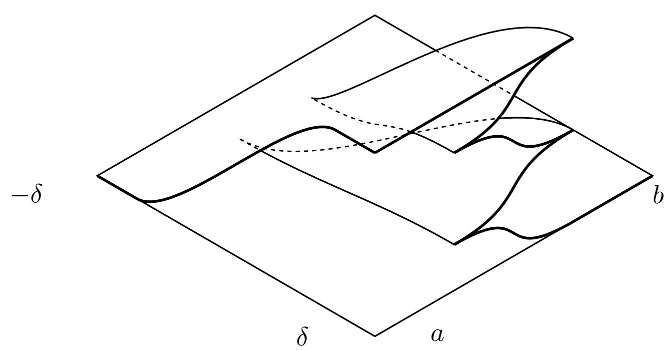

FIGURE 1. Schematic depiction of the graph of ρ_N for $N = 2$.

Conditions $(i) - (iii)$ imply that for each $N \in \mathbb{N}$ the function ρ_N defines an interpolation between the bump function of Definition 8.3, and the zero function. The last two conditions say that we can control the derivatives of the interpolation.

Proposition 8.10. *Asymptotically flat sequences of zig-zag bump functions with birth/death exists on any interval.*

Proof. Throughout the proof we think of ρ_N as a 1-parameter family of maps $\rho_{N,s} : [a,b] \to J^0([a,b],\mathbb{R})$. We first construct the bump function and show it satisfies conditions $(i)-(iv)$ of Definition 8.9. We treat the cases $N=1$ and $N>1$ separately, since the main idea is already contained in the $N=1$ case. Lastly we show that the derivatives can be controlled proving condition (v).

The case $N=1$. Consider the A_{2r}-swallowtail $\mathrm{Sw}_{2r} : \mathbb{R}^2 \to \mathbb{R}^3$, as in Definition 7.5, and restrict it to $[-\delta,\delta]\times[a,b]$. We identify this map with its image, which is a topologically embedded submanifold of $J^0(\mathbb{R}^2,\mathbb{R}) \simeq \mathbb{R}^3$. Modifying this submanifold we obtain a map interpolating between $\rho_1 : [a,b] \to J^0([a,b],\mathbb{R})$ from Definition 8.3, and the zero section.

In the coordinates (s,t), the map Sw_{2r} is given by

$$(s,t) \mapsto (s, -t^3 - st, \int_0^t (x^3 + sx - t^3 - st)^r \mathrm{d}x),$$

which we view as a 1-parameter family of maps $\mathrm{Sw}_{2r,s}$. It follows from this formula that

$$\mathrm{Sw}_{2r,\delta/2} : [a,b] \to J^0([a,b],\mathbb{R})$$

is a zig-zag conform Definition 7.14. Therefore, Proposition **??** provides an isotopy $\phi_s : [a,b] \to \mathrm{Diff}\left(J^0([a,b],\mathbb{R})\right)$, $s\in[\delta/2,\delta]$ satisfying:

(i) $\phi_\delta \circ \mathrm{Sw}_{2r,\delta} = \rho_1$ where ρ_1 is as in Definition 8.3 with $N=1$,
(ii) $\phi_s = \mathrm{id}$ for $s \in [\frac{3}{4}\delta,\delta]$ and $\phi_s = \phi_\delta$ for $s \in \mathcal{O}p(\delta)$,
(iii) $\phi_s \circ \mathrm{Sw}_{2r,s}$ is a zig-zag (Definition 7.14) for all $s \in [\delta/2,\delta]$.

Hence, the map defined by

$$\rho_{1,s} := \begin{cases} \mathrm{Sw}_{2r,s} & s \in [-\delta,\delta/2] \\ \phi_s \circ \mathrm{Sw}_{2r,s} & s \in [\delta/2\delta] \end{cases},$$

satisfies Conditions $(i),(ii)$ of Definition 8.9. A similar argument, modifying Sw_{2r} around $\{-\delta\}\times[a,b] \cup [-\delta,\delta]\times\partial[a,b]$, shows that conditions (iii) and (iv) can also be satisfied.

The case $N>1$. The proof is virtually the same as that for $N=1$. We choose N disjoint subintervals of $[a,b]$ by setting:

$$[a_\ell,b_\ell] := \left[\frac{2\ell-1}{2N+1},\frac{2\ell}{2N+1}\right], \quad \ell = 1,\dots,N.$$

We can assume that the restriction of the bump function $\rho_N : [a,b] \to J^0([a,b],\mathbb{R})$, constructed in Proposition 8.4, to the interval $[a_\ell,b_\ell]$ is equivalent to a zig-zag. Furthermore, the images of these restrictions are disjoint in $J^0([a,b],\mathbb{R})$, and ρ_N is graphical on the complement $[a,b]\backslash\bigcup_{\ell=1}^N[a_\ell,b_\ell]$.

Let $\mathrm{Sw}_{2r,N} : [-\delta,\delta]\times[a,b] \to J^0([-\delta,\delta]\times[a,b],\mathbb{R})$ be a multi-section such that:

(i) The restriction
$$\mathrm{Sw}_{2r,N}|_{[-\delta,\delta]\times[a_\ell,b_\ell]} : [-\delta,\delta]\times[a_\ell,b_\ell] \to J^0([-\delta,\delta]\times[a_\ell,b_\ell],\mathbb{R})$$
is equivalent to the A_{2r}-swallowtail $\mathrm{Sw}_{2r} : \mathbb{R}^2 \to \mathbb{R}^3$ from Definition 7.5.
(ii) $\mathrm{Sw}_{2r,N}$ is an honest section on the complement $[a,b]\backslash\bigcup_{\ell=1}^N[a_\ell,b_\ell]$.

Again, we think of this as a $[-\delta,\delta]$-family of multi-sections $\mathrm{Sw}_{2r,N,s} : [a,b] \to J^0([a,b],\mathbb{R})$.

By Proposition **??** we find isotopies $\phi_s : \mathcal{O}p\,[a,b] \to \mathrm{Diff}\left(J^0([a,b],\mathbb{R})\right)$, $s\in[\delta/2,\delta]$ satisfying:

(i) $\phi_\delta \circ \mathrm{Sw}_{2r,N,\delta} = \rho_N$ where ρ_N is as above,
(ii) $\phi_s = \phi_\delta$ for $s \in [\frac{3}{4}\delta,\delta]$ and $\phi_s = \mathrm{id}$ for $s \in \mathcal{O}p(\delta/2)$,
(iii) $\phi_s \circ \mathrm{Sw}_{2r,N,s}$ is a zig-zag (Definition 7.14) for all $s \in [\delta/2,\delta]$.

As before, this means that the map defined by

$$\rho_{N,s} := \begin{cases} \mathrm{Sw}_{2r,N,s} & s \in [-\delta, \delta/2] \\ \phi_s \circ \mathrm{Sw}_{2r,N,s} & s \in [\delta/2\delta] \end{cases},$$

satisfies Conditions $(i), (ii)$ and (iv) of Definition 8.9. Modifying Sw_{2r} around $\{-\delta\} \times [a,b] \cup [-\delta, \delta] \times \partial[a,b]$, shows that condition (iii) and (iv) can also be satisfied.

The flattening trick: It remains to show that $|z^{r'} \circ \rho_{N,s}| < \frac{1}{N}$ for all $0 < r' \leqslant r$. The above construction of $\rho_{N,s}$ depends on the (abstract) isotopy provided by Proposition **??**. Nevertheless, since the domain is compact, the derivatives of $\rho_{N,s}$ are bounded (although potentially very large). The key observation is that by "flattening "the graph of $\rho_{N,s}$ its derivatives can be made arbitrarily small.

For a small $\varepsilon > 0$, choose a strictly positive function $\lambda_\varepsilon : [-\delta, \delta] \to [0,1]$ satisfying:

(i) $\lambda_\varepsilon(s) = 1$ for $s \in \mathcal{O}p(\delta)$,
(ii) $0 < \lambda_\varepsilon(s) < \varepsilon$ for $s \in \mathcal{O}p([-\delta, \frac{3}{4}\delta])$.

This defines a "flattening diffeomorphism"

$$\psi_\varepsilon : [-\delta, \delta] \times J^0([a,b], \mathbb{R}) \to [-\delta, \delta] \times J^0([a,b], \mathbb{R})$$
$$(s, x, y) \mapsto (s, x, \lambda_\varepsilon(s) \cdot y).$$

Since ψ_ε is fibered over (s,x) it follows that:

$$z^{r'} \circ \psi_\varepsilon \circ \rho_{N,s} = \lambda_\varepsilon(s) \cdot (z^{r'} \circ \rho_{N,s}), \quad \forall 0 < r' \leqslant r.$$

The construction of $\rho_{N,s}$ shows that $z^{r'} \circ \rho_{N,s}$ is bounded for all s. Indeed, this follows immediately from the compactness of its domain and continuity of $\rho_{N,s}$. Therefore, choosing ε sufficiciently small the claim follows.

□

8.5.1. *Patching tears.* Proposition 8.10 provides a tool for gluing multisections in the foliated setting, analogous to Corollary 8.5.

Corollary 8.11. *Let $\varepsilon, \delta > 0$ be given and consider the foliated cylinder*

$$\Big(\mathbb{D}^k \times \mathbb{D}^{n-k}, \mathcal{F} := \bigcup_{x \in D^k} \{x\} \times \mathbb{D}^{n-k}\Big).$$

Given two sections $s_0, s_1 : D^n \to \underline{\mathbb{R}}^k$ there exist a foliated section with zig-zags (Definition 8.7) $s : \mathbb{D}^n \to J^r(\underline{\mathbb{R}}^k \to (\mathbb{D}^n, \mathcal{F}))$ satisfying:

(i) $(\pi_f \circ s)|_{\mathbb{D}^k_{1-\delta} \times \mathbb{D}^{n-k}_{1-\delta}} = s_1$;
(ii) $(\pi_f \circ s)|_{\mathcal{O}p(\partial \mathbb{D}^k \times \mathbb{D}^{n-k} \cup \mathbb{D}^k \times \partial \mathbb{D}^{n-k})} = s_0$;
(iii) $|J^r s_0 - s| < 4\varepsilon$.

Proof. The proof follows closely the argument of Proposition 8.4, but replacing the zig-zag bump fucntions from Proposition 8.4 by the zig-zag bump functions with birth/death from Proposition 8.10. The first parts of the proof of Proposition 8.4 go through word for word. Thus, after applying the "pushing trick" and the "first and second simplification" we can assume to be in the following situation:

- We have two sections $s_0, s_1 : \mathbb{D}^k \times \mathbb{D}^{n-k} \to J^r((\mathbb{D}^k \times \mathbb{D}^{n-k}, \mathcal{F}), \mathbb{R})$,
- s_0 is the zero-section and $s_1 = j^r 1$, is the lift of the constant function equal to one.

We identify these sections with their front projections.

Matching along the horizontal boundary:

The boundary of the cylinder splits into two parts:

$$\partial(\mathbb{D}^k \times \mathbb{D}^{n-k}) = \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \cup \mathbb{D}^k \times \mathbb{S}^{n-k-1}.$$

We refer to the first and second component as the vertical and horizontal boundary respectively.

We want to match s_0 and $\tilde{s}_1$ along the horizontal boundary while keeping control on the derivatives. In order to do this we introduce singularities. In the spirit of Proposition 8.4 we construct a suitable bump function "interpolating" between the constant functons s_0 and s_1.

Given a small $\delta' > 0$, consider the leafwise thickening of the horizontal boundary

$$T := \mathbb{D}^k \times [1-\delta, 1] \times \mathbb{S}^{n-k-1},$$

as in Figure 8.5.1.

A neighborhood of the corner (i.e. the boundary of the horizontal boundary) $\mathbb{S}^{k-1} \times \mathbb{S}^{n-k-1}$ is contained in the region where $\tilde{s}_1 = 0$. This will be important since our bump function will have birth/death events along $\mathbb{S}^{k-1} \times \mathbb{S}^{n-k-1}$.

We decompose the thickening into two regions:

$$T = [1-\delta, 1] \times \mathbb{S}^{k-1} \times [1-\delta, 1] \times \mathbb{S}^{n-k-1} \bigcup \mathbb{D}^{k-1}_{1-\delta} \times [1-\delta, 1] \times \mathbb{S}^{n-k-1}. \tag{8.5.1}$$

Now we define our bump function $\rho_N : T \to J^0((T, \mathcal{F}) \to \mathbb{R})$ as follows. On the first component we set ρ_N to be the $(\mathbb{S}^{k-1} \times \mathbb{S}^{n-k-1})$-stabilization of the bump function on $[1-\delta, 1] \times [1-\delta, 1]$ as constructed in Proposition 8.10.

Note that the restriction of ρ_N to $\{1-\delta\} \times [1-\delta, 1] \times \mathbb{S}^{k-1} \times \mathbb{S}^{n-k-1}$ equals the bumpfunction constructed in Proposition 8.4. Hence, we can smoothly extend ρ_N over the second component as the $(\mathbb{D}^k_{1-\delta} \times \mathbb{S}^{n-k-1})$-stabilization of the bump function from Proposition 8.4.

If $\delta > 0$ is chosen sufficiently small, then the first component of the decomposition in 8.5.1 is contained in the region where $\tilde{s}_1 = 0$. This implies, together with the "Checking the claimed properties" argument in the proof of Proposition 8.4 shows, that ρ_N defines a foliated section with zig-zags (Definition 8.7) satisfying the required conditions.

Matching along the vertical boundary: There are no restrictions on the derivatives of s in the parameter direction. Therefore any interpolation between s_0 and s_1 in the parameter direction suffices.

A thickening of (part of) the vertical boundary

$$V := [1-\delta] \times \mathbb{S}^{k-1} \times \mathbb{D}^{n-k}_{1-\delta},$$

intersects the thickening of Equation 8.5.1 in the region

$$V \cap T = [1-\delta] \times \mathbb{S}^{k-1} \times \mathbb{S}^{n-k-1}.$$

It follows from the way we defined ρ_N above that

$$\rho_N|_{V \cap T} = (\mathbb{S}^{k-1} \times \mathbb{S}^{n-k-1})\text{-stabilization of } \widetilde{\rho}_N|_{[1-\delta,1] \times \{1-\delta\}},$$

where $\widetilde{\rho}_N : [-\delta, \delta] \times [1-\delta, 1] \to J^0([1-\delta, 1], R)$ is constructed in Proposition 8.10. Thus, this restriction is a smooth function (without singularities) interpolating between s_0 and s_1. By extending it over V as the $\mathbb{S}^{k-1} \times \mathbb{D}^{n-k}$ stabilization of $\widetilde{\rho}_N$ defines we obtain an interpolation along the vertical boundary.

□

8.6. **Resolving singularities when r is odd.** The zig-zag bump function ρ with birth/death (Definition 8.9) allows us to interpolate between multisections with zig-zag singularities and non-singular sections. When r (the order of the jet-space we are working in) is even, the holonomic lift $j^r\rho$ defines an embedding into the jet bundle. Indeed, the metasymplectic projection (Section **??**) of the lift is given by the birth death of a double fold, see Section 4.4.

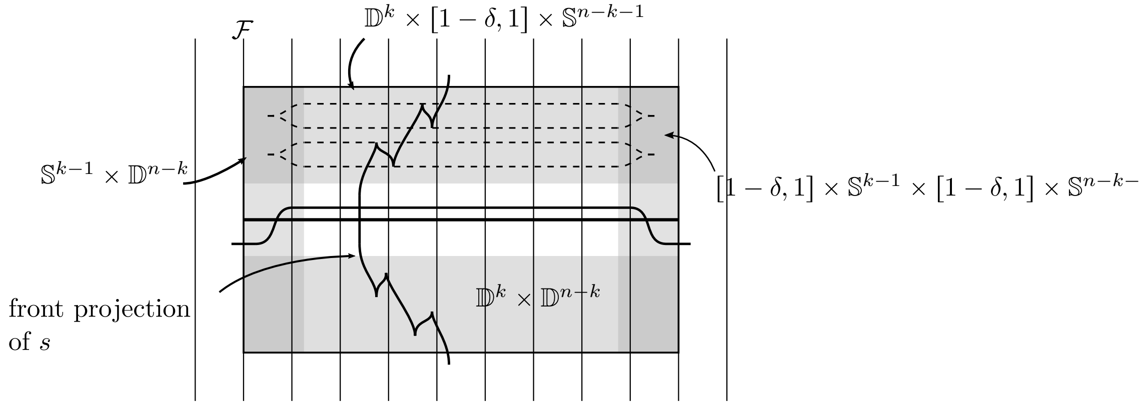

FIGURE 2. Schematic depiction of the graph of s together with the relevant regions in its domain. Note that the vertical and horizontal boundaries are connected if $k > 1$.

When r is odd, the situation is more complicated. Although a zig-zag still lifts to a smooth map (an immersion), its birth/death is singular. The reason for this is the open/closed switching from Section 7.2.6. The front projection of a zigzag is a pair of A_{2r+1} singularities in an open configuration. Hence, since r is odd, its metasymplectic projection is in closed configuration. As the following lemma shows this forces the birth/death to be singular.

This side comment needs to appear in the section on singularities

Lemma 8.12. *If r is odd, then the metasymplectic projection of a zig-zag cannot be homotoped, through immersions, to an embedding.*

Remark 8.13. *Since all of our singularities are stabilizations of the 1-dimensional zigzag the previous lemma suffices. For the general case we argue using the Maslov class? Is it clear that indepdently of the topology of the singular locus the zigzag can never be homotoped to the identity through immersions?*

Proof. Winding number ☐

To work around this problem we will change the singularities by a surgery to obtain a singularity whose birth/death is smooth. Before discussing the details let us first describe the birth/death of the resulting singularity.

8.6.1. *Birth/death of double folds.* Consider the trivial bundle $\pi : \underline{\mathbb{R}} \to \mathbb{R}$ which we think of as the front projection of $J^r(\underline{\mathbb{R}} \to \mathbb{R})$. We describe a 1-parameter family of multi-sections f_t, $t \in [0,1]$, (identified with their front projection) connecting the constant section $f_0 = 0$ and a multisection f_1 with two A_{2r+1}-folds. The family f_t is illustrated in Figure 8.6.1.

- Starting from the constant section f_0 we first add a stabilization (Section **??**). We denote by r_+ and r_- the right and left singularity respectively.
- Using an isotopy supported around r_-, we move r_- above the left branch of f_t. Note that this can be done is such a way that all the self-intersections of (the front of) f_t are transverse.
- Again using a locally supported isotopy, we move the right branch completely above the other branches. In this proces all the self-intersections are transverse, except for a single time where the right and middle branch intersect tangentially. At the tangential intersection points the curvature of the two branches can be chosen differently.

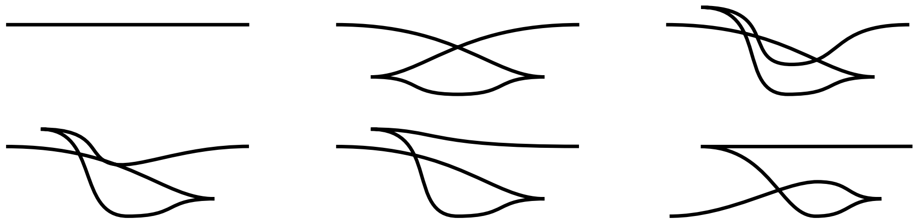

FIGURE 3. The 1-parameter family of multi-sections described in Proposition 8.14.

The self-intersections created in the homotopy above resolve when lifting to $J^r(\underline{\mathbb{R}} \to \mathbb{R})$ provided r is at least 2. Thus we obtain an embedding into the total space. On the other hand, for $r = 1$ it is known [] that it is impossible to remove self intersections. The downside to this construction is that the front projections of the resulting multi-sections are no longer topological embeddings (in contrast with the case when r is even).

Proposition 8.14. *If $r > 2$ the family of sections f_t, $t \in [0,1]$, described above lifts to a family of (embedded) multi-sections of $J^r(\underline{\mathbb{R}} \to \mathbb{R})$. Furthermore, for any $\varepsilon > 0$ we can arrange that $|j^r f_t|_{C^0} < \varepsilon$.*

Proof. The birth/death of a stabilization lifts to a smooth map in $J^r(\underline{\mathbb{R}} \to \mathbb{R})$, as shown in Section **??**. Moreover, the stabilization is a multi-section with two A_{2r+1} singularities. Thus, the above family lifts to a smooth family of maps.

Whenever two branches intersect, either their tangent spaces differ (transverse intersection) or their curvatures differ (non-transverse intersection). Hence, their lifts are disjoint and we obtain an embedded multi-section.

The domain of f_t and the parameter space of the family are compact. Hence the lift $j^r f_t$ is bounded. Denote by $\mu_C : \underline{\mathbb{R}} \to \underline{\mathbb{R}}$ the fiberwise multiplication by the constant $C \in \mathbb{R}$. Then for $C > 0$ sufficiently small (depending on ε) the composition $\mu_C \circ f_t$ satisfies the required properties. □

Remark 8.15. *In the above proposition we need $r > 2$ in order for the self-intersections to resolve. If the dimension of the fiber is $\geqslant 2$, that is if we consider sections of $\underline{\mathbb{R}}^k \to \mathbb{R}$ for $k \geqslant 2$, this condition is not necessary. Indeed, in this case the we can "push one branch in the extra direction" around each self-intersection. Also observe that in this case the front projection is a topological embedding.*

8.6.2. *Surgery.* The main purpose of our surgery result stated below is to use it in the proof of Theorem 8.8. Therefore, although the argument holds more generally, we state a somewhat specialized version which can be applied immediately.

