MORSE THEORY. EXERCISE SHEET

The point of these exercises is for you to get a *feeling* for the material. It is okay if you do not know how to answer rigorously some (or many) of the questions. What is important is for you to picture the different elements that appear in each of them.

1. First exercise session

Exercise 1. Let us look at the Morse theory of the circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$.

a. Consider the restriction of the height function $h(x, y) = y$ to $\mathbb{S}^1$. What are the critical points, level sets, and sublevel sets?
b. Let $f : \mathbb{S}^1 \to \mathbb{R}$ be a Morse function (i.e. a function with only maxima or minima as critical points). What can you say about the number #maxima − #minima?
c. Can you classify all Morse functions on $\mathbb{S}^1$ (up to a diffeomorphism of $\mathbb{S}^1$)?
d. Can you realise each of these Morse functions as the height function of an embedding of $\mathbb{S}^1$ into $\mathbb{R}^2$?

Exercise 2. Functions are interesting in *closed* manifolds because critical points always exist (a minimum and a maximum at least). However, for open manifolds this is not true anymore.

a. Find a function $\mathbb{R}^2 \to \mathbb{R}$ with no critical points.
b. Given any closed orientable surface S, construct a function $S \to \mathbb{R}$ with only isolated critical points.
c. Show that if you remove a disc D from S, there exists a function $f : S \setminus D \to \mathbb{R}$ with no critical points. Hint: use (b).
d. In items (a) and (c), what can you notice about the level sets of the functions constructed?

Exercise 3. The previous exercise tells us that if M is an open manifold, we should consider functions $M \to \mathbb{R}$ that are *proper*. I.e. functions such that the preimage of a compact is compact.

a. Observe that if $f : M \to \mathbb{R}$ is proper, the level sets do not escape to infinity.
b. Let S be a (possibly open) surface. Suppose $f : S \to \mathbb{R}$ is a proper function with no critical points. What is S?
c. What if f has just a minimum?
d. What if f has just a maximum?
e. What if f has a saddle point?
f. Construct a proper function on the cylinder with a minimum and a saddle.

Exercise 4. In the previous exercise we constructed open surfaces endowed with proper functions having at most one critical point. Think of these as elementary pieces: we can glue them to construct closed surfaces:

a. Let S be a closed surface. Suppose there exists a function $f : S \to \mathbb{R}$ with only a maximum and a minimum. What is S? Try to prove this claim rigorously.
b. Let S be a (possibly open) surface. Let $f : S \to \mathbb{R}$ be a proper function with minimum, a maximum, and a saddle point. What is S?
c. Let S be a closed surface. Let $f : S \to \mathbb{R}$ be a function with minimum, a maximum, and two saddle points. What is S?
d. Look at the possible pseudogradients in the previous examples. What kind of behaviours do they display?

Exercise 5. Pseudogradients can be extremely complicated.

a. Let f be a function on a closed surface S and let ∇f be a pseudogradient. Suppose there exists a flowline $\phi_t(p)$ such that $\lim_{t\to\infty} \phi_t(p)$ is not just a point. Show that this limit set is not made up of isolated points.
b. Find an example of S, f, and ∇f in which this condition holds.

Exercise 6. Let $f : [-1,1] \