Proposition 8.16. *For $r \geqslant 0$ let $f : M \to J^{2r+1}(\underline{\mathbb{R}} \to M)$ a foliated section with zigzags conform Definition 8.7. Then, for any $\varepsilon > 0$, there exists an embedded multi-section $\widetilde{f} : M \to J^{2r}(\underline{\mathbb{R}} \to M)$ satisfying:*

(i) $\Sigma(f) = \Sigma(\widetilde{f})$ *as subsets of M. However, the type of singularities differ.*
(ii) *The singularities of $\widetilde{f}$ consist of folds and birth/deaths of folds.*
(iii) $|\widetilde{f} - f|_{C^0} < \varepsilon$ *(inside $J^r(X \to M)$) and $\widetilde{f} = f$ away from the singularity locus.*

Proof. According to Definition 8.7 the singularities of f are stabilizations of wrinkly zig-zags. Since $2r+1$ is odd this means that they are regularized wrinkled (see Definition 7.24 and Definition **??**). Throughout the proof we identify these (multi-)sections with their front projections, as usual.

For our construction it is useful to think of a wrinkle along $\mathbb{S}^{n-1}$ as a pair of folds along the upper and lower hemisphere $D^{n-1}_{\pm}$ which come together and die along the equator $\mathbb{S}^{n-2}$. By applying a surgery we will change the wrinkle so that "the birth/death movie" matches that of Proposition 8.14.

We start by decomposing a neighborhood of the singular locus $\mathcal{O}p(\mathbb{S}^{n-1})$ of the wrinkle in the following way.

A neighborhood of the equator can be identified with $\mathbb{S}^{n-2}\times\mathbb{R}^2$. Let (t,x) denote coordinates on $\mathbb{R}^2$, then the intersection of the singular locus $\mathbb{S}^{n-1}$ is given by $\mathbb{S}^{n-2}\times\{x=t^2\}$. In particular, the equator is given by $\mathbb{S}^{n-2}\times\{0\}$. On this neighborhood the wrinkle equals the $\mathbb{S}^{n-2}$-stabilization of a swallowtail $\mathrm{Sw}_{4r+2}:\mathbb{R}^2\to\mathbb{R}^3$ (Definition 7.5).

We can then simply replace the graph of Sw_{4r+2} by the one from Proposition 8.14, as illustrated in Figure 8.6.2. Note that the graphs match along $\{t=1\}\cup\{x=\pm1\}$ and hence on these sides the new graph can be smoothly continued as f. Then, since the upper and lower hemisphere are disjoint, the new graph can be extended over $\mathbb{S}^{n-1}$. Thus, around the upper and lower hemisphere $\widetilde{f}$ equals a D^{n-1}-stabilization of an A_{4r+1}-cusp (Definition 7.4).

It remains to check that the resulting map satisfies the required properties. The first two properties follow immediately from the construction. For the third property observe that the surgery can be performed on an arbitrary small neighborhood of the equator $\mathbb{S}^{n-2}$. Here the lift of the original section is arbitrarily small. By Proposition 8.14 the same holds for the new map. Similarly the change of the fold locus can be performed on an arbitray small neighborhood of the upper hemisphere. Hence the lifts of f and $\widetilde{f}$ are C^0-close. ☐

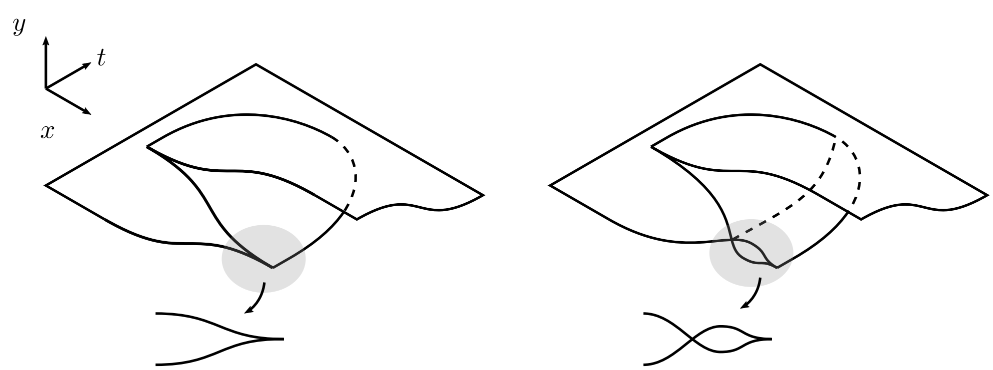

FIGURE 4. On the left: the swallowtail Sw_{4r+2} around the equator $\mathbb{S}^{n-2}$. On the right: the result after applying surgery.

8.7. **Proof of Theorem 8.8.** The main difference with the non-parametric proof originate from the fact that when r is odd the birth/death of a zig-zag does not lift to a (smooth, embedded) multi-section. Therefore, when r is odd, we apply apply the proof for $r+1$ which is even. Then, in the last step we use the surgery construction of Proposition 8.16 to obtain a multi-section of the r-th Jet bundle. When r is even, proof is mostly the same as that of Theorem 8.2 and so we only point out the differences.

Firstly, now we are in the foliated setting. Hence, when we choose a triangulation $\mathcal{T}$ of X we need to ensure it is compatible with the foliation $\mathcal{F}$. This follows from Theorem 5.5, stating that the triangulation can be chosen in general position with respect to $\mathcal{F}$.

In the reduction step of the proof we apply Theorem 5.2 wiggeling the codimension-one skeleton. This results in a new trianguliation of X, and we want it to be in general position with respect to $\mathcal{F}$. To this end we choose a vector field $V \in \mathcal{O}p(\mathcal{T}^{n-1})$ tangen to the leaves of $\mathcal{F}$. Then (as stated in Theorem 5.2) we can arrange that the wiggled skeleton stays transverse to V (and thus to $\mathcal{F}$).

Next, we apply the extension step, using Corollary 8.11 instead of Corollary 8.5. Each top dimensional cell Δ of $\widetilde{\mathcal{T}}$ is tangent to $\mathcal{F}$ along a sphere $\mathbb{S}^{k-1}$, see Figure 8.7. It has a small neighborhood (inside $\mathbb{D}^n$) diffeomorphic to the upper half ball:

$$\mathbb{D}^n_+ := \{x \in \mathbb{D}^n \mid x_n \geqslant 0\},$$

and such that the leaves of $\mathcal{F}$ are tangent to the bottom boundary

$$\partial_-\mathbb{D}^n_+ := \{x \in \mathbb{D}^n_+ \mid x_n = 0\}.$$

Hence, removing this neighborhood from $\mathbb{D}^n$ we obtain a cylinder $\mathbb{D}^k \times \mathbb{D}^{n-k}$ satisfying the conditions of Corollary 8.11. Thus we can extend the section around the $(n-1)$-skeleton over each of the top dimensional cells.

If r is even this concludes the proof. In the odd case (having applied the previous steps with $r+1$) it remains to apply Proposition 8.16.

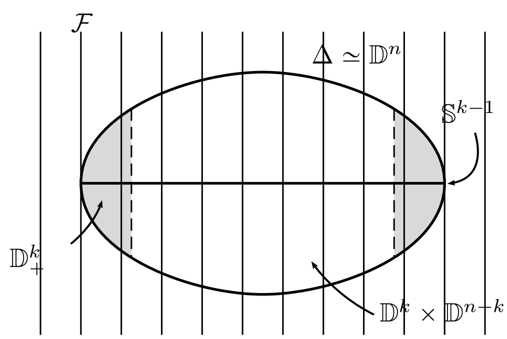

FIGURE 5. Top dimensional cell Δ of $\widetilde{\mathcal{T}}$.

9. HIGHER WRINKLED EMBEDDINGS

As an application of Theorem 8.2 we prove in this section a generalization of the h-principle for wrinkled embeddings from [21].

At the end of the Section we explain various surgery results for wrinkles, including the passage from wrinkles to double folds and viceversa.

9.1. **Statement of the result.** Two submanifolds $N_1, N_2 \subset M$ have the same r-jet at $x \in N_1 \cap N_2$ if N_2 is graphical over N_1 (around x) and the induced section of the normal bundle vanishes up to order r, see Section 3.4.2. As before we denote by $J^r(M, n)$ the space of r-jets of submanifolds of dimension n in M.

Theorem 9.1. *Let the following data be given:*

- *Two smooth manifolds N and M of dimensions n and m, respectively.*

- *A connected, compact manifold pair* (K, K'), *with* K' *possibly empty, playing the role of the parameter space.*
- *A family of smooth embeddings* $f_k : M \to N$, *with* $k \in K$.
- *A family* $F_{k,s}$ *of lifts of* f_k*:*

$$\begin{array}{ccc} & & J^r(N,m) \\ & \overset{F_{k,s}}{\nearrow} & \downarrow \pi \\ M & \xrightarrow{f_k} & N \end{array} \qquad (k,s) \in K \times [0,1]$$

with $F_{k,s} = j^r f_k$ *whenever* $k \in K'$ *or* $s = 0$.
- *A constant* $\varepsilon > 0$.

Then there exists a family of A_r*-folded embeddings* $f_{k,s} : M \to N$ *parametrized by* $(k,s) \in K \times [0,1]$, *possibly containing embryos, and satisfying:*

- $f_{k,s} = f_k$ *whenever* $k \in K'$ *or* $s = 0$.
- $|f_{k,s} - f_k|_{C^0} < \varepsilon$.
- $|j^r f_{k,s} - F_{k,s}|_{C^0} < \varepsilon$.

We need to add a proof of the following two statements, since they are used in the proof of the h-principle for integral submanifolds modelled on jet space.

- If $P \subset M$ is a polyhedron of positive codimension, then the fold locus of $f_{k,s}$ can be assumed to be disjoint from P.
- Given a line bundle $V \subset TN$ we can arrange that the folds of $f_{k,s}$ are "along V". That is, in the local model of the folds, V can be chosen as the vertical direction. Also point out that such line bundles need to be chosen locally on the top dimensional cells of a triangulation, so there is no obstruction to their existence.

By surgery of the singularities we can pass from folded maps to wrinkled maps, see Section **??**. This immediately implies the following:

Proposition 9.2. *Given the same data as in Theorem 9.1, there exists a family of* A_r*-wrinkled embeddings* $f_{k,s} : M \to N$ *parametrized by* $(k,s) \in K \times [0,1]$, *possibly containing embryos, and satisfying:*

- $f_{k,s} = f_k$ *whenever* $k \in K'$ *or* $s = 0$.
- $|f_{k,s} - f_k|_{C^0} < \varepsilon$.
- $|j^r f_{k,s} - F_{k,s}|_{C^0} < \varepsilon$.

Should add the parametric arguments

9.2. **Reduction to sections.** Consider an embedded submanifold $f : M \to N$. The normal bundle $\pi : \mathcal{N} \to M$ provides the structure of a fibration on an open neighborhood, also denoted by $\mathcal{N}$, of M. In turn this gives an embedding

$$J^r(\mathcal{N} \to M) \hookrightarrow J^r(\mathcal{N}, m) \subset J^r(N, m). \tag{9.2.1}$$

If the homotopy F_s in Theorem 9.1 is sufficiently small (in the C^0-norm on $J^r(N,m)$) then we can interpret it as a section of $J^r(\mathcal{N} \to M)$.

For the proof of Theorem 9.1 we will also need a fibration structure around folded embeddings (Definition **??**). Let $f : M \to N$ be a folded embedding, which we identify with its image. Recall that the singular locus $\Sigma := \Sigma(f)$ consists of a disjoint union of codimension one spheres. Even though f has singularities, it has a well-defined Gauss map (Remark **??**) $\mathrm{Gr}(f) : M \to \mathrm{Gr}(TN, m)$.

Given a thickening of the singular locus:

$$\mathcal{S} := \mathcal{O}p(\Sigma) \simeq (-1,1) \times \Sigma \subset M,$$

we can find an embedding $g : \mathcal{S} \hookrightarrow N$, such that $j^1_x g = j^1_x f$ for all points $x \in \Sigma$.

Similarly, for any top-dimensional cell $U_i \in \mathcal{T}^m$ we find a thickening

$$\mathcal{U}_i := \mathcal{O}p(U_i) \simeq U_i \cup_{\partial U_i} [0,1) \times \partial U_i \subset M,$$

and an embedding $g_i : \mathcal{U}_i \to N$ extending $f|_{\text{int}(U_i)}$.

Both $\mathcal{U}_i$ and $\mathcal{V}$ are embedded submanifolds. We choose normal bundles, denoted by $\mathcal{N}_i \to \mathcal{U}_i$ and $\mathcal{N} \to \mathcal{V}$ respectively, such that their restriction to intersection $\mathcal{U}_i \cap \mathcal{V}$ agrees. This provides the structure of a fibration as in Equation 9.2.1.

9.3. **The graphical case.** In this section we prove a special case of Theorem 9.1. Let $f : M \to N$ be a wrinkled embedding and $F_s : M \to J^r(N,m)$, $s \in [0,1]$ a lift of f such that

$$|j^r f - F_0|_{C^0} < \varepsilon.$$

Choose $\mathcal{N}_i \to \mathcal{U}_i$ and $\mathcal{N} \to \mathcal{S}$ as in the previous section. We assume that the image of F_s is contained in the image of the induced coordinates, see Equation 9.2.1. Thus, we only have to work with jet spaces of fibrations and we can think of F_s as a family of sections. We show how to extend f to a family of wrinkled embeddings $f_s : M \to N$ satisfying

$$|j^r f_s - F_s|_{C^0} < \varepsilon.$$

9.3.1. *Holonomic approximation around the folds.* Using the bundle $\mathcal{N} \to \mathcal{S}$ we interpret $f|_{\mathcal{S}}$ as a section of $\mathcal{N}$ and the family $F_s|_{\mathcal{S}}$, $s \in [0,1]$, as sections of $J^r(\mathcal{N} \to \mathcal{S})$. Since $\Sigma \subset \mathcal{S}$ has positive codimension we can apply Theorem 5.2 to find:

- a family of isotopies $\phi_{s,t} : \mathcal{S} \to \mathcal{S}$, $s,t \in [0,1]$,
- a family of sections $\sigma_s : \mathcal{S} \to J^r(\mathcal{N} \to \mathcal{S})$,

satisfying:

- σ_0 is the zero-section corresponding to $j^r g$,
- σ_s is holonomic on $\mathcal{O}p(\widetilde{\Sigma}_s)$ where $\widetilde{\Sigma}_s := \phi_{s,1}(\Sigma)$,
- $|\sigma_s - F_s|_{C^0} < \varepsilon$.

This data induces an isotopy in the front projection of $J^r(\mathcal{N} \to \mathcal{S})$ which we use to translate the folds of f as follows. Choose a trivialization $\mathcal{N} \simeq \mathcal{S} \times \mathbb{R}^k$ such that g corresponds to the zero-section, and an isomorphism $\mathcal{O}p(\Sigma) \simeq (-1,1) \times \Sigma$ inducing coordinates (r,x). We define

$$\psi_{s,t} : \mathcal{O}p(\Sigma) \times \mathbb{R}^k \to \mathcal{S} \times \mathbb{R}^k$$

$$(r,x,y) \mapsto \left(\phi_{s,t}(x) + r, y + \sigma_s\left(\phi_{s,1}^{-1}(x)\right)\right).$$

To extend $\psi_{s,t}$ to a global isotopy choose a family of bump functions

$$\tau_s : \mathcal{S} \to [0,1], \quad s \in [0,1], \tag{9.3.1}$$

supported in $\mathcal{S}$ and satisfying $\tau_s|_{\mathcal{O}p(\tilde{\Sigma}_s)} = 1$. Then the extension is given by $\psi_s : \mathbb{N} \to \mathbb{N}$, $x \mapsto \psi_{s,\tau_s(x)}(x)$, see Figure **??**. Finally, $f_s := \phi_s \circ f$ defins the desired translation of the fold locus.

Since ψ_s is supported in $\mathcal{O}p(\Sigma)$ it follow that $f_s = f$ away from Σ. We claim that the fold locus of f_s "follows F_s as s varies". More precisely we have $\Sigma(f_s) = \Sigma(f)$ and

$$|j^r f_s - F_s|_{C^0} < \varepsilon \quad \text{on } \mathcal{O}p(\Sigma).$$

To see this note that $j^r(f_s|_\Sigma) = \sigma_s|_\Sigma$ implying that the above inequality holds when restricted to Σ. Furthermore, recall that the folds of f are equal to the Σ-stabilization of the 1-dimensional fold whose r-jet is C^0-close to that of the zero function. Together with the fact that ϕ_s is linear in the r and y coordinates (as in Equation 9.3.1) on an open neighborhood of Σ this implies the claim

9.3.2. *Extending over M.* To extend the solution f_s on $\mathcal{O}p(\Sigma)$ defined above to the whole M we want to apply Theorem 8.8 relative to Σ. This cannot be done immediately since $j^r f_s$ does not agree (or is close to) F_s away from Σ. We solve this problem by interpolating between $j^r f_s$ and F_s while keeping the front projection fixed.

Choosing a trivialization of $\mathcal{N} \to \mathcal{S}$ induces the usual coordinates $(x, y, z) \in J^r(\mathcal{N} \to \mathcal{S})$ where $z := (z^1, \dots, z^r)$ denotes the collection of coordinates corresponding to formal derivatives (see Definition 3.1). Then, using the bumpfunction τ_s from Equation 9.3.1 we define a (formal) multi-section $\widetilde{f}_s : \mathcal{S} \to J^r(\mathcal{N} \to \mathcal{S})$ by defining its front projection to be equal to f_s and

$$z \circ \widetilde{f}_s := \tau_s(x) j^r f_s(x) + (1 - \tau_s(x)) F_s(x).$$

Near the boundary of $\mathcal{S}$ this map is equal to F_s and so we can extend it to a map $\widetilde{f}_s : M \to J^r(N, m)$.

On each connected component of $M\backslash$ the map $\widetilde{f}_s$ can be interpreted as an honest section of $\mathcal{N} \to (M \backslash \Sigma)$ which is holonomic near the boundary. Hence applying Theorem 8.8 relative to the boundary yields the desired extension.

9.4. **Proof of Theorem 9.1.** In the non-parametric case (i.e. when K is a point) the proof follows from inductively applying the graphical case from the previous section. More precisely assume that the required family f_s has been constructed for $s \in [0, s_0] \subset [0, 1]$. Then, for sufficiently small $s_1 > s_0$, the pair (f_{s_0}, F_s) for $s \in [s_0, s_1]$, satisfies the assumptions of Section **??**. Hence, we can extend f_s to $[0, s_1]$.

The proof of the parametric case follows exactly the same argument. Indeed both the classical and the wrinkled holonomic approximation, Theorem 5.2 and Theorem 8.8 hold parametrically.

9.5. **Modifying singularities.** In this section we explain two surgery constructions on the singularities of maps into jet space. The first construction allows us to pass between (double) fold and wrinkle type singularities. Recall that singular locus of a wrinkle is a sphere whose hemisphere consists of birth death Secondly we show how a wrinkle can be replaced by many smaller ones. Together with the previous construction this also allows us to replace double folds by smaller ones.
The value of these surgeries as technical lemmas, comes from the fact that they are C^0-small in nature. By this we mean that the difference between original and the modified map is arbitarily small in the C^0-norm on maps to jet space. Thus these results can be applied at will simplifying many arguments. Moreover, they also serve as an illustration of the flexibility of wrinkle and fold singularities.

Most of the singularities we have encountered so far are fibered in nature. They are essentially just higher parametric versions of (double) folds with birth/deaths. For example, the standard wrinkle (Definition 4.3) is fibered over its membrane D (isomorphic to a disk $\mathbb{D}^n$). Thus we can think of a wrinkle as a family of double folds indexed by D, and with birth/deaths at ∂D. It is straightforward to generalize this, allowing the membrane to be any domain.

Definition 9.3. *Let M be a manifold and D a hypersurface (possibly with smooth boundary). Assume that D admits a product neighborhood $\mathbb{D} \times \mathbb{R}$ with coordinates (q, x). Then we define a A_{2r}-**wrinkle along** D (or just a D-wrinkle) as the map*

$$w_D : D \times \mathbb{R} \to D \times \mathbb{R}^2$$
$$(q, x) \mapsto (q, -x^3 - \rho(q)x, \int_0^x (s^3 + \rho(q)s - x^3 - \rho(q)x)^r ds)$$

where $\rho : D \to \mathbb{R}$ is a distance function to the boundary ∂D. If $\partial D = \varnothing$ then we take $\rho = 1$ everywhere.

It is instructive to compare this definition with Definition 7.5. Although the precise definition (and its singular locus) depends on the choice of distance function and product neighborhood, different choices yield equivalent maps.

If $\partial D \neq \varnothing$ then the singular locus of W_D is isomorphic to the double of D. More precisely,

$$\Sigma(W_D) = \Sigma^1(W_D) = D \cup_{\partial D} D,$$

where ∂D (given by the boundary of the membrane) consists of birth/death points. The other points in the singular locus are folds. If $\partial D = \emptyset$ then the singular locus is the disjoint union of two (parallel) copies of D. In this case $\Sigma(W_D)$ consists of fold points only. For example, taking $D = \mathbb{D}^{n-1}$ gives Definition 4.3, while taking $D = \mathbb{S}^{n-1}$ recovers Definition 7.7. There are several surgery operations we can perform to change the properties of (the singular locus of) an A_{2r}-wrinkle.

9.5.1. *Moving the singular locus.* The most elementary operation consists of changing the singular locus of a given map. As we are usually considering (front projections of) maps into jet space of order r, we want the difference between the initial and final map to be C^r-small.

To explain the construction in its most elementary form consider the map

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad (q, x) \mapsto (q, x^2). \tag{9.5.1}$$

It has fold singularities along its singular locus $\Sigma(f) = \{x = 0\}$. Let $\gamma : [-1, 1] \to \mathbb{R}^2$ be an embedded curve such that $\gamma(\pm 1) = (\pm 1, 0)$. Suppose we want to change the singular locus to

$$\widetilde{\Sigma} := (\Sigma(f)\backslash\{0\} \times [-1, 1]) \cup \gamma([-1, 1]).$$

Although this can easily be arranged by precomposing f with a suitable diffeomorphism mapping Σ to $\widetilde{\Sigma}$ this changes f in a way that is not C^r-small. Instead, by adding more singularities we can achieve a C^r-small change.

Lemma 9.4. *We use the same notation as above. For any $\varepsilon > 0$ there exists a map $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$, depicted in Figure 9.5.1, with the following properties:*

(i) The singular locus of $\widetilde{f}$ has two connected components:

$$\Sigma(\widetilde{f}) = (\Sigma(f)\backslash\{0\} \times [-1, 1] \cup \gamma([-1, 1])) \sqcup \mathbb{S}^1.$$

Furthermore, it consists only of fold and birth/death points.

(ii) $\widetilde{f} - f|_{C^r} < \varepsilon$.

Additionally it is not hard to see that the above maps f and $\widetilde{f}$ are homotopic.

Proof. Consider the function $\widetilde{f}$ whose graph is depicted in Figure 9.5.1. □

The general case is similar. The map $\widetilde{f}$ can be interpreted as a 1-parameter family of maps $\widetilde{f}_q : \mathbb{R} \to \mathbb{R}$. Hence it suffices to understand how to deal with more general parameter spaces. The key point is that on a collar neighborhood of the boundary $\partial D \times [0, 1)$ the general case is just a stabilization of the 1-dimensional case. The statement is as follows:

Lemma 9.5. *Let $f : M \to J^r(X \to M)$ be a multisection, and assume we have the following data:*

(i) A domain $D \subset \Sigma^{10}(f)$, possibly with non-empty (smooth) boundary;

(ii) A product neighborhood $D \times \mathbb{R} \subset M$ with coordinates (q, x), such that $D = D \times \{0\}$ and f equals:

$$f : D \times \mathbb{R} \to D \times \mathbb{R}^{k+1}$$

$$(q, x) \mapsto (q, x^2, x^{2r+1}, 0, \dots, 0).$$

(iii) A smooth function $\Delta : D \to \mathbb{R}$ satisfying $j^\infty(\Delta)|_{\partial D} = 0$. We denote its graph by $\widetilde{D} \subset D \times \mathbb{R}$.

Then for any $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists a multi-section $\widetilde{f} : M \to J^r(X \to M)$ satisfying:

(i) $|j^r\widetilde{f} - j^r f|_{C^0} < \varepsilon$

*(ii) The singular locus $\Sigma(\widetilde{f})$ is obtained from $\Sigma(f)$ by cutting out D (from the fold locus), gluing in $\widetilde{D}$, and adding a $\widetilde{D}$-wrinkle (Definition **??**). That is:*

$$\Sigma(\widetilde{f}) = (\Sigma(f)\backslash D) \cup_{\partial \widetilde{D}} \widetilde{D} \sqcup \left(\widetilde{D} \cup_{\partial \widetilde{D}} \widetilde{D}\right).$$

Note that by definition of a wrinkle, a product neighborhood $D \times \mathbb{R}$ as above always exists. However, in general it is not true that any hypersurface $\widetilde{D}$ graphical over $\Sigma(f)$ is contained in such a neighborhood. The same arguments also allow for vertical singularities/singularities of mapping?

Proof. By assumption we may assume that

$$f : D \times \mathbb{R} \to D \times \mathbb{R}^{k+1}$$

$$(q, x) \mapsto (q, x^2, x^{2r+1}, 0, \dots, 0).$$

Forgetting the coordinates which are constant, f is the D-stabilization of the map

$$\hat{f} : \mathbb{R} \to \mathbb{R}^2, \quad x \mapsto (x^2, x^{2r+1}).$$

Then, by another change of coordinates (preserving f), we may assume that there exists a collar neighborhood

$$\partial D \times [0, 1) \subset D, \tag{9.5.2}$$

and that Δ satisies the following properties:

(i) On the complement of the collar Δ is constant and equal to 1.
(ii) On the collar Δ depends only on the interval direction, and the induced function $\Delta_\partial : [0, 1) \to \mathbb{R}$ is strictly increasing.

To construct the required map we use the 1-parameter family of maps $F : [0, 1] \times \mathbb{R} \to \mathbb{R}^2$, depicted in Figure 9.5.1.

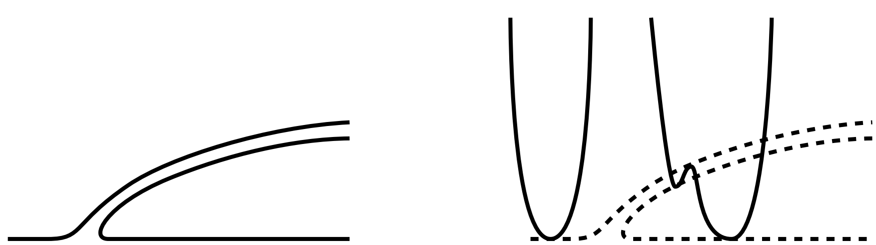

FIGURE 6.

It has the following properties:

(i) The upper branch is given by the graph of Δ_∂, the lower branch is contained in $\mathbb{D} \times \{0\}$ and the middle branch is arbitrary close to the upper branch.
(ii) Along the upper branch F has A_{2r}-cusps (Definition 7.4). In particular $F_t := F|_{\{t\} \times \mathbb{R}}$ is equal to f for $t \in \mathcal{O}p(0)$.
(iii) Along the two bottom branches (and the domain between them) the map

$$F : \mathbb{R}^2 \to \mathbb{R}^3, \quad (t, x) \mapsto (t, F_t(x)),$$

is equivalent to an A_{2r}-swallowtail (Definition 7.5). The orientation of the cusps is indicated in Figure 9.5.1.

Then on the collar neighborhood from Equation 9.5.2 we define

$$\widetilde{f} : \partial D \times [0,1) \times \mathbb{R} \to \partial D \times [0,1) \times \mathbb{R}^2,$$
$$(q,t,x) \mapsto (q,t,F_t(x)).$$

We can smoothly extend this over the interior as the stabilization of $F_1 : \mathbb{R} \to \mathbb{R}^2$. It follows immediately from the definition that the resulting $\widetilde{f} : D \times \mathbb{R} \to D \times \mathbb{R}^2$ satisfies the claimed properties. □

9.5.2. *cutting wrinkles?*

Lemma 9.6. *Consider a multi-section $f : M \to J^r(X \to M)$. Let $D \times \mathbb{R} \subset M$ be a product neighborhood of a hypersurface D, and $\varepsilon > 0$ a small constant such that:*

(i) f has a double fold on $D \times \mathcal{O}p([0,1])$ (Definition 7.7).
(ii) $|j^r f(x,0) - j^r f(x,1)|_{C^0} < \varepsilon$

Then for any (separating) hypersurface $\Delta \subset D$ there exists a multi-section $\widetilde{f} : M \to J^r(X \to M)$ satisfying:

(i) $|j^r \widetilde{f} - j^r f|_{C^0} < \varepsilon$;
(ii) $\Sigma^{10}(\widetilde{f}) = \Sigma^{10}(f) \cap \Delta$, and $\Sigma^{110}(\widetilde{f}) = \Sigma^{110}(f) \sqcup \Delta$.

Informally, the above conditions say that the fold locus of f is cut into two along Δ. Since the double fold of f is small the resulting map is C^0-close to f.

Proof. □

Theorem 9.7. *Let $f : M \to J^r(X \to M)$ be a multi-section without nested singularities, and $\{U_i\}_{i \in I}$, and $\{V_j\}_{j \in J}$ be covers of M and X respectively. Given any $\varepsilon > 0$ there exists a multi-section $\widetilde{f} : M \to J^r(X \to M)$ satisfying:*

(i) $|j^r \widetilde{f} - j^r f|_{C^0} < \varepsilon$;
*(ii) $\widetilde{f}$ has only wrinkle singularities (Definition **??**);*
(iii) The membrane, and image of each wrinkle are contained in some U_i and V_j respectively.

Proof. By choosing refinements of the covers we can assume that for any U_i there exists a V_j such that $f(U_i) \subset V_j$.

Let us first consider the case that the singular locus of f consists of a single double fold. Thus we may assume (see Definition 7.7) that $M = D \times \mathbb{R}$ and that f has folds of opposite Maslov coorientation along $D \times \{0\}$ and $D \times \{1\}$.

If we choose $0 < \delta < 1$ sufficiently small we can find a (finite) cover $\{\widehat{U}_i\}_{i=1,\dots,N}$ of D with the following properties:

(i) the sets $\widehat{U}_i \times [0,\delta]$ cover $D \times [0,\delta]$;
(ii) each of the $\widehat{U}_i \times [0,\delta]$ is contained in some U_i.

Let $\delta = \sum_{i=1}^N \rho_i$ be a partition of the constant function $\delta : D \to \mathbb{R}$, subordinate to $\{\widehat{U}\}_{i \in I}$. We denote the partial sums by

$$\delta_j := \sum_{i=1}^{j} \rho_i, \quad j = 1, \dots, N$$

so that $\delta_0 = 0$ and $\delta_N = \delta$.

We can interpret δ_{j+1} as a function on (i.e. whose domain is) the graph of δ_j. Hence, we can inductively apply Lemma 9.5 taking (in the notation of the lemma) D to be the graph of δ_j and $\widetilde{D}$ the graph of δ_{j+1}. The resulting multisection has N additional wrinkles and the singular locus $D \times \{0\}$ has moved to $\Delta \times \{\delta\}$.

Repeating the above argument we reduce to the case that f has a double fold along $D \times [1-\delta, 1]$ where $\delta > 0$ is arbitrarily small. Hence removing this double fold only induces a C^0-small perturbation. The resulting multi-section $\widetilde{f} : M \to J^r(X \to M)$ satisfies the required properties. Indeed, each of its wrinkles is contained in some U_i and hence is mapped into some V_j. Moreover, the induction proces consists of finitely many steps. Hence making suitable choices in each step it follows that $|j^r \widetilde{f} - j^r f|_{C^0} < \varepsilon$.

Next, consider the case that f has a single wrinkle. On the complement of the birth/death locus a wrinkle defines a double fold. Hence, we can apply the argument above. That is, in complement of the birth/death locus, we can move the fold loci of the original wrinkle arbitrarily close to each other. Then, as before, removing this double fold and its birth/death induces only a C^0-small perturbation.

For the general case observe that the above arguments only change f in a neighborhood of the double fold or wrinkle. Thus, since the singularities of f are assumed not to be nested, we can apply the construction to one singularity at a time. □

9.5.3. *Passing between wrinkles and folds.*

PAPER II:

10. Introduction

Consider a manifold M endowed with a geometric structure, e.g. a distribution $\xi \subset TM$. A common problem in this setting is to find embeddings $f : N \to M$ which are in some way compatible with the geometric structure, e.g. are in general position with respect to ξ. In this paper we study a particular example of such a problem; the case of embeddings which are tangent to a distribution modelled on jet-spaces.

Recall that any jet bundle $J^r(B, F)$ comes equipped with a Cartan distribution ξ which measures whether a section is holonomic. More precisely, the graphs of holonomic sections are integral submanifolds, i.e. submanifolds tangent to ξ.

More generally, (M, ξ) is a Cartan-Goursat manifold if it is locally isomorphic to $(J^r(B, F), \xi_{\text{can}})$ for some B, F and r. In this setting we are interested in (the homotopy type of) the space of integral submanifolds $N \subset (M, \xi)$.

Unlike (the image of) a section, an arbitrary submanifold can be tangent to the fibers of $J^r(B, F) \to B$. An extreme example is the fiber F which is itself an integral submanifold. Except when (M, ξ) is a contact structure (i.e. when the model jet space is $J^1(B, \mathbb{R})$), the fiber direction can be recovered directly from ξ and defines a global foliation $\mathcal{V}_{\text{can}}$ on M.

To avoid some of the complications which arise in the general case we restrict ourselves to $\text{Emb}_{\Sigma^2}(N, (M, \xi))$, the space of Σ^2-free embeddings. This means that the dimension of the intersection of N with the leaves of $\mathcal{V}_{can}$ is at most one. The formal counterpart of a Σ^2-free embedding is a pair $(f, F_t) : N \to (M, \xi)$, $t \in [0, 1]$, consisting of a smooth embedding $f : N \to M$ and a family of bundle monomorphisms $F_t : TN \to TM$, such that $F_0 = df$ and $\Sigma^2(F_1, \mathcal{V}_{can}) = \varnothing$.

Apart from the ideas of [**?**] we consider three types of techniques. In Section 15 we consider the problem in the smooth (i.e. non-integrable) case. That is, we consider (non-integral) embeddings in (M, ξ) which have prescribed singularities with $\mathcal{V}_{\text{can}}$. The simplest version of the main result is stated as follows:

Theorem 10.1. *Let ξ be a distribution on M and $(f, F_t) : N \to (M, \xi)$ a formally Σ^2-free embedding. Then for any $\varepsilon > 0$ there exist an isotopy $f_s : N \to M$ satisfying:*

- *(i) $f_0 = f$ and $|f_s - f_0|_{C^0} < \varepsilon$;*
- *(ii) $\Sigma(f_1, \mathcal{V}_{can})$ consists of $\Sigma(F_1, \mathcal{V}_{can})$ plus an arbitrary collection of (potentially nested) wrinkles.*

11. Cartan-Goursat manifolds and singularities

11.1. **Cartan-Goursat distributions.** Consider a manifold M endowed with a distribution ξ. Recall from Section 3.4 that we say that ξ is **modelled on the jet space** $(J^r(B, F), \xi_{can})$ if around each $p \in M$ there exists coordinates (x, y, z), whose domain is a subset of $J^r(B, F)$, in which $\xi = \xi_{can}$. We also refer to such distributions as **Cartan-Goursat distributions** when the precise model is not important.

Although the identification with a (trivial) jet bundle exists only locally, many of the useful properties of jet bundles can be encoded purely in terms of the canonical distribution and hence make sense globally on (M, ξ). In particular associated to any Cartan-Goursat distribution are the numbers $n = \dim(B)$, $k = \dim(F)$ and r, the number of steps in which ξ is bracket generating.

As most of our techniques are based on manipulations in the front projection of jet bundles, we would like to have a global analogue of the front projection. If $k > 1$, the fibers of the (local) front projections can be recovered from the Cartan-Goursat distribution, see Section **??**. Thus we obtain a well-defined foliation $\mathcal{F}_{can}$ on (M, ξ), called the **characteristic foliation**, whose leaves (locally) correspond to the fibers of the front projection. In general (when $k = 1$ and $r > 1$) this foliation does not exist. However by Corollary **??** we can still recover the vertical distribution

$$V_{can} = \ker d\pi_{r,r-1} : TJ^r(B, F) \to TJ^{r-1}(B, F),$$

from ξ. Again this defines a foliation $\mathcal{V}_{can}$ on (M, ξ), called the **vertical foliation**. Lastly note that in the contact case, when ξ is modelled on $J^1(\mathbb{R}^n, \mathbb{R})$, neither the characteristic foliation nor the vertical foliation exists globally on (M, ξ). This is a consequence of the fact that contact structures have a much bigger symmetry group than the other Cartan distributions.

11.2. **Integral Submanifolds.** The objects of interest in this section are embedded **integral submanifolds** of Cartan-Goursat distributions, i.e. submanifolds $N \subset (M, \xi)$ everywhere tangent to ξ. Note that as distributions $\mathcal{V}_{can}$ is contained in ξ. Hence an integral submanifold can have singularities of tangency (Definition **??**) with respect to $\mathcal{V}_{can}$, i.e. $\Sigma(N, \mathcal{V}_{can}) \neq \varnothing$. In general, complicated singularities of tangency with $\mathcal{V}_{can}$ cause rigidity for integral submanifolds. Example; Engel structures?. However, if the singularities of tangency are not too complicated, integral submanifolds satisfy the h-principle. Thus we focus on the following class of submanifolds.

Definition 11.1. *An integral submanifold of a Cartan-Goursat distribution $N \subset (M, \xi)$ is called Σ^2**-free** if $\Sigma^2(N, \mathcal{V}_{can}) = \varnothing$. The space of all such embeddings of a manifold N is denoted by* $\mathrm{Emb}_{\Sigma^2}(N, (M, \xi))$.

As usual, the formal counterpart of an integral submanifold decouples the embedding from its derivative. Observe that the definition of singularity of tangency still makes sense for (injective) bundle maps. Furthermore, it is not hard to see that there are injective bundle maps which are not homotopic to a Σ^2-free one. Thus we need to require the existence of such a homotopy in the definition of an formal integral submanifold.

Definition 11.2. *An Σ^2**-free formal integral submanifold** of a Cartan-Goursat distribution (M, ξ) is a pair (f, F_t) consisting of*

(i) An embeddeding $f : N \to M$;
(ii) A homotopy of injective bundle maps $F_t : TN \to TM$ covering f and satisfying:
 - *(i) $F_0 = df$;*
 - *(ii) the image of F_1 is contained in ξ, and F_1 is Σ^2-free, i.e. $\Sigma^2(F_1, \mathcal{V}_{can}) = \varnothing$.*

The space of formal integral submanifolds with domain N is denoted by $\mathrm{FEmb}_{\Sigma^2}(N, (M, \xi)))$.

Note that in the contact case, i.e. when ξ is modelled on $J^1(B, \mathbb{R})$, any integral submanifold is Σ^2-free since $\mathcal{V}_{can}$ is not defined in this case. Given $(f, F_t) \in \mathrm{FEmb}_{\Sigma^2}(N, (M, \xi))$ it can happen that (f, F_t) is already an integral embedding on a domain $D \subset N$. By this we mean that when restricted to D we have $F_t = F_0 = \mathrm{d}f$. In this case we also say that (f, F_t) is **holonomic** on D.

The projection $\pi : \mathrm{FEmb}_{\Sigma^2}(N, (M, \xi)) \to \mathrm{Emb}(N, M)$ is a fibration. In particular to define a homotopy of (f, F_s) it suffices to specify an isotopy of f.

Lemma 11.3. *let $(f, F_s) : N \to (M, \xi)$ be a Σ^2-free formal integral submanifold, and $f_q : N \to M$ a $\mathbb{D}^k$-parameter family of embeddings with $f_0 = f$. Then there exists a $\mathbb{D}^k$-parameter family $(f_q, F_{q,s}) : N \to (M, \xi)$ such that $(f_0, F_{0,s}) = (f, F_s)$.*

Proof. Since f_q is an embedding, $\mathrm{d}f_q$ is an injective bundle map for all $q \in \mathbb{D}^k$. Hence we can define

$$F_{q,s} := \begin{cases} \mathrm{d}f_{(1-2s)q} & 0 \leqslant s \leqslant 1/2 \\ F_{2s} & 1/2 \leqslant s \leqslant 1. \end{cases}$$

□

11.3. **Homotopically essential singularities.** This section should probably be included somewhere else Let us start by assuming that the formal integral submanifold $(f, F_s) : N \to (M, \xi)$ is generic. That is, both f and F_s, $s \in [0,1]$ are generic. It then follows from the Thom-Boardmann stratification theorem that the singular locus $\Sigma^1 := \Sigma^1(F_1, \mathcal{V}_{can})$ is a (codimension-1) submanifold. Furthermore, it comes with a stratification by submanifolds

$$\Sigma^1 \supset \Sigma^{11} \supset \ldots \Sigma^{1^n 0} \supset \varnothing,$$

where each $\Sigma^{1^{i+1}} \subset \Sigma^{1^i}$ is a codimension-1 submanifold. Note that general these singularities cannot be removed even homotopically. Indeed, otherwise we could choose a different family $\widetilde{F}_s$, satisfying the conditions of Definition 11.2 and with $\Sigma(\widetilde{F}_1, \mathcal{V}_{can}) = \varnothing$.

Along Σ^1 the image of F_1 intersects $\mathcal{V}_{can}$ which defines a line bundle:

$$V := F_1^{-1}\left(F_1(TN) \cap \mathcal{V}_{can}\right) \subset TN|_{\Sigma^1}.$$

The Maslov coorientation (Section ??) can be encoded in a trivialization of V. More explicitely, we can choose a nowhere vanishing vector field $v \in \mathfrak{X}(\mathcal{O}p(\Sigma^1))$ such that $V = \operatorname{Span}(v|_{\Sigma^1})$. This vector field satisfies the following transversality conditions:

$$v|_{\Sigma^{1^i}} \in \Gamma(T\Sigma^{1^{i-1}}|_{\Sigma^i}), \quad \text{and,} \quad v|_{\Sigma^{1^i 0}} \pitchfork \Sigma^{1^i 0}, \quad i = 1, \ldots, n.$$

Here we used the convention that $\Sigma^{1^0} = \mathcal{O}p(\Sigma^1)$. For an illustration of such a vector field see Figure ??.

Definition 11.4. *The pair (Σ^1, v) is the* ***essential singular locus*** *of the almost Σ^2-free integral submanifold (f, F_s).*
Two such pairs (Σ, v) and $(\widetilde{\Sigma}^1, \widetilde{v})$ are ***equivalent*** *if there exists a germ of diffeomorphism $\phi : \mathcal{O}p(\Sigma^1) \xrightarrow{\sim} \mathcal{O}p(\widetilde{\Sigma}^1)$ preserving the stratification and taking v to $\widetilde{v}$.*

Up to equivalence the extension of $v|_{\Sigma^1}$ to $\mathcal{O}p(\Sigma^1)$ does not matter. Hence we will usually not distinguish between v and its restriction $v|_{\Sigma^1}$.

12. Σ^2-free integral submanifolds

Evidently any (Σ^2-free) integral submanifold determines a formal integral submanifold. This gives rise to a canonical inclusion map,

$$\iota : \operatorname{Emb}_{\Sigma^2}(N, (M, \xi)) \hookrightarrow \operatorname{FEmb}_{\Sigma^2}(N, (M, \xi)). \tag{12.0.1}$$

The main goal of this section is to show that the above map satisfies the h-principle, i.e. is a weak homotopy equivalence.

In order to prove that Equation 12.0.1 is a weak homotopy equivalence we will produce integral submanifolds whose singularity locus contains the singularity locus of the formal data and, in addition, has extra wrinkles/double folds. We introduce the former using explicit Whitney singularity models. Their complement is transverse to the vertical foliation and can then be handled using the previous sections; this step is where the additional singularities are introduced.

Theorem 12.1. *Let the following data be given:*

- *A non-contact Cartan-Goursat distribution ξ on a manifold pair (M, M').*
- *A manifold N.*
- *A connected and compact manifold pair (K, K'), with K' possibly empty, playing the role of parameter space.*
- *A K-parametric family $(f_k, F_{k,s}) \in \operatorname{FEmb}_{\Sigma^2}(N, (M, \xi))$ which is holonomic when $x \in M'$ or $k \in K'$.*
- *A constant $\varepsilon > 0$.*

Then there exists a family of formal Σ^2-free integral submanifolds $(f_{k,t}, F_{k,t,s}) \in \operatorname{FEmb}_{\Sigma^2}(N, (M, \xi))$ indexed by $K \times [0,1]$ satisfying:

(i) $(f_{k,0}, F_{k,0,s}) = (f_k, F_{k,s})$ and $(f_{k,1}, F_{k,1,s})$ is holonomic.

(ii) $|f_{k,t} - f_{k,0}|_{C^0} < \varepsilon$.
(iii) $(f_{k,t}, F_{k,t,s}) = (f_k, F_{k,s})$ *whenever* $k \in K'$ *or* $x \in M'$.

Corollary 12.2. *The canonical inclusion*

$$\mathrm{Emb}_{\Sigma^2}(N, (M, \xi)) \hookrightarrow \mathrm{FEmb}_{\Sigma^2}(N, (M, \xi))$$

is a weak homotopy equivalence.

12.1. **Proof of main theorem.**

12.1.1. *The case when* $M' = K' = \varnothing$. We start by applying Theorem 15.2 to $(f, F_s) : N \to (M, \mathcal{V}_{can})$. Thus we may assume that f is a smooth embedding whose singularity locus with respect to $\mathcal{V}_{can}$ satisfies:

$$\Sigma(f, \mathcal{V}_{can}) = \Sigma(F_1, \mathcal{V}_{can}) \cup \bigcup_{i \in I} S_i,$$

where $\{S_i\}_{i \in I}$ is a finite collection of (nested) spheres along which f has singularities of fold type. Furthermore, we may assume that $|\mathrm{d}f - F_s|_{C^0} < \varepsilon$ on an open neighborhood of the essential singularity locus $\Sigma(F_1, \mathcal{V}_{can})$.

Next we want to make f integral with respect to ξ, while preserving the essential singularity locus. This is a two step process. On $\mathcal{O}p(\Sigma(F_1, \mathcal{V}_{can})$ we change the singularities of f by hand (using a local model) to become integral. Then, we use general h-principle arguments to make f integral on the complement of the singularity locus. We state the first part as the following lemma:

Are we using the assumption $|\mathrm{d}f - F_s|_{C^0} < \varepsilon$ somewhere in the proof of the following lemma?

Lemma 12.3. *Consider a Cartan − Goursat manifold* (M, ξ) *and let* $f : N \to (M, \xi)$ *be a smooth embedding which is* Σ^2*-free with respect to* $\mathcal{V}_{can}$*, i.e.*

$$\Sigma(f, \mathcal{V}_{can}) = \Sigma^1(f, \mathcal{V}_{can}).$$

Then, for any $\varepsilon > 0$*, there exists an isotopy* $f_t : N \to (M, \xi)$*,* $t \in [0, 1]$ *such that:*

(i) $f_0 = f$ *and* $|f_t - f|_{C^0} < \varepsilon$*;*
(ii) $\Sigma(f_t, \mathcal{V}_{can}) = \Sigma(f, \mathcal{V}_{can})$ *and* f_1 *is integral with respect to* ξ *on* $\mathcal{O}p(\Sigma(f_1, \mathcal{V}_{can})$.

Remark 12.4. *It is not hard to see that the conclusion of the above lemma can be arranged directly in the proof of Theorem 15.2. However, since we want to use that theorem as a blackbox we do not include the result there.*

Proof of Lemma 12.3. For ease of notation we denote $\Sigma := \Sigma(f, \mathcal{V}_{can})$ and we (often) identify N with its image under f. Let us start by assuming that there exists a global fibration chart $(M, \xi) = J^r(B, F)$. Recall that $\mathcal{V}_{can}$ is the tangent space of the fibers of the fibration $\pi : J^r(B, F) \to B$. Hence the assumptions of the lemma imply that $f : N \to J^r(B, F)$ defines a (non-holonomic) multi-section, i.e. the image of each component of $N \backslash \Sigma(f, \mathcal{V}_{can})$ is graphical over B. The idea of the proof is to homotope f to an integral multi-section on $\mathcal{O}p(\Sigma(f, \mathcal{V}_{can}))$.

By the following lemma there is a C^0-small homotopy making Σ integral with respect to ξ.

Lemma 12.5. *Let* $f : N \to J^r(B, F)$ *be a (non-holonomic) multi-section with singularity locus* $\Sigma := \Sigma(f, \mathcal{V}_{can})$*, and let* $\varepsilon > 0$ *be a constant. Then there exists a homotopy (through multi-sections)* $f_t : N \to J^r(B, F)$*,* $t \in [0, 1]$ *satisfying:*

(i) $f_0 = f$ *and* $|f_t - f|_{C^0} < \varepsilon$*;*
(ii) $\Sigma(f_t, \mathcal{V}_{can}) = \Sigma$*, for all* $t \in [0, 1]$*;*
(iii) The restriction $f_1|_\Sigma$ *is integral.*

Proof of Lemma 12.5. Use holonomic approximation to make the singularity locus integral. Use isotopy extension theorem applied to the original section to produce the homotopy. Then an isomorphism in the tangent space along Σ allows us to map $F(TN) \cap \xi$ to a principal direction. By the implicit function theorem we obtain an isotopy of the ambient manifold realizing this.

□

Recall (see Section 16.4) that since F_1 is Σ^2-free and maps into ξ, the line

$$F_1(TN) \cap \mathcal{V}_{can} = \langle V \rangle \subset TM$$

is a principal subspace. This implies that Whitney singularities in the direction of V are lifts of sections. More precisely, let Σ^{1^i} be a stratum of $\Sigma(f, \mathcal{V}_{can})$, which by the previous lemma we assume to be integral. Then the corresponding Whitney map with singularity locus Σ^{1^i} in the direction of V is the r-th order lift of a multi-section. We use this observation to make f holonomic in the normal direction to Σ. Again this is done by induction on the strata of Σ.

Assume f is holonomic on $\mathcal{O}p(\Sigma^{1^{i+1}})$. More precisely assume $f = j^r\sigma_i$ where $\sigma_{i+1} : \mathcal{O}p(\Sigma^{1^i}) \to J^0(B, F)$ is a multi-section satisfying:

(i) $j^r\sigma(\Sigma^{1^{i+1}} \backslash \Sigma^{1^{i+2}}) = f(\Sigma^{1^{i+1}} \backslash \Sigma^{1^{i+2}})$ and $\mathrm{d}j^r\sigma(v) = V$;
(ii) $\Sigma(j^r\sigma, \mathcal{V}_{can}) = \Sigma^{1^{i+1}}(j^r\sigma, V_{can})$ and contains $\Sigma^{1^{i+1}}(f, \mathcal{V}_{can})$.

Note that we start the induction at $i = \dim N$. In this case $\Sigma^{1^{i+1}} = \varnothing$ so that the base of the induction is trivially satisfied. We extend σ to a multi-section $\sigma_i : \mathcal{O}p(\Sigma^{1^i})$ with the same properties as above. Such a map is easily defined by taking the $\Sigma^{1^{i+1}}$-stabilization of a suitable Whitney singularity (Definition 7.2) in the direction of $V = F_1(v)$, using that V is a principal direction as explained above.

Next we homotope f to agree with $j^r\sigma_i$ on $\mathcal{O}p(\Sigma^{1^i})$. Observe that by assumption we have that $|\mathrm{d}f - F_1|_{C^0} < \varepsilon$ implying $|\mathrm{d}f(v) - V|_{C^0} < \varepsilon$, while $\mathrm{d}j^r\sigma(v) = V$. Thus, $j^r\sigma$ and f agree on Σ and locally around Σ their images are graphical over each other. A simple linear interpolation in a neighborhood of Σ^{1^i} deforms f into $j^r\sigma$. Note that this does not change f on $\mathcal{O}p(\Sigma^{1^{i+1}})$ and on Σ^{1^i} (although it does change $\mathrm{d}f$ at points in $\Sigma^{1^i} \backslash \Sigma^{1^{i+1}}$). Furthermore, since both f and $j^r\sigma_i$ are Σ^2-free so is the interpolation. Inductively repeating the above argument yields a multi-section $\sigma : \mathcal{O}p(\Sigma) \to J^0(B, F)$. The r-th order lift of σ agrees with f along Σ and satisfies $\Sigma(j^r\sigma, \mathcal{V}_{can}) = \Sigma(f, \mathcal{V}_{can})$ as stratified sets.

To finish the proof it remains to make f integral on the complement of the essential singularity locus. Here f is graphical over B. By applying Theorem 8.8, relative to $\mathcal{O}p(\Sigma)$, we can homotopy f into an integral submanifold at the cost of adding double folds (in the complement of the essential singularity locus). Recall that, according to Section **??**, these can be added relatively by a homotopy through integral embeddings as long as we are not in the contact case. □

13. Eating wrinkles

In this Section we explain various surgery procedures for integral submanifolds in Cartan-Goursat manifolds. The main goal is proving that a suitable zig-zag (the "loose chart") can absorb other singularities with respect to the vertical. This will be used in the next Section to prove an h-principle for integral submanifolds with prescribed singularities.

13.1. **Merging wrinkles.** The solutions constructed using h-principle arguments are often rather abstract. One of the main causes is that the amount of singularities needed in these arguments is never made explicit. In this section we show that it is fact sufficient to have a single wrinkle. More precisely, given a wrinkled multisection we show that it can be modified to have only a single wrinkle. This provides a moral inverse to the operation from the previous section. Moreover, just like the chopping construction, reducing the number of wrinkles only requires in a C^0-small change of the map.

The mental picture to have in mind for the construction is as follows. As we have seen in the previous section the birth/death locus can be propagated in the domain of a multi-section (through a C^0-small perturbation). In this way a wrinkle can 'grow tentacles' connecting to the other wrinkles. After connecting the wrinkles can be merged into a single one. We note that although the (image of) the multi-section stays C^0-close to the original one, the singularity locus changes a lot.

Let us start giving the details by formalizing the notion of tentacles.

Definition 13.1. *Let $X \to M$ be a fibration and $f : M \to J^r(X \to M)$ be a multi-section. A **tentacle system** for f is a pair (B, γ) where*

$$B := \bigcup_{i=1}^{N} B_i, \quad \textit{and}, \quad \gamma := (\gamma_0, \dots, \gamma_N),$$

are collections of (embedded) balls and curves in M such that:

(i) $f|_{B_i}$ is equivalent to a wrinkle and $f|_{M\setminus B}$ has only fold singularities;
(ii) γ_i is a curve from B_0 to B_i starting and ending in the singular locus of $f|_{B_0}$ and $f|_{B_i}$ respectively;
(iii) the γ_i are disjoint and each of them intersects the fold locus of f transversely.

Remark 13.2. *Using the passing between wrinkles and folds, the above theorem gives a similar result for folded maps. The construction is local in the sense that the map is only changed on a neighborhood of the wrinkles and the connecting curves.*

It is not hard to see that if M is connected and f has only folds and wrinkle singularties then a tentacle system always exists. In general it can happen that the wrinkles of f are nested. In this case we will always assume that the indices are **compatible** with the nesting in the sense that

$$B_i \subset B_j \quad \text{implies} \quad i > j.$$

Here $B_i \subset B_j$ means that the membrane of B_i is contained in the membrane of B_j. Note that the wrinkle $f|_{B_0}$ is distinguished since the curves connect all the other wrinkles to this one. Furthermore, the compatibility condition implies that this wrinkle is never nested inside another one.

Then we move the wrinkles giving the following statement. It is important to realize that although the (image of) the map stays C^0-close the singularity locus of f changes a lot!

The precise statement is as follows:

Theorem 13.3. *Let $X \to M$ be a fibration and $f : M \to J^r(X \to M)$ a multi-section and (B, γ) a tentacle system (Definition 13.1). Then, for any $\varepsilon > 0$ there exists a multi-section $\widetilde{f}$ satifying:*

(i) $\widetilde{f}$ has a single wrinkle;
(ii) $|j^r\widetilde{f} - j^r f|_{C^0} < \varepsilon$ and $\widetilde{f} = f$ on $M \setminus \mathcal{O}p(B \cup \gamma)$.

Add remark that the proof mostly works also for double folds instead of wrinkles. The only difficulty is that making a double fold into an inside out fold is not C^0 small unless the double fold is. Furthermore since we can pass between folds and wrinkles it also follows from the above theorem.
How far are we from proving everything here for submanifolds of jet space, analogous to emmys paper?

Remark 13.4. *Altough it not relevant to our setup, the above condition that $f|_{M\setminus B}$ has only fold singularities is not strictly necessary. The proof below still goes through if f has higher singularities provided that we can choose a tentacle system disjoint from them. This happens for example if they form a subset of M which has codimension $\geqslant 2$.*

The proof uses two observations which are presented in the next two subsections. First, since the curves of a tentacle system can intersect the fold locus, we need to show that a double fold can 'pass through another fold'. This allows the tentacles to grow 'through' the fold locus. Second, we show how two wrinkles connected by a curve (which does not intersect any other folds) can be merged to a single wrinkle. Using these ingredients the proof of the theorem is given in Section 13.3.

13.1.1. *Moving wrinkles through folds.* We start with a special case in which we have local coordinates and the sections are in normal form. Consider the trivial fibration $\underline{\mathbb{R}} \to \mathbb{R}^2$. Let (q, x) be coordinates for the base and y for the fiber. We use our usual identification of multi-sections $f : \mathbb{R}^2 \to J^r(\underline{\mathbb{R}} \to \mathbb{R}^2)$ with their front projection. That is, we identify them with surfaces in $\mathbb{R}^3$, or with 1-parameter families of curves in $\mathbb{R}^2$.

There are two different operations to pass a double fold 'through' a fold, depending on the Maslov coorientation of the folds relative to each other. Recall that, since our fiber bundle is trivial, this amounts to whether the front has a positive of negative singularity of tangency with the fiber. That is, away from the singularities of tangency, the orientation of the front (coming from the base) and the orientation of the fiber determine an orientation of the total space. At the singularity of tangency this orientation changes, and we say the singularity is positive (resp. negative) if the orientation changes from negative to positive (resp. positive to negative).

For the first case, when the folds have alternating Maslov coorientations, our starting section is denoted by $f_{-,0} : \mathbb{R}^2 \to J^r(\underline{\mathbb{R}} \to \mathbb{R}^2)$. Its front projection is depicted on the left in Figure 13.1.1. It has fold singularities at $\{x = -2\}$, $\{x = -1\}$ and $\{x = 1\}$ which have negative, positive, and negative Maslov coorientation respectively. We think of the first two folds as a doublefold which we want to pass through the third fold. As in Figure 13.1.1 this fold can be moved arbitrarily close to the third fold. Then, unpairing the first two folds and pairing tge last two the double fold 'jumps over' the fold and can then continue moving. Note that the fold locus of a single fold cannot be moved without introducing a C^0-large change. Therefore, after the second step the position left most fold is fixed and cannot be moved back to its starting position.

This proves the following lemma:

Lemma 13.5. *For any $\varepsilon > 0$ there exists a 1-parameter family of multi-sections $f_{-,t} : \mathbb{R}^2 \to J^r(\underline{\mathbb{R}} \to \mathbb{R}^2)$, $t \in [0,1]$, satisfying the following conditions:*

(i) $|j^r f_{-,t}|_{C^0} < \varepsilon$ *for all* $t \in [0,1]$*;*
(ii) $f_{-,t}(q,x) = f_{-,0}(q,x)$ *for all* $t \in [0,1]$ *and* $(q,x) \in \mathcal{O}p(\{q = \pm 2\} \cup \{x = \pm 2\})$*;*
(iii) the front projection and singular locus of $f_{-,t}$ are as depicted in Figure 13.1.1. To be precise, for all t, the singular locus of f_t consists of 3 connected components of fold points with alternating Maslov coorientation. For $t = 0$ the first two fold loci are paired together forming a double-fold while for $t = 1$ and $q \in [-1,1]$ the last two folds are paired.

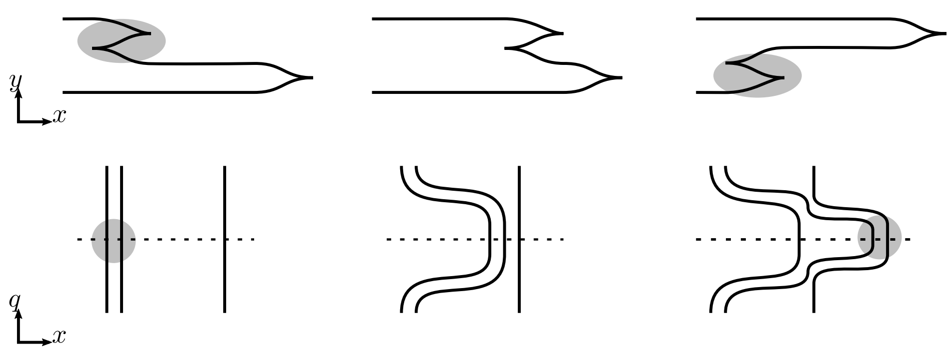

FIGURE 7. The top and bottom depict respectively the front and fold locus of the family of multi-sections f_t, $t \in [0,1]$ defined in Lemma 13.5. To be precise, the top depicts the restriction of f_t to the dashed line in the domain. As t varies from 0 to 1 (from left to right in the figure), the double fold moves to the third fold, the pairing switches to the last two folds forming a new double-fold which then moves further.

In the second case, when the last two folds have the same Maslov coorientation, the homotopy is slightly more complicated. Consider the 1-parameter family of multi-sections depicted in Figure **??**.

Add in the premilinaries somewhere a remark pointing out that Reidemeister moves also work in J^r using A_{2r}-folds.

TODO, important: the procedure to pass the zig-zag to the other side, without surgery, is missing. This is key to preserve the singularity locus. The point is that it appears in the other side as a zig-zag with "fishes"

Since the pairing trick does not work we need to apply surgery to the singular locus. This amounts to first applying a Reidemeister II move to create a self-intersection in the front and then cancel the last two folds (which have the same Maslov coorientation) by an inverse Reidemeister I move. In terms of the singular locus this amounts (locally around $\{q = 0\}$) to cancelling the two folds against each other. Thus, we create a 'hole' in the singular locus through which we would like to move the first fold. However, recall that only double folds can be moved while preserving the C^0-norm of the map. Hence we first apply a Reidemeister I move, creating a double fold. This puts us in the setting of Lemma 13.5 allowing us to move two of the folds through the hole. Again we state this as a lemma for later reference.

Lemma 13.6. *For any $\varepsilon > 0$ there exists a 1-parameter family of multi-sections $f+,t : \mathbb{R}^2 \to J^r(\underline{\mathbb{R}} \to \mathbb{R}^2)$, $t \in [0,1]$, satisfying the following conditions:*

(i) $|j^r f_{+,t}|_{C^0} < \varepsilon|$ *for all* $t \in [0,1]$;
(ii) $f_{+,t}(q,x) = f_{+,0}(q,x)$ *for all* $t \in [0,1]$ *and* $(q,x) \in \mathcal{O}p(\{q = \pm 2\} \cup \{x = \pm 2\})$;
(iii) the front projection and singular locus of $f_{+,t}$ *are as depicted in Figure 13.1.1.*

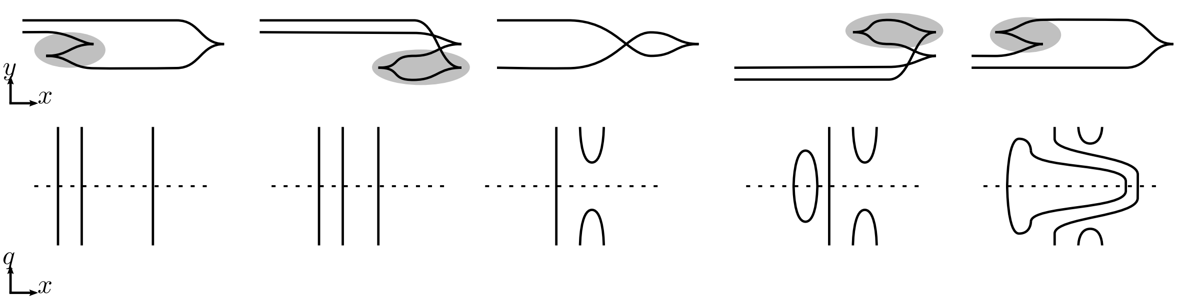

FIGURE 8. The top and bottom depict respectively the front and fold locus of the family of multi-sections f_t, $t \in [0,1]$ defined in Lemma 13.6. From left to right (as t varies) the front projection changes by a Reidmeister II, an inverse Reidemeister I, a Reidemeister I, and an inverse Reidemeister II move. On the fold locus this has the effect of locally canceling the last two folds against each other, then creating a wrinkle and moving the resulting double fold through the gap.

13.2. **Inside out wrinkles.** Next we describe how to wrinkles can be merged into a single one. Recall from Section 7.1.3 that the A_{2r}-Swallowtail is given by the map

$$\begin{aligned} \mathrm{Sw}_{2r} : \mathbb{R}^n &\to \mathbb{R}^{n+k} \\ (q,x) &\to (q, -x^3 - q_1 x, \int_0^x (s^3 + q_1 s - x^3 - q_1 x)^r ds). \end{aligned} \tag{13.2.1}$$

We usually interpret it as a 1-parameter family of maps, indexed by q_1, interpolating between a double fold and a regular map. In other words, the q_1-coordinate controls the birth/death of the double fold. Replacing q_1 with another function depending on the q coordinates we obtain different configurations of the birth/death locus.

Definition 13.7. *The A_{2r}-**inside out wrinkle** is the germ around $\{(q,x) \in \mathbb{R}^2 \mid x = 0, -1 \leqslant q \leqslant 1\}$ of the map*

$$\begin{aligned} \mathrm{IWr}_{2r} : \mathbb{R}^n &\to \mathbb{R}^{n+k} \\ (q,x) &\to (q, -x^3 - (q_1^2 - 1)x, \int_0^x (s^3 + (q_1^2 - 1)s - x^3 - (q_1^2 - 1)x)^r ds, 0, \ldots, 0). \end{aligned} \tag{13.2.2}$$

There is another configuration of the birth/death locus that is relevant to us. Let $(\phi, \rho) \in \mathbb{S}^{n-2} \times \mathbb{R}$ denote spherical coordinates on $\mathbb{R}^{n-1}$.

Definition 13.8. *The* A_{2r}*-**Spherical inside out wrinkle** is the germ around* $\{(\phi,\rho,x)\in\mathbb{R}^n \mid \rho\leqslant 1,\ x=0\}$ *of the map*

$$\begin{aligned} \mathrm{IWr}_{2r}:\mathbb{R}^n &\to \mathbb{R}^{n+k} \\ (\phi,\rho,x) &\to (\tilde{q},-x^3-(\rho^2-1)x,\int_0^x (s^3+(\rho^2-1)s-x^3-(\rho^2-1)x)^r ds,0,\dots,0). \end{aligned} \tag{13.2.3}$$

Observe that up to equivalence, only the qualitative behavior of the function ρ^2-1 in the above definition matters. That is, if $\lambda:\mathbb{R}\to\mathbb{R}$ is any function which is negative on $[0,1)$, positive on $(1,\infty)$, and transverse to zero then replacing ρ^2-1 by $\lambda(\rho)$ yields an equivalent map. In particular the value of λ does not matter.

Using a suitable family of functions depending on ρ we can homotope the spherical inside out wrinkle into a double fold. This will be our model for the merging of two wrinkles. Thus, for a fixed $\varepsilon>0$ let $\lambda_t:\mathbb{R}\to\mathbb{R}$, $t\in[0,1]$, be a smoothing of the following piecewise linear map, see Figure **??**:

(i) $\lambda_t(\rho)=\lambda_t(-\rho)$ for all $t\in[0,1]$ and $q\in\mathbb{R}$;

(ii) $\lambda_t(\rho):=\begin{cases}\varepsilon & \rho\in[(1+\varepsilon(1-t),\infty)\\ -\varepsilon & \rho\in[0,(1-\varepsilon)(1-t))\\ \rho-1+t & \text{everywhere else on } [0,\infty)\end{cases}$

Then, the desired map is defined as:

$$\begin{aligned} \mathrm{SIWr}_{2r,t}:\mathbb{R}^n &\to \mathbb{R}^{n+k} \\ (\phi,\rho,x) &\to (\phi,\rho,-x^3-\lambda_t(\rho)x,\int_0^x (s^3+(\rho^2-1)s-x^3-\lambda_t(\rho)x)^r ds,0,\dots,0). \end{aligned} \tag{13.2.4}$$

Note that the homotopy can be made arbitrarily small by a suitable choice of $\varepsilon>0$. Have to do emmy style calculation to show that the perturbation is C^r-small not jst C^0-small

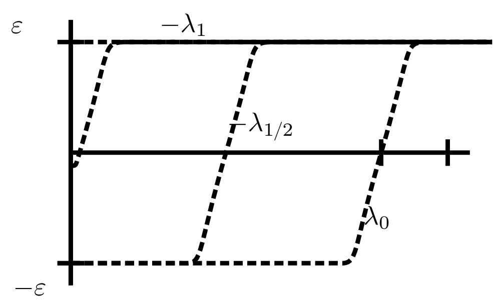

FIGURE 9.

Should be careful that the coorientations of the folds of each wrinkle might not match

Lemma 13.9. *Suppose* $f:M\to J^r(X\to M)$ *is a multi-section whose singular set consists of two disjoint wrinkles. Furthermore, let* γ *be a path from (he singular locus of) one wrinkle to the other and which does not intersect the singular locus away from its endpoints. Then, for any* $\varepsilon>0$ *there exists a multi-section* $\tilde{f}:M\to J^r(X\to M)$ *satisfying:*

(i) The singular set of $\tilde{f}$ *consists of a single wrinkle;*
(ii) $|j^rf-j^r\tilde{f}|_{C^0}<\varepsilon$ *and* $\tilde{f}=f$ *on* $M\backslash\mathcal{O}p(\Sigma(f)\cup\gamma))$.

Proof. Recall that although a wrinkle has a local model arounds its singular locus $\mathbb{S}^n$ (where $n=\dim M$), in general this local model does not extend over the ball $\mathbb{D}^n$. In particular, the folds at corresponding points in the northern and southern hemisphere of $\mathbb{S}^n$ need not be in cancelling position. On the other hand as the equator consists of birth/death points, the folds near the equator are in cancelling position.

We start by modifying γ so that its endpoints lie in the fold locus of the wrinkle sufficiently close to the birth/death locus. Note that this only requires modifying γ in an (arbitrary small) neighborhood of the wrinkle. This ensures that the resulting map $\widetilde{f}$ satisfies $\widetilde{f} = f$ on $M \backslash \mathcal{O}p(\gamma \cup \Sigma(f))$.

Around the endpoints of the modified curve (still denoted by γ) there there exists a coordinate neighborhood $(\phi, \rho, x) \in \mathbb{R}^n$ where f equals $\mathrm{SIWr}_{2r,1}$ (see Equation 13.2.4) and the (image of the) curve γ is given by $\{(\phi, \rho, x) \mid \rho = 0, x \geqslant \varepsilon\}$. Here $\varepsilon > 0$ will be chosen sufficiently small at the end of the proof.

We start modifying f by replacing $\mathrm{SIWr}_{2r,1}$ with $\mathrm{SIWr}_{2r,0}$. Then we can find a cilinder $C \simeq \S^{n-2} \times I$ around γ whose boundary equals the birth/death locus of $\mathrm{SIWr}_{2r,0}$ as in Figure **??**. To be precise C satisfies the following conditions:

(i) Let g be a metric on M which agrees with the standard Euclidean metric in the coordinate charts around the endpoints of γ. Then C is contained in an ε-thickening (with respect to g) $\mathbb{D}^{n-1} \times \gamma$ of γ.
(ii) Near its boundary C is tangent to the hyperplane $\{(\phi, \rho, x) \mid x = 0\}$.

The second condition implies that there exists a coordinate neighborhood $\S^{n-2} \times \mathbb{R}^2$ of C on which f (after the first modification) looks like the $\mathbb{S}^{n-2}$ stabilization of $\mathrm{IWr}_{2r} : \mathbb{R}^2 \to \mathbb{R}^{2+k}$ from Definition 13.7.

□

13.3. **proof of Theorem 13.3.** The proof is essentially a straightforward applicatication of the results from the previous sections. The main technicallity is that the wrinkles of the multi-section $f : M \to J^r(X \to M)$ might be nested. Recall (Definition 13.1 that given a tentacle system (B, γ) the indices of the singular loci are compatible with the nesting in the sense that if $B_i \subset B_j$ then $i > j$. This means that if we remove the wrinkles one by one, starting from the highest index, we can assume without loss of generality that the wrinkle we are removing does not contain any other wrinkle.

Thus let $\gamma_i(t) : [0,1] \to M$ be the curve from B_0 to B_i as in Definition 13.1. By modifying γ_i we can arrange that its endpoints lie in the fold locus of the wrinkle arbitrarily close to the birth/death locus. Then, slightly extending γ_i, we may assume that it intersects both the upper and lower hemispheres of B_0 and B_i.

We associate to γ_i a partitition of the unit interval $t_0 = 0 < t_1 < \cdots < t_{\ell-1} < t_\ell = 1$, for some $\ell \in \mathbb{N}$, so that each t_j corresponds to a point $\gamma_i(t_j)$ where γ_i intersects the fold locus of f. By the preceding remark

$$f|_{\gamma_i([t_0,t_1])}, \quad \text{and,} \quad f|_{\gamma_i([t_{\ell-1},t_\ell])},$$

are equivalent to a double fold. Depending on the Maslov coorientation of the fold at $\gamma_i(t_1)$ we use Lemma 13.5 or Lemma 13.6 to "pass B_0 through the fold at $\gamma_i(t_1)$ ". In terms of the above partition this means that $f_{\gamma_i([t_1,t_2])}$ becomes equivalent to a double fold. Continuing like this we end up with a modified f for which $f|_{\gamma_i([t_{\ell-3},t_{\ell-2}])}$ and $f|_{\gamma_i([t_{\ell-1},t_\ell])}$ are both equivalent to double folds, and the latter lies on the wrinkle B_i.

By applying Lemma 13.9 we can absorb the latter wrinkle into the double fold. Finally we can undo the modifications we made using Lemma 13.5 and Lemma 13.6 so that $f|_{\gamma_i([t_0,t_1])}$ are again paired as a double fold. Note that the last step is essential as it removes the extra wrinkles indroduced when applying Lemma 13.6.

13.4. **Flexible double-folds.** Recall that Theorem 13.3 tells us that any multi-section can be simplified to have a single wrinkle. In this section we aim to prove a similar simplification result for (double) fold singularities. This result will play a key role in proving a general h-principle for integral submanifolds (with simple singularities) of jet space, given in Section **??**, generalizing the main result from [**?**].

Let us start by observing that the proof of Theorem 13.3 never uses the whole wrinkle, only a small piece of it consisting of a double fold. However, being part of a wrinkle, this region can be chosen close to the birth/death singularities, so that the double fold is arbitrarily small (see the proof of Lemma 13.9). Ultimately, this is the reason that allows the homotopy of Theorem 13.3 to be C^0-small. In turn this avoids any potential problems with self-intersections of the perturbed map.

In general an integral submanifold might have large double folds so that we cannot expect to simplify the fold locus by a C^0-small perturbation. In fact, sometimes the fold locus cannot be simplified at all. Do we have a simple/explicit example of this?To deal with these problems we follow the approach of [**?**], and define a model double-fold whose existence allows us to simplify the fold locus. As indicated by the above observations, the main property of this model is that it has the correct "size ", which will be made precise below.

Recall that the jet bundle $J^r(\underline{\mathbb{R}} \to \mathbb{R}^n)$ can be trivialized using a trivialization $\underline{\mathbb{R}} \simeq \mathbb{R}^n \times \mathbb{R}$. Explicitely, given coordinates $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ let z^I, $I = (i_1, \ldots, 1_n)$ denote induced holonomic coordinates (Definition 3.1). We define the following subsets in jetspace:

$$C_0^r := \{(x, y, z) \in J^r(\underline{\mathbb{R}} \to \mathbb{R}) \mid |x| \leqslant 1,\ |y| \leqslant 1,\ |z^i| \leqslant 1,\ \forall i \leqslant r\}.$$

The corresponding subset of the base is denoted by

$$B_0 := \pi_b(C_0^r) = [-1, 1].$$

Let $N(n, r) := \dim J^r(\underline{\mathbb{R}} \to \mathbb{R}^n)$ then for any $\rho > 0$ we also define

$$C_\rho^r := C_0^r \times [-\rho, \rho]^{N(n,r)-N(1,r)} \subset J^r(\underline{\mathbb{R}} \to \mathbb{R}^n).$$

As before we denote the projection to the base by:

$$B_\rho := \pi_b(C_0^r) = [-1, 1] \times [-\rho, \rho]^{n-1}.$$

Note that a point $(x, y, z) \in J^r(\underline{\mathbb{R}} \to \mathbb{R}^n)$ is contained in C_ρ^r if and only if $x \in [-1, 1] \times [-\rho, \rho]^{n-1}$, all the pure derivatives in x_1 are in $[-1,]$ and all other derivatives in $[-\rho, \rho]$. Note that if $n = 1$ then C_0^r and C_ρ^r coincide.

Let $\phi_r : B_0 \to J^r(\underline{\mathbb{R}} \to B_0)$ be a multi-section with the following properties:

(i) ϕ_r is constant equal to 0 around $x = -1$ and constant equal to 1 around $x = 1$;
(ii) the image of its lift $j^r\phi_r$ is contained in C_0^r;
(iii) ϕ_r is equivalent to an A_{2r}-double fold.

Lastly we define

$$\phi_{r,\rho} : B_\rho \to J^r(\underline{\mathbb{R}} \to B_\rho),$$

as the $[\rho, \rho]^{n-1}$ stabilization of ϕ_r (i.e. extending ϕ_r to be constant in the $[-\rho, \rho]^{n-1}$-coordinates). The pair $(C_\rho^r, \phi_{r,\rho})$ is called an A_{2r}**-double fold chart**.

We emphasize that it is not only the map $\phi_{r,\rho}$, but the pair $(\phi_{r,\rho}, C_\rho^r)$ that is important for our discussion. Indeed, as a map $\phi_{r,\rho}$ is equivalent to any A_{2r}-double fold, and hence any integral submanifold with a double-fold contains it. However, the definition of equivalence requires only the germ (around the image) of $\phi_{r,\rho}$ to be contained. Here, we ask the whole cube C_ρ^r to be contained.

Going back to the proof of Theorem 13.3, we used that double folds can be homotoped to inside out wrinkles. The key point is that this homotopy cannot always be contained inside C_ρ^r, unless ρ is sufficiently large. Hence we make the following definition:

Definition 13.10. *The double fold $\phi_{r,\rho}$ is said to be* ***flexible*** *if it is homotopic, within C_ρ^r, to a spherical inside-out wrinkle (Definition 13.8). In this case the pair $(C_\rho^r, \phi_{r,\rho})$ is called a* ***flexible double fold chart****.*

Of course we still need to understand for which ρ the above model is flexible. For small r this can be computed explicitely while for general r we have a (non sharp) bound given by the following lemma:

Lemma 13.11. *The minimal length ρ for which the A_{2r}-double fold chart $(C_\rho^r, \phi_{r,\rho})$ is flexible, denoted by ρ_r, satisfies:*

$$\rho_1 = 1, \quad \rho_2 = \frac{3}{2}, \quad \rho_2 = 2, \quad \rho_r < 3^{r-1}.$$

Proof. As shown by Equation 13.2.4, it is possible to homotope a double fold to an inside out wrinkle. Furthermore, it is not hard to see that if the starting double fold is sufficiently small (with respect to the C^0-norm on the r-th order jet bundle) then the homotopy stays inside C^r_ρ. To see why this is true first observe that, by compactness and continuity, the r-th order lift of $\mathrm{Sw}_{2r,t}$ is bounded. Hence, we can apply a "scaling trick "to make the lift arbitrarily close to the zero section as follows.

Let (x, y, z^I), $I = (i_1, \dots, i_n)$ denote the usual coordinates on $J^r(\underline{\mathbb{R}} \to \mathbb{R}^n)$. In these coordinates we define a scaling diffeomorphism of the front of $J^r(\underline{\mathbb{R}} \to \mathbb{R}^n)$ by:

$$\mu_\lambda : \underline{\mathbb{R}} \to \mathbb{R}^n \to \underline{\mathbb{R}} \to \mathbb{R}^n, \quad (x, y) \mapsto (\lambda x_1, x_2, \dots, x_n, \lambda^{r+1} y),$$

for any $\lambda > 0$. Recall that any diffeomorphism of the front can be lifted to a contact transformation (Definition 3.6). We still denote the lift by $\mu_\lambda : J^r(\underline{\mathbb{R}} \to \mathbb{R}^n) \to J^r(\underline{\mathbb{R}} \to \mathbb{R}^n)$ which in coordinates is given by

$$\mu_\lambda(x, y, z^I) = \left(\lambda x_1, x_2, \dots, x_n, \lambda^{r+1} y, \lambda^{r+1-i_1} z^I\right), \quad I = (i_1, \dots, i_n). \tag{13.4.1}$$

Since $0 \leqslant i_1 \leqslant r$ it follows that μ_λ, for $\lambda \to 0$, contracts C^r_ρ into an arbitrarily small neighborhood of the zero section in $J^r(\underline{\mathbb{R}} \to \mathbb{R}^n)$. Therefore, if λ is sufficiently small, we can homotope $\mu_\lambda \circ \phi_{r,\rho}$ to an inside out wrinkle within C^r_ρ.

Of course, we want our homotopy to be relative to the boundary of B_ρ. Thus, it remains to find an interpolation between $\mu_\lambda \circ \phi_{r,\rho}$ and $\phi_{r,\rho}$, and this is where the size of ρ becomes crucial. Let $\lambda : [-\rho, \rho]^{n-1} \to [0, 1]$ be a function which is 1 around the boundary of the domain, and equal ε to around the origin. Then, denote $\widehat{x} := (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$, and consider the map

$$\widetilde{\phi}_{r,\rho} : B_\rho \to J^r(\underline{\mathbb{R}} \to \mathbb{R}^n), \quad x \mapsto \mu_{\lambda(\widehat{x})} \circ \phi_{r,\rho}(x).$$

Observe that $\phi_{r,\rho}$ does not depend on the $\widehat{x}$ coordinates. Together with Equation 13.4.1 this implies that the image of $\widetilde{\phi}_{r,\rho}$ is contained in C^r_ρ provided the derivatives of λ satisfy

$$|\frac{\mathrm{d}}{\mathrm{d}x^I}\lambda| \leqslant 1, \quad \forall |I| \leqslant r.$$

To find λ it suffices to find a function $f_r : [0, \rho] \to [0, 1]$ satisfying:

$$f_r(0) = 0, \quad f_r(\rho) = 1, \quad \left|\frac{\mathrm{d}^k}{\mathrm{d}x^k} f_r\right| \leqslant 1, \quad \forall 0 \leqslant k \leqslant r, \tag{13.4.2}$$

and which can be smoothly extended as the constant function. In fact, since scaling by a constant preserves the above conditions, it is enough to find a function satisfying $f_r(\rho) > 1$. Then, taking

$$\lambda(x) := f_r(|x_2|) \cdot \dots \cdot f_r(|x_n|),$$

yields the desired function. Therefore, ρ_r is the minimal number such that for any $\rho > \rho_r$ there exists a function as above.

For $r = 1, 2, 3$ explicitely define the corresponding piecewise smooth function as in Figure **??**. If $\rho > \rho_r$ these functions satisfy $f_r(\rho) > 1$. Hence, a smoothening of f will satisfy the required conditions.

For the general case we can use the following inductive algorithm. Suppose f_r satisfies the properties above, then we define f_{r+1} in the following way:

$$F(x) := \begin{cases} f_r(x) & x \in [0, \rho_r] \\ 1 & x \in [\rho_r, \rho_r + 1] \\ f_r(\rho_r + \delta - x) & x \in [\rho_r + 1, 2\rho_r + 1]. \end{cases}$$

Observe that the (piecewise smooth) function

$$f_{r+1}(x) := \int_0^x F(y)\mathrm{d}y,$$

satisfies the conditions in Equation 13.4.2. Hence a suitable smoothing and rescaling yields the desired function. Furthermore, since $\rho_r \geqslant 1$ for all r, we obtain the bound $\rho_r < 3^{r-1}$. $\square$

Lemma 13.12. *If a multi-section contains a birth/death singularity then it contains a flexible double-fold.*

13.4.1. *Fishes: an alternative approach.*

13.4.2. *The parametric case.*

14. Σ^2-free integral submanifolds with prescribed singularities

14.1. **Setup and statement of the theorem.** We consider now integral submanifolds whose locus of tangencies with respect to the vertical foliation is fixed. As before, we impose for the singularities to be of Whitney type.

Our goal is to prove an h-principle. For this, we need a certain local model to provide sufficient flexibility. This is reminiscent of the existence of a zig-zag in the study of S-immersions [], the existence of an "overtwisted disk" in the study of contact structures [9] and Engel structures [14], or the existence "loose chart" in the study of Legendrians [37].

In our setting we need a flexible doublefold chart as in Definition 13.10. Given a ball $D \subset N$ we denote by $\mathrm{Emb}_{\Sigma_2}(N, (M, \xi); D, \Delta)$ (resp. $\mathrm{FEmb}_{\Sigma^2}(N, (M, \xi); D, \Delta)$) the space of all Σ^2-free integral submanifolds (resp. the space Σ^2-free formal integral submanifolds) whose singularity locus is Δ and have D as a flexible doublefold chart. The precise nature of Δ, as a stratified subset, and D, as a chart, has to be explained.

Theorem 14.1. *The canonical inclusion map*

$$\mathrm{Emb}_{\Sigma^2}(N, (M, \xi); D, \Delta) \hookrightarrow \mathrm{FEmb}_{\Sigma^2}(N, (M, \xi); D, \Delta),$$

is a weak homotopy equivalence.

The above theorem follows immediately from the following more detailed (and technical) statement. Furthermore, it shows that the above h-principle is in fact C^0-close and relative both in domain and parameter. The theorem states that any (K-parametric) family of formal Σ^2-free integral free embeddings can be connected by a C^0-small path (of such K-parametric families) to a family of Σ^2-free integral embeddings. Furthermore, if the family is already holonomic (in domain or parameter) then the path can be chosen constant.

Theorem 14.2. *Let the following data be given:*

- *A Cartan-Goursat distribution ξ on a manifold pair (M, M'), and a manifold N containing a ball D;*
- *A connected and compact manifold pair (K, K'), with K' possibly empty, playing the role of parameter space;*
- *A K-parametric family $(f_k, F_{k,s}) \in \mathrm{FEmb}_{\Sigma^2}(N, (M, \xi); D, \Delta)$ which is holonomic when $x \in M'$ or $k \in K'$;*
- *A constant $\varepsilon > 0$.*

Then there exists a family of formal Σ^2-free integral submanifolds $(f_{k,t}, F_{k,t,s}) \in \mathrm{FEmb}_{\Sigma^2}(N, (M, \xi); D, \Delta)$ indexed by $K \times [0, 1]$ satisfying;

(i) $(f_{k,0}, F_{k,0,s}) = (f_k, F_{k,s})$ *and* $(f_{k,1}, F_{k,1,s})$ *is holonomic;*
(ii) $|f_{k,t} - f_{k,0}|_{C^0} < \varepsilon$*;*
(iii) $(f_{k,t}, F_{k,t,s}) = (f_k, F_{k,s})$ *whenever* $k \in K'$ *or* $x \in M'$.

15. Appendix. Smooth embeddings with prescribed singularities

15.1. **Setup and statement of the result.** The simplest case of Theorem 9.1 (when $r = 1$) states that any formal deformation of a submanifold can be C^0-approximated by an isotopy. That is, given an embedding $f : N \to M$ and a tangential rotation $F_s : TN \to TM|_N$, for any ε we can find an isotopy $f_s : N \to M$ such that

$$|f_s - f|_{C^0} < \varepsilon, \quad |\mathrm{d}f_s - F_s|_{C^0} < \varepsilon.$$

In many interesting cases M comes equipped with a distribution ξ, and we would like to additionally control the singularities of tangency of f_s with respect to ξ.

Throughout this section we fix distribution ξ of rank $\leqslant k$ on a manifold M of dimension $n+k$, and we consider embeddings $f : N \to (M, \xi)$ where $\dim N = n$.

Definition 15.1. *A* ***formally*** Σ^2***-free submanifold*** *of* (M,ξ) *is a pair* (f, F_s) *consisting of:*

(1) An embedding $f : N \to M$*;*
(2) A homotopy of injective bundle maps $F_s : TN \to TM$ *covering* f *such that* $F_0 = \mathrm{d}f$ *and* $\Sigma^2(F_1, \xi) = \varnothing$*.*

If we can take $F_s = F_0 = \mathrm{d}f$ then we say that f is Σ^2**-free** or that (f, F_s) is **holonomic**.

The following h-principle is the main result of this section. It states that, up to additional folds, any formally Σ^2-free submanifold is isotopic to a holonomic Σ^2-free submanifold with the same singularity locus.

Theorem 15.2. *Let* $(f, F_s) : N \to (M, \xi)$ *be a formally* Σ^2*-free submanifolds. Then, for any* $\varepsilon > 0$ *there exists an isotopies* $f_s : N \to M$ *satisfying:*

(i) $f_0 = f$ *and*
$$|f_s - f_0|_{C^0} < \varepsilon;$$
(ii) On $\mathcal{O}p(\Sigma(F_1,\xi)$ *we have*
$$|\mathrm{d}f_1 - F_s|_{C^0} < \varepsilon;$$
(iii) the singularity locus satisfies
$$\Sigma(f_1,\xi) = \Sigma(F_1,\xi) \cup \bigcup_{i\in I} S_i,$$
where $\{S_i\}_{i\in I}$ *is a finite collection of (nested) codimension-one spheres along which* f_1 *has singularities of fold type. Here the equality is as stratified sets.*

15.2. **Proof of the result.** To explain the idea of the proof, first suppose that $F_s \pitchfork \xi$ for all $s \in [0,1]$. In this case we can arrange that ξ is contained in the fibers of the normal bundle $\pi : \mathcal{N} \to N$. Therefore, F_t can be interpreted as a section of $J^1\mathcal{N} \to N$ and we can use holonomic approximation to isotope N to follow F_t along its codimension-one skeleton (containing Σ). Then, we modify f in the normal direction to Σ so that it has the correct singularities of tangency with ξ. Finally, appealing to Theorem 8.8, we can extend the isotopy over the top dimensional cells of N, at the cost of introducing additional (double) fold singularities.

In general both f and F_s (for $s<1$) can have extremely bad singularities of tangency with ξ. However, knowing that f is formally Σ^2-free, we can replace ξ with a family of distributions ξ_s such that $\mathcal{F}_s \pitchfork^2 \xi_s$ for all s. To be precise, since $F_s : TN \to TM|_{f(N)}$ is an injective bundle map we can find a family of bundle isomorphisms $\widehat{F_s} : TM \to TM$ such that
$$\widehat{F_s} \circ F_0 = F_s : TN \to TM|_{f(N)}, \quad \forall s \in [0,1].$$
Using this map define a family of distributions ξ_s, $s \in [0,1]$ on M by:
$$\xi_s := \widehat{F}_s \circ \widehat{F}_1^{-1}(\xi).$$
We observe that $\xi_1 = \xi$, and $F_s \pitchfork^2 \xi_s$ for all $s \in [0,1]$. In particular $\xi_0 \pitchfork^2 \mathrm{d}f$. Indeed, transversality is preserved by isomorphisms, and since $F_1 \pitchfork^2 \xi$ we obtain
$$F_s = (\widehat{F}_s \circ \widehat{F}_1^{-1}) \circ F_1 \pitchfork^2 (\widehat{F}_s \circ \widehat{F}_1^{-1})\xi = \xi_s.$$

Recall that singularities of tangency of F_s with ξ_s are encoded by the stratified submanifold
$$\Sigma(F_s,\xi_s) = \Sigma^1 \supset \Sigma^{11} \cdots \supset \Sigma^{1^n} \supset \varnothing,$$
where Σ^{1^i} has codimension-i in N. Note that (by construction of ξ_s) the singularity locus does not depend on s. Furthermore, along Σ we have a splitting
$$\xi_s|_\Sigma = \widehat{\xi}_s \oplus \langle V_s \rangle, \tag{15.2.1}$$

where $V_s \in TN|_\Sigma$ satisfies

(i) $F_s^{-1}(TN \cap \xi_s) = \mathrm{Span}(v)$;
(ii) Orientation determined by maslov coorientation.

Probably fine to just say that F_s maps v to V. Don't need to fix orientations so that V is defining the orientation or not.

Notation got mixed up? There is a difference between v and $V := F_1(v)$. The splitting of ξ is with respect to V not v.

Next we isotope N to have the correct singularities around Σ. This done using a double induction argument. First choose a partition $t_0 = 1 < t_2 < \cdots < t_k = 1$ of the interval $[0,1]$. If the partition is sufficiently fine then F_s, $s \in [t_i, t_{i+1}]$ is graphical over F_{t_i} for each $1 \leqslant i \leqslant k-1$. Moreover we can arrange the maximal angle between F_s and F_{t_i} to be as small as we want. Then the first induction statement is as follows:

Induction on time: Suppose there exists an isotopy f_s, $s \in [0, t_i]$ satisfying the conditions in the statement of Theorem 15.2. Furthermore, suppose that F_s, for $s \in [t_i, t_{i+1}]$ is graphical over F_{t_i} on $\mathcal{O}p(\Sigma)$. Then there is an extension f_s, for $s \in [0, t_{i+1}]$ with the same properties.

To prove the induction statement it suffices to construct the extension satisfying $|df_{t_{i+1}} - F_{t_{i+1}}|_{C^0} < \varepsilon$ on $\mathcal{O}p(\Sigma)$. Since the angle between $F_{t_{i+1}}$ and F_s, $s \in [t_{i+1}, t_{i+2}]$, is small the graphically condition then follows. The extension is constructed inductively on the strata of Σ. To be precise we prove:

For the induction step its important that $|\mathrm{d}f_1 - F_1|_{C^0} < \varepsilon$ on a neighborhood of $\mathcal{O}p(\Sigma^i)$. This follows from the proof. Away from Σ this is not true at all.

Lemma 15.3. *Consider $(f, F_s) : N \to (M, \xi)$ be a formally Σ^2-free submanifold such that:*

(i) $\Sigma^2(F_s, \xi) = \varnothing$ *for all* $s \in [0,1]$*;*
(ii) F_s*,* $s \in [0,1]$ *is graphical over* N*.*

Suppose that $\Sigma(f|_{\mathcal{O}p(\Sigma^{1^{i+1}})}) = \Sigma(F_1, \xi)$*. Then for any* $\varepsilon > 0$ *there exist an isotopy* $f_s : N \to M$ *satisfying:*

(i) $f_0 = f$*,* $|f_s - f_0|_{C^0} < \varepsilon$*;*
(ii) $f_s = f$ *on* $\mathcal{O}p(\Sigma^{i+1})$
(iii) $|\mathrm{d}f_s - F_s|_{C^0} < \varepsilon$ *on* $\mathcal{O}p(\Sigma^i)$*;*
(iv) $\Sigma(f_1) \cap \mathcal{O}p(\Sigma^{1^i}) = \Sigma^{1^i}$*;*

Before proving the lemma let us see how to complete the proof of the theorem. By the double induction procedure (and Lemma 11.3) we obtain a homotopy of formally Σ^2-free submanifolds $(f_t, F_{t,s})$, $s,t \in [0,1]$ such that $(f_0, F_{0,s}) = (f, F_s)$ and
$$\Sigma(f_1|_{\mathcal{O}p(\Sigma)}, \xi) = \Sigma(F_{1,1}, \xi).$$

We apply Theorem 9.1 to $(f_1, F_{1,s})$, relative to $\mathcal{O}p(\Sigma)$. This yields another isotopy, $\widetilde{f}_s$ satisfying:

(i) $\widetilde{f}_0 = f_1$, and $\widetilde{f}_s = f_1$, for all s on $\mathcal{O}p(\Sigma)$;
(ii) $|\widetilde{f}_s - f_1|_{C^0} < \varepsilon$.
(iii) $\widetilde{f}$ is a folded embedding (with respect to ξ) away from Σ.

Observe that since we apply Theorem 9.1 in the smooth case, the condition $|\mathrm{d}\widetilde{f}_1 - F_{1,1}|_{C^0} < \varepsilon$ does not hold close to the fold locus. Instead we have that $F_{1,1} \pitchfork \xi$ so that by the moreover part of Theorem 9.1 we can arrange the folds of $\widetilde{f}$ to be tangent to ξ. Moreover, away from the folds, $|\mathrm{d}\widetilde{f} - F_{1,1}| < \varepsilon$ which implies that $\widetilde{f}_1$ is transverse to ξ. Thus, up to additional fold singularities, the singularity loci of f_1 and $\widetilde{f}_1$ agree, concluding the proof.

Proof of Lemma 15.3. Using F_1 obtain a splitting

$$\xi|_\Sigma = \widehat{\xi} \oplus \langle v \rangle,$$

as in Equation 15.2.1. We can extend this splitting to an open neighborhood U of Σ in M. Furthermore, since $\widehat{\xi} \pitchfork N$ we may assume that it is contained in the fibers of the normal bundle $\pi : \mathcal{N} \to N$ to N in M.

Recall that v is transverse to $\Sigma^{1^i} \backslash \Sigma^{1^{i+1}}$. Hence we can thicken Σ^{1^i} to a hypersurface $\widehat{\Sigma}^{1^i} \subset U$ transverse to v. A neighborhood of $\widehat{\Sigma}^{1^i}$ can be identified with the total space of the bundle

$$\pi : \mathcal{N}|_{\widehat{\Sigma}^{1^i}} \times \mathbb{R} \to \widehat{\Sigma}^{1^i} \times \mathbb{R},$$

in which v is identified with the unit vector field on the $\mathbb{R}$-factor. Furthermore, these coordinates f is identified with a section (which by slight abuse of notation we still denote by) $f : \widehat{\Sigma}^{1^i} \times \mathbb{R} \to \mathcal{N}_{\widehat{\Sigma}^{1^i}} \times \mathbb{R}$.

Having the correct singularity should be made into a precise statement (in a lemma?). This should mean that you have a fold singularity wrt a fixed vector field direction and the correct corientation. Because this is already enough (by stability of folds etc) to conclude that the interpolation is through fold singularities. We dont need to actually have coordinates where the map looks like the standard model. Also for the parametric case this is better since obtaining models parametrically is problematic; there can be some noise which is small enough so that the interpolation is through folds.
Approximating F_s: By the graphicallity assymption F_1 defines a section σ of the bundle $J^1(\mathcal{N}|_{\widehat{\Sigma}^{1^i}} \times \mathbb{R})$. Furthermore, we may assume that $\sigma = j^1 f$ on the neighborhood $U := \mathcal{O}p(\Sigma^{1^{i+1}})$ where f is assumed to have the correct singularities. Indeed, let $\rho : N \to [0,1]$ be a bump function supported in U and equal to one on a smaller open $V \subset U$ containing $\Sigma^{1^{i+1}}$. Then,

$$\widetilde{F}_s := F_{(1-\rho)s},$$

agrees with F_s outside U, and proving the lemma for F_s is equivalent to proving it for $\widetilde{F}_s$.

Thus we can apply holonomic approximation (Theorem 5.2) to σ along the hypersurface $\widehat{\Sigma}^{1^i}$ relative to U. This provides a homotopy of sections $\sigma_t : \widehat{\Sigma}^{1^i} \times \mathbb{R} \to J^1(\mathcal{N}|_{\widehat{\Sigma}^i} \times \mathbb{R})$ whose front projection we denote by f_t, as well as a family of isotopies $\phi_t : \widehat{\Sigma}^{1^i} \times \mathbb{R} \to \widehat{\Sigma}^{1^i} \times \mathbb{R}$ satisfying:

(i) σ_1 is holonomic on $\mathcal{O}p(\phi_1(\widehat{\Sigma}^{1^i})$, i.e. $\sigma_1 = j^1 f_1$;
(ii) $|f_t - f|_{C^0} < \varepsilon$, $|df_1 - F_1|_\varepsilon$, $f_0 = f$ and $f_t = f$ on $\mathcal{O}p(\Sigma^{1^{i+1}})$;
(iii) $\phi_t(\widehat{\Sigma}^{1^i} \times \{0\})$ is transverse to v.

To extend $\widetilde{f}_t$ to N, note that we can make the identification

$$\mathcal{O}p(\phi_{s,1}(\widehat{\Sigma}^{1^i}) = \phi_{s,1}(\widehat{\Sigma}^{1^i}) \times (-\delta, \delta) \subset \widehat{\Sigma}^{1^i} \times \mathbb{R},$$

for $\delta > 0$ sufficiently small. Choose a smooth bump function $\rho : (-\delta, \delta) \to [0,1]$ which is 1 around zero and 0 near the boundary. Then the desired extension is given by:

$$f_s := \widetilde{f}_{\rho s}.$$

Placing singularities: By the previous step, we can assume that N has the correct singularities on $\mathcal{O}p(\Sigma^{i+1})$ and that Σ^i is transverse to ξ. It remains to place the correct singularities around Σ^i.

Recall from Section 4, that Σ^{1^n} can be realized by the Whitney singularities. More precisely write $(x, q) = (x_1, \dots, x_{n-1}, q)$ for the coordinates on $\mathbb{R}^n$ and consider the embedding:

$$\widetilde{\mathrm{Whit}}_n : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$$
$$(x, q) \mapsto (x, q^{n+1} + x_1 q^{n-1} + \dots + x_{n-1} q, q).$$

This map expresses the n-th Whitney singularity $\mathrm{Whit}_n : \mathbb{R}^n \to \mathbb{R}^n$ (see Section 4) as the singularity of tangency with respect to the fibers of the projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$. Recall from Remark ??that $\widetilde{\mathrm{Whit}}_n$ describes the birth/death of two copies of $\widetilde{\mathrm{Whit}}_{n-1}$. In particular, away from the origin $\widetilde{\mathrm{Whit}}_n$ is equivalent to the $\mathbb{R}$-stabilization of $\widetilde{\mathrm{Whit}}_{n-1}$.

Next we look for the correct coordinates in M to place the singularities. First observe that normal bundle of each connected component of $\Sigma^{1^i}\backslash\Sigma^{1^{i+1}}$ inside N is trivial. Indeed, this follows inductively since v is transverse to $\Sigma^{1^{i+1}}$ inside Σ^{1^i} (and since we can can always slightly extend the trivialization over a connected component of $\Sigma^{1^i}\backslash\Sigma^{1^{i+1}}$ over $\Sigma^{1^{i+1}}$). Thus a neighborhood of Σ^i in N can be identified with

$$\Sigma^i \times \mathbb{R} \times \mathbb{R}^{n-i-1},$$

with coordinates (p,q,x), where the q coordinate corresponds to the flow lines of v. In these coordinates we have $\widehat{\Sigma}^i = \{q=0\}$.

Similarly, inside M we know that Σ^i is transverse to $V = F_1(v)$. By assumption V is graphical over N (although we cannot assume it to be tangent). Hence we can choose the normal bundle $\mathcal{N}$ of N to be transverse to V. Thus we identify a neighborhood of Σ^i in M with the total space of the bundle

$$\pi : \mathcal{N}|_{\widehat{\Sigma}^i} \times \mathbb{R} \to \widehat{\Sigma}^i \times \mathbb{R}, \quad (p,q) \mapsto (\pi(p), q), \tag{15.2.2}$$

where $\pi : \mathcal{N} \to N$ is the usual projection, and the $\mathbb{R}$ factor in the total space (resp. the base) corresponds to integral curves of V (resp. v). As such we think of v as the projection of V onto N.

In these coordinates N is identified with the image of a section f, as constructed in the previous step. Moreover, we can choose a splitting $\mathcal{N} = \widehat{\mathcal{N}} \oplus \mathbb{R}$ such that f is identified with a map into the $\mathbb{R}$-factor. Now consider the section

$$f_1 : \Sigma^i \times \mathbb{R} \times \mathbb{R}^{i-1} \to \widehat{\mathcal{N}}|_{\Sigma^i} \oplus \mathbb{R} \times \Sigma^i \times \mathbb{R} \times \mathbb{R}^{i-1}$$
$$(p,q,x) \mapsto (0, \mathrm{Whit}_i(q,x)), p, q, x).$$

Note that this is just the Σ^{1^i}-stabilization of $\widetilde{\mathrm{Whit}}_i$. By the construction in the previous step, f_1 agrees with f on $\mathcal{O}p(\Sigma^{1^{i+1}})$ (possibly after a fiberwise change of coordinates of the bundle in Equation 15.2.2). Since both f and f_1 are sections, we can simply interpolate between them to obtain the desired isotopy of N. □

OTHER SECTIONS:

16. The integral Grassmannian

Let B and F be vector spaces of dimensions $n = \dim(B)$ and $k = \dim(F)$. We are interested in l-dimensional integral submanifolds of $(J^r(B,F), \xi_{\mathrm{can}})$. Our goal in this Section is to understand their linear counterpart, i.e. the corresponding integral elements.

We will do this step by step, looking first at the horizontal elements (Subsection 16.2), then at the elements that intersect the vertical distribution in a given dimension (Subsection 16.3), and finally at how these different pieces glue together (Subsections 16.4 and 16.5).

Let us provide some context about integral manifolds and integral elements: the first to regard general integral submanifolds of jet space as "generalised solutions" seems to have been R. Thom in [11], where he sketched the proof of his famous "homological h-principle". Later, A.M. Vinogradov brought attention to them, in the context of Geometry of PDEs, in [42]. Several works have followed in this direction [6, 7, 43].

It is within the Geometry of PDEs literature [27, 28] that the integral Grassmannian has been studied. As far as the we are aware, the majority of what is currently known can be found in the works of V. Lychagin [30, 29, 31, 32]. Despite containing beautiful results, these articles follow an announcement format and proofs are often missing or just outlined. One of our goals in this Section is to provide a detailed account of Lychagin's work. We will not attempt to discuss the relation with commutative algebras; this will be done in future work.

We note that our homotopy type computations for the integral Grassmannian in Subsection 16.5 seem to be new.

16.1. **Decomposing the integral Grassmannian.** Following subsection **??**, we identify the tangent space of $J^r(B,F)$ at any point with the vector space

$$\mathfrak{g} = B \oplus F \oplus \mathrm{Hom}(B,F) \oplus \mathrm{Sym}^2(B^*,F) \oplus \cdots \oplus \mathrm{Sym}^r(B^*,F).$$

In Definition 3.4 we endowed $\mathfrak{g}$ with a natural graded Lie algebra structure given by the contraction of vectors with tensors. We called this the jet Lie algebra with parameters n, k, and r. It was then proven in Proposition **??** that $\mathfrak{g}$ models the nilpotentisation of ξ_{can}. Under this isomorphism, integral elements (of a given dimension l) correspond to Lie subalgebras lying in the zero degree part

$$\mathfrak{g}_0 = B \oplus \mathrm{Sym}^r(B^*,F).$$

The space of integral elements is denoted by $\mathrm{Gr}_{\mathrm{integral}}(\mathfrak{g},l)$. It decomposes into several pieces, depending on how integral elements intersect the vertical component. We define:

$$\mathrm{Gr}_{\Sigma^i}(\mathfrak{g},l) := \{W \in \mathrm{Gr}_{\mathrm{integral}}(\mathfrak{g},l) \mid \dim(W \cap \mathrm{Sym}^r(B^*,F)) = i\},$$

where the subscript Σ^i is inspired by the Thom-Boardman notation.

The piece $\mathrm{Gr}_{\Sigma_0}(\mathfrak{g},l)$ is precisely the horizontal Grassmannian, as introduced in subsection 3.3. We also call it the **regular cell** even though it is, in general, not dense in $\mathrm{Gr}_{\mathrm{integral}}(\mathfrak{g},l)$. This is shown in subsection 16.3.3 below. We will describe the spaces $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g},l)$ in Subsections 16.2 and 16.3.

16.1.1. *The grassmannian of multi-sections.* In Section 6 we will introduce *multi-sections*, i.e. integral submanifolds that are horizontal in a dense set. These are submanifolds that one can manipulate through their front projection. Any integral element tangent to a multi-section must be in the closure $\overline{\mathrm{Gr}_{\Sigma^0}(\mathfrak{g},n)}$ of the horizontal elements; we call this space the **Grassmannian of multi-section elements**.

Furthermore, we are interested in multi-sections with mild singularities of tangency, which will be, in particular, of corank 1. Therefore, we content ourselves with describing how the two strata $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g},n)$ and $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g},n)$ glue together.

Definition 16.1. *The Σ^2**-free integral Grassmannian**, is the union*

$$\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g},n) := \mathrm{Gr}_{\Sigma^0}(\mathfrak{g},n) \cup \mathrm{Gr}_{\Sigma^1}(\mathfrak{g},n).$$

We will study its topology in Subsection 16.5.

We will study $\mathrm{Gr}_{\mathrm{integral}}(\mathfrak{g},l)$ as a whole in the future. In particular, in the present work we do not look at the closures $\overline{\mathrm{Gr}_{\Sigma^i}(\mathfrak{g},n)}$ with $i > 1$.

16.2. **Horizontal elements.** We now prove Lemma 16.3: the Grassmannians of horizontal elements are vector bundles with (standard) Grassmannian base. This description appeared already in the recent work [7].

16.2.1. *Maximal horizontal elements.* A maximal horizontal element W is graphical over B. We can represent it (uniquely) as the graph of a homomorphism $A \in \mathrm{Hom}(B, \mathrm{Sym}^r(B^*,F))$. Then:

Lemma 16.2. *Let $W = \mathrm{graph}(A)$ be a n-dimensional subspace of $\mathfrak{g}_0$ graphical over B. Then, W is integral if and only if $A \in \mathrm{Sym}^{r+1}(B^*,F)$.*

Proof. The Lie subalgebra condition for W means that for any pair $w_0 + A(w_0), w_1 + A(w_1) \in W$ we have:

$$0 = [w_0 + A(w_0), w_1 + A(w_1)] = \iota_{w_0}A(w_1) - \iota_{w_1}A(w_0)$$

which implies that A is symmetric with respect to the first variable as well. The claim follows. □

This Lemma realises the correspondence between horizontal elements at a point $p \in J^r(Y \to X)$ and points in the fibre of $J^{r+1}(Y \to X)$ over p.

16.2.2. *General dimension.* More generally, if W is horizontal and of dimension $l \leqslant n$, it projects down to some l-dimensional subspace $H \subset B$, defining a map

$$\pi_b : \mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, l) \to \mathrm{Gr}(B, l)$$

to the l-Grassmannian of the base. We claim that this is a vector bundle which can be explicitly described in terms of the tautological bundle γ over $\mathrm{Gr}(B, l)$.

Lemma 16.3. *There is a canonical isomorphism of vector bundles over* $\mathrm{Gr}(B, l)$*:*

$$\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, l) \cong \frac{\mathrm{Sym}^{r+1}(B^*, F)}{\mathrm{Sym}^{r+1}(\gamma^\perp, F)},$$

where $\gamma^\perp$ is the annihilator of the tautological bundle γ.

Proof. We look at all the graphical l-subspaces in $\mathfrak{g}_0$, not necessarily integral: given $H \subset B$, its possible lifts correspond to the elements of $\mathrm{Hom}(H, \mathrm{Sym}^r(B^*, F))$. Packaged all together, for varying H, they are elements of the total space of the vector bundle:

$$\mathrm{Hom}(\gamma, \mathrm{Sym}^r(B^*, F)) \to \mathrm{Gr}(B, l).$$

We want to determine which of these are horizontal.

To do so, we use the auxiliary trivial vector bundle $\underline{\mathrm{Sym}^{r+1}(B^*, F)} \to \mathrm{Gr}(B, l)$. We look at the bundle map given by evaluation on each l-subspace:

$$\mathrm{ev}_\gamma : \underline{\mathrm{Sym}^{r+1}(B^*, F)} \subset \underline{\mathrm{Hom}(V, \mathrm{Sym}^r(B^*, F))} \mapsto \mathrm{Hom}(\gamma, \mathrm{Sym}^r(B^*, F)).$$

The image of this map is necessarily contained in $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, l)$. We claim that the map is an epimorphism: this follows from the fact that any horizontal W, projecting to $H \subset B$, may be extended to a maximal horizontal element by direct summing with the complement of H in B.

The kernel of ev_γ is, by definition, the subspace of those elements of $\mathrm{Sym}^{r+1}(B^*, F)$ which vanish when a vector in γ is plugged in. By symmetry, we deduce that there is a exact sequence

$$0 \to \mathrm{Sym}^{r+1}(\gamma^\perp, F) \to \underline{\mathrm{Sym}^{r+1}(B^*, F)} \to \mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, l) \to 0$$

of vector bundles, proving the claim. □

16.2.3. *The subspace filtration.* Let $H \subset B$ be a linear subspace. In the proof above we looked at those elements in $\mathrm{Sym}^{r+1}(B^*, F)$ which vanish when an element of H is plugged in. One can, more generally, consider those tensors that vanish when a collection of elements in H is used. This leads us to define the following filtration:

$$\mathrm{Sym}^{r+1}(B^*, F)^{(H,j)} := \{A \in \mathrm{Sym}^{r+1}(B^*, F) \mid \iota_{v_j} \cdots \iota_{v_1} A = 0, \text{ for any } v_i \in H\},$$

$$\cdots \subset \mathrm{Sym}^{r+1}(B^*, F)^{(H,j)} \subset \mathrm{Sym}^{r+1}(B^*, F)^{(H,j+1)} \subset \ldots$$

By the discussion in the previous subsection, we have that

$$\mathrm{Sym}^{r+1}(B^*, F)^{(H,1)} = \mathrm{Sym}^{r+1}(H^\perp, F).$$

In general, by choosing a direct summand of H, we can identify:

$$\frac{\mathrm{Sym}^{r+1}(B^*, F)^{(H,j)}}{\mathrm{Sym}^{r+1}(B^*, F)^{(H,j-1)}} \cong \mathrm{Sym}^{j-1}(H^*, F) \otimes \mathrm{Sym}^{r+2-j}(H^\perp, F).$$

yielding the dimension formula:

$$\dim \left(\frac{\mathrm{Sym}^{r+1}(B^*, F)^{(H,j)}}{\mathrm{Sym}^{r+1}(B^*, F)^{(H,j-1)}} \right) = k \binom{n+j-2}{n-1} \binom{n+r+1-j}{n-1}.$$

In Subsection 16.4 we will study the *principal cone* in $\mathrm{Sym}^{r+1}(B^*, F)$, i.e. the space of tensors A of the form $A \in \mathrm{Sym}^{r+1}(H^\perp, F)$, for some $H \subset B$.

16.2.4. *Aside: the conormal.* We finish this Subsection presenting the *conormal construction*. Given a horizontal submanifold of $J^r(Y \to X)$, it produces a maximal integral submanifold containing it. This will not be needed later on, but it helps us emphasise that maximal integral submanifolds are often exotic looking (compared to those integral submanifolds that are almost everywhere horizontal).

We first present the linear analogue of this phenomenon:

Definition 16.4. *Let $W \subset \mathfrak{g}_0$ be l-dimensional and horizontal. Denoting its projection to B by H, we define the* ***conormal*** *of W to be the subspace:*

$$\mathrm{conormal}(W) := W \oplus \mathrm{Sym}^r(H^\perp, F) \subset \mathfrak{g}_0.$$

The space $\mathrm{Sym}^r(H^\perp, F)$ is the intersection of the polar space of H with the vertical component. Hence, the conormal is a maximal integral element.

In the contact case, $\mathrm{conormal}(W)$ is middle-dimensional and therefore a lagrangian subspace of ξ_{can}. In the general case, $\mathrm{Sym}^r(H^\perp, F)$ has dimension $k\binom{(n-l)+r-1}{n-l-1}$ which is often (much) larger than $n-l$. For instance:

- If $l = n-1$, we have $\dim(\mathrm{Sym}^r(H^\perp, F)) = k$.
- If $l = n-2$, we have $\dim(\mathrm{Sym}^r(H^\perp, F)) = k(r+1)$.
- If $l = n-3$, we have $\dim(\mathrm{Sym}^r(H^\perp, F)) = k\frac{(r+2)(r+1)}{2}$.

Therefore, the conormal construction produces integral elements which are tangent to the fibre along a large subspace, and whose dimension is often much larger than n.

Now for the manifold version:

Definition 16.5. *Let $N \subset J^r(Y \to X)$ be a l-dimensional, integral submanifold with immersed projection $\pi_b(N) \subset X$. We define its* ***conormal*** *to be the manifold:*

$$\mathrm{conormal}(N) := \{p \in J^r(Y \to X) \mid \pi_{r,r-1}(p) \in \pi_{r,r-1}(N),\, p \supset T_{\pi_{(r,r-1}(p)}\pi_{r,r-1}(N)\}.$$

In the last inclusion we think of $p \in J^r(Y \to X)$ as a maximal horizontal element in $\pi_{r,r-1}(p) \in J^{r-1}(Y \to X)$.

To see how this corresponds to the linear version, we choose a trivialisation so we may work with $J^r(B, F)$, where B and F are vector spaces. Then the conormal is precisely the space

$$\{p \in J^r(B, F) \mid \pi_{r,r-1}(p) \in \pi_{r,r-1}(N),\, p \in \mathrm{conormal}(T_{\pi_{r,r-1}(p)}\pi_{r,r-1}(N))\}.$$

Here we use the fact that both the base B and the fibre F are vector spaces to canonically identify the fibre of r-jet space with $\mathrm{Sym}^r(B^*, F)$ and therefore invoke the linear definition.

16.3. **Integral elements of given corank.** Having understood the horizontal case (which we will have to invoke repeatedly), we may look now at more general integral elements. Namely, those intersecting the vertical component in a subspace of dimension i.

16.3.1. *The setup.* The space $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)$ is endowed with two canonical maps. The first is simply the restriction of the base projection; we denote it by:

$$\pi_b : \mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l) \mapsto \mathrm{Gr}(B, l-i).$$

The second one intersects an integral element with the vertical component. We write:

$$\cap \mathrm{Sym}^r(B^*, F) : \mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l) \to \mathrm{Gr}(\mathrm{Sym}^r(B^*, F), i).$$

Given $W \in \mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)$, the subspaces $H = \pi_b(W)$ and $W_v = W \cap \mathrm{Sym}^r(B^*, F)$ must be orthogonal with respect to the curvature/Lie bracket. This means that W_v must be, in fact, an element of $\mathrm{Gr}(\mathrm{Sym}^r(H^\perp, F), i)$. Reasoning in this fashion for all W simultaneously leads us to look at the total space of the bundle $\mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), i) \to \mathrm{Gr}(B, l-i)$. We write ν for the tautological bundle over it.

The two canonical maps defined above yield a projection $\pi : \mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l) \to \mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), i)$. It is immediate that π is a vector bundle in which a natural choice of zero section is:

$$(H, W_v) \to H \oplus W_v, \tag{16.3.1}$$

where $H \in \mathrm{Gr}(B, l-i)$ and $W_v \in \mathrm{Gr}(\mathrm{Sym}^r(H^\perp, F), i)$.

16.3.2. *The result.* We may describe $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)$ explicitly:

Lemma 16.6. *There is a canonical isomorphism of vector bundles:*

$$\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l) \cong \frac{\mathrm{Sym}^{r+1}(B^*, F)}{\mathrm{Sym}^{r+1}(\gamma^\perp, F) \oplus \mathrm{Hom}(\gamma, \nu)}$$

over the total space of $\mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), i) \to \mathrm{Gr}(B, l-i)$.

Proof. As before denote by $\underline{\mathrm{Sym}^{r+1}(B^*, F)} \to \mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), i)$ the trivial vectorbundle, with fiber $\mathrm{Sym}^{r+1}(B^*, F)$. We define a vector bundle epimorphism

$$\oplus : \underline{\mathrm{Sym}^{r+1}(B^*, F)} \mapsto \mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)$$

which, at a point $W_v \in \mathrm{Gr}(\mathrm{Sym}^r(H^\perp, F), i)$, is given by

$$A \quad \mapsto \quad \oplus_{H,W_v}(A) := \mathrm{graph}(A|_H) \oplus W_v.$$

The tensor A is in the kernel of $\oplus_{H,W_v}$ (i.e. gets mapped to the zero section from Equation 16.3.1) if and only if the associated quotient map

$$\tilde{A} : H \mapsto \mathrm{Sym}^r(B^*, F)/W_v$$

is zero. I.e. $\iota_v A \in W_v$ for every $v \in H$. Therefore, after choosing a direct summand for H, we can identify:

$$\ker(\oplus_{H,W_v}) \cong \mathrm{Sym}^{r+1}(H^\perp, F) \oplus \mathrm{Hom}(H, W_v),$$

which is a vector subspace of $\mathrm{Sym}^{r+1}(B^*, F)^{(H,2)} \cong \mathrm{Sym}^{r+1}(H^\perp, F) \oplus \mathrm{Hom}(H, \mathrm{Sym}^r(H^\perp, F))$. □

16.3.3. *Dimension counting.* From the previous proof, we deduce that:

Corollary 16.7. *The fibre of* $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)$*, as a vector bundle over* $\mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), i)$*, has dimension*

$$\left[\binom{n+r}{n-1} - \binom{n-l+i+r}{n-l+i-1}\right] k - i(l-i).$$

Similarly, we deduce:

Corollary 16.8. *The manifold* $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)$ *has dimension*

$$\begin{aligned}\dim(\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)) = \quad & (l-i)(n-l+i) + \\ & \left[\binom{r+(n-l+i)-1}{n-l+i-1} k - i\right] i + \\ & \left[\binom{n+r}{n-1} - \binom{n-l+i+r}{n-l+i-1}\right] k - i(l-i).\end{aligned}$$

Proof. The space $\mathrm{Gr}(B, l-i)$ has dimension $(l-i)(n-l+i)$. The fibre of $\mathrm{Sym}^r(\gamma^\perp, F)$ has dimension $\binom{r+(n-l+i)-1}{n-l+i-1}$, so it follows that the fibre of $\mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), i)$ has dimension:

$$\left[\binom{r+(n-l+i)-1}{n-l+i-1} k - i\right] i.$$

Putting all these computations together, we deduce the claim. □

We are particularly interested in comparing $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)$ with the regular cell $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, l)$, which we want to regard as the "generic" ones. To do so with define a number, which we call the **codimension**, as follows:

$$\mathrm{codim}(r, n, k, l, i) := \dim(\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, l)) - \dim(\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)).$$

We particularise to the case $n = l$ and we compute:

$$\begin{aligned} \mathrm{codim}(r, n, k, n, i) &= i^2 + kr\frac{1-i}{1+r}\binom{r+i-1}{i-1}, \\ \mathrm{codim}(r, n, k, n, 1) &= 1, \\ \mathrm{codim}(r, n, k, n, 2) &= 4 - kr, \\ \mathrm{codim}(r, n, k, n, 3) &= 9 - kr(r+2). \end{aligned}$$

So we deduce:

Corollary 16.9. *The space* $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ *has codimension* 1 *in* $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)$.

In the contact setting $k = r = 1$, *the space* $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, n)$ *has codimension* $\frac{i(i+1)}{2}$ *in* $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)$.

That is: with the exception of a few cases in which r and k are small, the strata $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, n)$, $i > 1$, are often larger than the regular cell.

The most interesting component, from a PDE perspective, is the closure $\overline{\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, l)}$ of the horizontal cell. We will not attempt to look at it in depth. As pointed out in the introduction, it is enough that we understand how $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, l)$ sits inside; we will do so in Subsection 16.5.

16.4. **Principal subspaces.** It is convenient that we introduce some auxiliary concepts before we look at $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, l) \subset \overline{\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, l)}$. The main definition of interest in this Subsection is:

Definition 16.10. *A horizontal element* $A \in \mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n) \cong \mathrm{Sym}^{r+1}(B^*, F)$ *is* ***principal*** *if*

$$A = f^{r+1} \otimes \alpha,$$

for some (unique) $f \in B^*$ *and* $\alpha \in F$. *The span of a principal element is said to be a* ***principal subspace***.

Any non-zero principal element defines a kernel subspace $\ker(A) := \ker(f) \subset B$ which is of codimension 1, and an image subspace $\mathrm{Image}(A) \subset \mathrm{Sym}^r(B^*, F)$ which is by definition the 1-dimensional space spanned by $f^r \otimes \alpha$.

Remark 16.11. *As points in* $(r+1)$*-jet space, principal elements correspond precisely to* pure derivatives *(i.e. derivatives of order* $r + 1$ *along a single direction in the base).*

16.4.1. *The principal cone.* We claim that the set of all principal subspaces in $\mathrm{Sym}^{r+1}(B^*, F)$ is the cone of an algebraic subvariety in the projectivisation. Let us recall two constructions from classic algebraic geometry.

Let V and W be vector spaces. We define the **Veronese mapping**:

$$\begin{aligned} \mathbb{P}(V) &\mapsto \mathbb{P}(\mathrm{Sym}^{r+1}(V)), \\ [v] &\mapsto [v^{r+1}]. \end{aligned}$$

Similarly, the **Segre mapping** is defined by the expression:

$$\begin{aligned} \mathbb{P}(V) \times \mathbb{P}(W) &\mapsto \mathbb{P}(V \otimes W), \\ ([v], [w]) &\mapsto [v \otimes w]. \end{aligned}$$

Both of them are algebraic maps.

In our setting, we can put them together to define the **principal mapping**:

$$\begin{aligned} \mathbb{P}(B^*) \times \mathbb{P}(F) &\mapsto \mathbb{P}(\mathrm{Sym}^{r+1}(B^*, F)), \\ ([f], [\alpha]) &\mapsto [f^{r+1} \otimes \alpha]. \end{aligned}$$

We are interested in the cone it defines. It is given by the image of the map:

$$\begin{aligned} B^* \times F &\mapsto \mathrm{Sym}^{r+1}(B^*, F), \\ (f, \alpha) &\mapsto f^{r+1} \otimes \alpha. \end{aligned}$$

We will abuse notation and still call this map the *principal mapping*, as long as no confusion may arise. Its image, which we denote by $\mathcal{V}_0$ and we call the **principal cone**, is an algebraic subvariety. By construction, a horizontal element is principal if and only if it is contained in $\mathcal{V}_0$.

16.4.2. *The closure of the principal cone.* Fix $A_0, A_1 \in \mathrm{Sym}^{r+1}(B^*, F)$, with A_1 principal, and consider the linear combinations $(A_0 + sA_1)_{s\in\mathbb{R}}$. We can see that

$$(A_0 + sA_1)|_{\ker(A_1)} = A_0|_{\ker(A_1)},$$

i.e. the graph over $\ker(A_1)$ does not depend on s. However, $A_0 + sA_1$ explodes in the complement of $\ker(A_1)$ as s goes to infinity. This implies that the sequence of horizontal elements $(A_0 + sA_1)_{s\in\mathbb{R}}$ has well-defined limit in $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$: the integral element

$$\mathrm{graph}(A_0|_{\ker(A_1)}) \oplus \mathrm{Image}(A_1).$$

In terms of r-jet space, this phenomenon corresponds to an explosion of a pure derivative of order $r+1$. Any element in $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ may be written as such a limit, so we deduce:

Lemma 16.12. $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ *is contained in the closure of* $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)$.

Applying this reasoning with $A_0 = 0$, we are effectively looking at the closure $\mathcal{V} := \overline{\mathcal{V}_0}$ in $\overline{\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)}$ of the principal cone:

Lemma 16.13. *The **principal subvariety** $\mathcal{V}$ is the union of two pieces $\mathcal{V}_0$ and $\mathcal{V}_1$. The latter piece is the zero section of* $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ *as a bundle over* $\mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), 1) \to \mathrm{Gr}(B, n-1)$.

Proof. Any element in the closure of $\mathcal{V}_0$ can be realised as the limit of a path $(sA)_{s\in\mathbb{R}}$, with A principal. As reasoned above, its limit is then the direct sum $\ker(A) \oplus \mathrm{Image}(A)$, where the first term is a hyperplane in B and the second one is a line in $\mathrm{Sym}^r(\ker(A)^\perp, F)$. This concludes the claim. □

Lastly, we remark that $\mathcal{V}_1 = \mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), 1)$, as a bundle over $\mathrm{Gr}(B, n-1)$, is trivial. Indeed, an element in the fibre is a line in $\mathrm{Sym}^r(\gamma^\perp, F)$, which can be uniquely identified with its image in F, which is again a line. This shows that:

Corollary 16.14. *There is an identification*

$$\mathcal{V}_1 = \mathrm{Gr}(B, n-1) \times \mathrm{Gr}(F, 1) = \mathbb{P}(B^*) \times \mathbb{P}(F).$$

16.4.3. *The topology of the principal subvariety.* We want to determine the homotopy type of $\mathcal{V}$ by putting its pieces together. This is relevant because, as we will see in Subsection 16.5.2, $\mathcal{V}$ is homotopy equivalent to $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$.

Let us make a preliminary remark. We write $\widetilde{\mathcal{V}}$ for be the blow-up of $\mathcal{V}$ at the origin. We denote the tautological bundles over $\mathbb{P}(B^*)$ and $\mathbb{P}(F)$ by γ_{B^*} and γ_F, respectively. We then look at the forgetful map

$$\widetilde{\mathcal{V}} \to \mathbb{P}(B^*) \times \mathbb{P}(F).$$

One can check that it is a fibration with $\mathbb{RP}^1$ fibres and, in fact, it is the fibrewise compactification of the real line bundle $\gamma_{B^*}^{\otimes r+1} \otimes \gamma_F$. From this expression we see that there is a certain asymmetry depending on the parity of r, so we must tackle each case separately.

Write $\widehat{B^*} \cong \mathbb{RP}^n$ for the compactification of B^* by adding $\mathbb{P}(B^*)$ at infinity. Denote by $\mathbb{S}(F)$ the unit sphere (with respect to some scalar product). Then:

Lemma 16.15. *Let r be even. Then, there is a fibration*

$$\mathbb{Z}_2 \to \widehat{B^*} \times \mathbb{S}(F) \to \mathcal{V}.$$

In particular, if $k = \dim(F) = 1$, we have that $\mathcal{V}$ is homotopy equivalent to $\widehat{B^*} \cong \mathbb{RP}^n$.

Proof. We define maps

$$\begin{array}{rcl} B^* \times \mathbb{S}(F) & \mapsto & \mathcal{V}_0, \\ (f,\alpha) & \mapsto & f^{r+1} \otimes \alpha; \\ \mathbb{P}(B^*) \times \mathbb{S}(F) & \mapsto & \mathcal{V}_1, \\ ([f],\alpha) & \mapsto & ([f],[\alpha]). \end{array}$$

Their composition defines a continuous map $\widehat{B^*} \times \mathbb{S}(F) \mapsto \mathcal{V}$, as claimed. For the second claim we note that the bundle is trivial because $\mathbb{S}(F) = \mathbb{Z}_2$. □

Similarly:

Lemma 16.16. *Let r be odd. Then $\mathcal{V}$ is homotopy equivalent to the quotient*

$$\frac{\mathbb{P}(B^*) \times \widehat{F}}{\mathbb{P}(B^*) \times 0}.$$

Proof. Regard $\mathbb{P}(B^*)$ as the quotient of the unit sphere (for some scalar product) under the antipodal map. Consider the map:

$$\begin{array}{rcl} \mathbb{P}(B^*) \times F & \mapsto & \mathcal{V}_0, \\ ([f],\alpha) & \mapsto & f^{r+1} \otimes \alpha, \end{array}$$

which is well-defined because r is odd. Together with the identity map $\mathbb{P}(B^*) \times \mathbb{P}(F) \mapsto \mathcal{V}_1$, this defines a mapping

$$\mathbb{P}(B^*) \times \widehat{F} \mapsto \mathcal{V}$$

which is surjective, maps $\mathbb{P}(B^*) \times \{0\}$ to the origin in $\mathcal{V}$, and is a homeomorphism in the complement; quotienting we deduce the claim. □

16.4.4. *The tangent variety of the principal cone.* Lastly, being a subvariety of a vector space, we can look at the tangent variety $T\mathcal{V}_0 \subset \mathrm{Sym}^{r+1}(B^*, F)$ associated to $\mathcal{V}_0$.

To determine $T\mathcal{V}_0$, we look at the map $\psi(f,\alpha) = f^{r+1} \otimes \alpha$. Its differential at a covector $f \in B^*$ and a vector $\alpha \in F$ is readily computed:

$$d_{f,\alpha}\psi : B^* \times F \to \mathrm{Sym}^{r+1}(B^*, F),$$

$$d_{f,\alpha}\psi(g,\beta) = f^{r+1} \otimes (\alpha + \beta) + (r+1) g \cdot f^r \otimes \alpha.$$

Equivalently, if we set $H = \ker(f) \subset B$, we see that the tangent space to $\mathcal{V}_0$ at $f^{r+1} \otimes \alpha \neq 0$ is the subspace:

$$\mathrm{Sym}^{r+1}(H^\perp, F) \oplus H^* \otimes \mathrm{Sym}^r(H^\perp, \langle\alpha\rangle) \subset \mathrm{Sym}^{r+1}(B^*, F)^{(H,2)}.$$

This identifies the normal fibre to $\mathcal{V}_0$ at (f,α) with the quotient

$$\frac{\mathrm{Sym}^{r+1}(B^*, F)}{\mathrm{Sym}^{r+1}(H^\perp, F) \oplus H^* \otimes \mathrm{Sym}^r(H^\perp, \langle\alpha\rangle)},$$

as we would expect from our description of $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ as a bundle over $\mathcal{V}_1$.

16.5. **The Σ^2-free integral Grassmannian.** In this last Subsection we state some structural results about $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$ and we provide sketches of proofs. A more comprehensive account will appear in future work.

16.5.1. *Smoothness.* According to Subsections 16.2 and 16.3, the pieces $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)$ and $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ are smooth manifolds. The first is a vector space. The second one is a vector bundle over a smooth bundle with grassmannian base and fibre. The computations in subsection 16.3.3 show that the later has dimension one less than the former. One can put together these facts to show:

Proposition 16.17. $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$ *is a smooth open manifold, embedded in* $\mathrm{Gr}(\mathfrak{g}_0, n)$*. Furthermore,* $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ *sits inside as a smooth hypersurface.*

Proof. It is sufficient to describe, at each point $W \in \mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$, a chart that is simultaneously a submanifold chart of $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$ inside of $\mathrm{Gr}(\mathfrak{g}_0, n)$ and a submanifold chart of $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ inside $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$. We will just provide the latter.

Let W be presented as $\lim_{s\to\pm\infty} \mathrm{graph}(A_0 + sA_1)$, with $A_0, A_1 \in \mathrm{Sym}^{r+1}(B^*, F)$ and A_1 principal. We write L for a neighbourhood of A_0 within the normal fibre to the principal cone at A_0. Additionally, we fix a $(n+k-1)$-dimensional family U of rank-1 maps whose projectivisations are a neighbourhood of $[A_1]$ in the space of principal subspaces.

Then, the map

$$\begin{aligned} \Phi : L \times U \times (-\delta, \delta) &\to \mathrm{Gr}(\mathfrak{g}, n) \\ (A, A', s) &\to A + \frac{1}{s}A' \end{aligned}$$

is a smooth embedding with image a neighbourhood of W in $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$. Further, the map $\Phi|_{U\times L\times\{0\}}$ parametrises the hypersurface $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$. ☐

We remark that we do not know whether $\overline{\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)}$ is smooth in general. In the contact case it is known that it is.

16.5.2. *Homotopy type.* We can put together Proposition 16.17 with the work we did in the previous Subsection about the principal subvariety to show that:

Proposition 16.18. *The* Σ^2*-free Grassmannian* $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$ *is homotopy equivalent to the principal subvariety* $\mathcal{V}$*.*

Proof. We just provide a sketch of proof.

Let us fix a metric in $\mathfrak{g}_0$ making the horizontal and vertical components orthogonal. This immediately defines a distance function $\measuredangle$ between lines in $\mathfrak{g}_0$, given as the sine squared of the angle they make. We can readily extend this function to $\mathrm{Gr}(\mathfrak{g}_0, n)$ as follows:

$$\measuredangle(A, A') := \max_{L\subset A, L'\subset A'} \measuredangle(L, L').$$

We restrict $\measuredangle$ to $\overline{\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)}$.

Note that the horizontal cell $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)$ is the set of points at distance strictly less that 1 from the zero map. Similarly, $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$ is the set at distance strictly less than 1 from $\mathcal{V}$. We may then define the distance function

$$\begin{aligned} d : \mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n) &\to [0, 1) \\ d(A) &:= \inf_{B\in\mathcal{V}} \measuredangle(A, B), \end{aligned}$$

whose zero set is $\mathcal{V}$.

The function d is smooth. It can be seen that its restriction to $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ is Morse-Bott and its critical set is precisely $\mathcal{V}_1$. The situation in $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)$ is more delicate because d is not Morse-Bott: its zero locus is the principal cone, which is singular, and the additional critical points (corresponding to the cut locus of d) form a conical algebraic subvariety S.

We may then proceed as follows: we modify d by adding a perturbation $h(A) = |A|^2\rho$; here $\rho : \mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n) \to \mathbb{R}$ is a bump function supported in the intersection of a neighbourhood of S and the complement of a ball around zero. In particular, this perturbation is zero in the hypersurface $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$. The effect of this is that minus the gradient flow of $d + h$ retracts everything to a neighbourhood of $\mathcal{V}$, which itself retracts onto $\mathcal{V}$. ☐

16.5.3. *The Maslov hypersurface.* In the Lagrangian Grassmannian, the complement of the regular cell is usually called the *Maslov cycle*. As studied by V. Maslov and V. Arnol'd [33, 3], it is a two-sided (i.e. cooriented) and non-separating hypersurface and, it defines a first homology class through the intersection pairing. Let us study this phenomenon in general jet spaces. We will henceforth denote:

Definition 16.19. $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n) \subset \mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$ *is called the **Maslov hypersurface**.*

The Maslov hypersurface is non-separating in general. Furthermore:

Proposition 16.20. *The Maslov hypersurface is two-sided if and only if one of the following conditions holds:*

- $\dim(F) = 1$ *and* r *is odd, or*
- $\dim(B) = \dim(F) = 1$.

These are not mutually exclusive.

Proof. According to Proposition 16.18, it is sufficient that we prove that $\mathcal{V}_1$ is coorientable within $\mathcal{V}$. Then, we refer back to subsection 16.4.3, where it was explained that $\widetilde{\mathcal{V}}$ (the blow-up at the origin of $\mathcal{V}$) is the fibrewise compactification of the tautological bundle $\gamma_{B^*}^{\otimes r+1} \otimes \gamma_F$ over $\mathbb{P}(B^*) \times \mathbb{P}(F)$. Here the zero section corresponds to the blow-up of the origin and the infinity section is precisely $\mathcal{V}_1$, but their roles are symmetric.

Now we observe that $\gamma_{B^*}^{\otimes r+1} \otimes \gamma_F$ is isomorphic to the normal bundle of $\mathcal{V}_1$ in $\widetilde{\mathcal{V}}$, and therefore isomorphic to the normal bundle of $\mathcal{V}_1$ in $\mathcal{V}$. Furthermore, this bundle is trivial if and only if the terms γ_F and $\gamma_{B^*}^{\otimes r+1}$ are individually trivial. This proves the claim. □

Furthermore:

Corollary 16.21. *Let* $\dim(F) = 1$ *and* r *be odd. Then a choice of orientation for* F *determines a coorientation for the Maslov hypersurface.*

Proof. Indeed, as computed in the proof of Proposition 16.20, the normal bundle to $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ is precisely γ_F, which is canonically identified with F. □

Similarly:

Corollary 16.22. *Let* $\dim(B) = \dim(F) = 1$ *with* r *even. Then, a choice of orientation for* $B^* \oplus F$ *determines a coorientation for the Maslov hypersurface.*

Proof. The normal bundle to $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ is $\gamma_{B^*} \otimes \gamma_F$, which is identified with $\det(B^* \oplus F)$. □

In both cases, once we have oriented either F or $B^* \oplus F$, we will call the resulting coorientation the **Maslov coorientation**.

16.5.4. *The Maslov class.* Under the assumptions of Proposition 16.20, the Maslov hypersurface is non-separating, cooriented, and closed as a subset. This is enough to have a well-defined cohomology class using the intersection pairing:

Definition 16.23. *Suppose one of the following conditions holds:*

- $\dim(F) = 1$ *and* r *is odd, or*
- $\dim(B) = \dim(F) = 1$,

and that a Maslov coorientation has been fixed.

*Then, the **Maslov index** or **Maslov class** is the non-zero, non-torsion element*

$$\mathrm{Ind} \in H^1(\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n), \mathbb{Z})$$

defined by:

$$\mathrm{Ind}([\gamma]) := |\gamma \cap \mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)| \in \mathbb{Z} \quad ,$$

where γ is a curve representative intersecting the Maslov hypersurface transversally. The count of intersection points takes into account signs, comparing the orientation of γ with the Maslov coorientation.

16.5.5. *The local Maslov class.* Even if the Maslov hypersurface is not two-sided, it still makes sense to talk about a **local Maslov coorientation**: indeed, let $W \in \mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ and consider a ball $U \subset \mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$ containing W. In U, the intersection $U \cap \mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ is two-sided, so a coorientation can be chosen.

Given a local Maslov coorientation for $U \cap \mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$, we can reason as before to define a **local Maslov class** for oriented curves

$$([0,1], \{0,1\}) \to (U, (\partial U) \backslash (U \cap \mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)))$$

using the intersection pairing. It can only take the values $\{0, 1, -1\}$.

This will play a role in Subsection 7.2.

References

[1] D. Alvarez-Gavela. The simplification of singularities of lagrangian and legendrian fronts. *Inventiones Mathematicae*, 214(2):641–737, 2016.

[2] D. Alvarez-Gavela. Refinements of the holonomic approximation lemma. *Algebr. Geom. Topol.*, 18(4):2265–2303, 2018.

[3] V. Arnol'd. Characteristic class entering in quantization conditions. *Funct. Anal. Appl.*, 1(1):11–13, 1967.

[4] V. Arnol'd. Normal forms of functions near degenerate critical points, Weyl groups a_k, d_k, e_k, and lagrangian singularities. *Funct. Anal. Appl.*, 6(4):3–25, 1972.

[5] V. Arnol'd. Lagrangian manifolds with singularities, asymptotical rays and open swallowtail. *Funct. Anal. Appl.*, 15(4):1–14, 1981.

[6] M. Bachtold. *Fold-type solution singularities and characteristic varieties of nonlinear PDEs*.

[7] M. Bachtold and G. Moreno. Remarks on non-maximal integral elements of the Cartan plane in jet spaces. *Journal of Geometry and Physics*, 85:185–195, 2014.

[8] J. Boardman. Singularities of differentiable maps. *Publ. Math. IHES*, 33:383–419, 1967.

[9] M. Borman, Y. Eliashberg, and E. Murphy. Existence and classification of overtwisted contact structures in all dimensions. *Acta Math.*, 215(2):281–361, 2015.

[10] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths. *Exterior differential systems*, volume 18 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1991.

[11] E. Cartan. Remarques sur les problèmes comportant des inéquations différentielles globales. *Bull. Soc. Math. France*, 87:455–461, 1959.

[12] R. Casals and A. del Pino. Classification of Engel knots. *Math. Ann.*, 210(1-2):391–404, 2018.

[13] A. del Pino and F. Presas. Flexibility for tangent and transverse immersions in Engel manifolds. *Rev. Mat. Comp.*, 32(1):215–238, 2019.

[14] A. del Pino and T. Vogel. The Engel-Lutz twist and overtwisted Engel structures. *Submitted*, 2017.

[15] Y. Eliashberg. Surgery of singularities of smooth mappings. *Izv. Akad. Nauk. SSSR Ser. Mat.*, 36:1321–1347, 1972.

[16] Y. Eliashberg, S. Galatius, and N. Mishachev. Madsen–Weiss for geometrically minded topologists. *Geom. Topol.*, 15(1):411–472, 2011.

[17] Y. Eliashberg and N. Mishachev. Wrinkling of smooth mappings and its applications I. *Invent. Math.*, 130:345–369, 1997.

[18] Y. Eliashberg and N. Mishachev. Wrinkling of smooth mappings III. Foliations of codimension greater than one. *Topol. Methods in Nonlinear Analysis*, 11:321–350, 1998.

[19] Y. Eliashberg and N. Mishachev. Wrinkling of smooth mappings II. Wrinkling of embeddings and K. Igusa's theorem. *Topology*, 39:711–732, 2000.

[20] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle*, volume 48 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[21] Y. Eliashberg and N. Mishachev. Wrinkled embeddings. *Foliations, geometry, and topology; Comtemp. Math.*, 498:207–232, 2009.

[22] Y. Eliashberg and N. Mishachev. Topology of spaces of S-immersions. *Perspectives in analysis, geometry, and topology. Progr. Math.*, 296:147–167, 2012.

[23] V. Gershkovich and A. Vershik. *Geometry of distributions and variational problems*. Nonholonomic dynamical systems. 1987.

[24] A. Givental. Whitney singularities of solutions of partial differential equations. *Journal of Geometry and Physics*, 15:353–368, 1995.

[25] M. Gromov. *Partial differential relations*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1986.

[26] M. Gromov. *Carnot-Carathéodory spaces seen from within*, volume 144 of *Sub-Riemannian geometry. Progr. Math.* Birkhäuser, 1996.

[27] I. S. Krasil'shchik, V. V. Lychagin, and A. M. Vinogradov. *Geometry of jet spaces and nonlinear partial differential equations*, volume 1 of *Adv. Stud. Cont. Math.* Gordon and Breach Science Publishing, New York, 1986.

[28] B. Kruglikov and V. Lychagin. Geometry of differential equations. *Handbook of Global Analysis*, 210:725–771, 2008.

[29] V. Lychagin. Geometric singularities of the solutions of nonlinear differential equations. *Dokl. Akad. Nauk. SSSR*, 261, 1980.

[30] V. Lychagin. On singularities of the solutions of differential equations. *Dokl. Akad. Nauk. SSSR*, 251, 1980.

[31] V. Lychagin. Singularities of multivalued solutions on nonlinear differential equations, and nonlinear phenomena. *Acta Appl. Math.*, 135(3), 1985.

[32] V. Lychagin. Geometric theory of singularities of solutions of nonlinear differential equations. *Journal of Soviet Mathematics*, 51(6):2735–2757, 1990.

[33] V. Maslov. Theory of perturbations and asymptotic methods. *Izd. MGU*, 1965.

[34] R. Montgomery. *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2002.

[35] R. Montgomery and M. Zhitomirskii. *Points and Curves in the Monster Tower*, volume 956 of *Memoirs of the AMS*. American Mathematical Society, 2009.

[36] B. Morin. Formes canoniques des singularités d'une application différentiable. *C.R. Acad. Sci. Paris*, 260:5662–5665, 6503–6506, 1965.

[37] E. Murphy. Loose legendrian embeddings in high dimensional contact manifolds. *arXiv e-prints*, 2012.

[38] C. Sevenheck. *Lagrangian singularities*. 2003.

[39] R. Thom. Les singularités des applications différentiables. *Ann. Inst. Fourier*, 6:43–87, 1955/6.

[40] W. Thurston. The theory of foliations of codimension greater than one. *Comm. Math. Helv.*, 87:214–231, 1974.
[41] W. Thurston. Existence of codimension-one foliations. *Ann. Math.*, 104(2):249–268, 1976.
[42] A. Vinogradov. Many-valued solutions, and a principle for the classification of nonlinear differential equations. *Dokl. Akad. Nauk SSSR*, 210:11–14, 1973.
[43] L. Vitagliano. Characteristics, bicharacteristics, and geometric singularities of solutions of PDEs. *International Journal of Geometric Methods in Modern Physics*, 11, 2014.
[44] H. Whitney. On singularities of mappings of euclidean spaces I. Mappings of the plane into the plane. *Ann. of Math.*, 62:374–410, 1955.
[45] V. Zakalyukin. Generating ideals of lagrangian varieties. *Advances in Soviet Mathematics. American Mathematical Society*, 1:333, 1990.

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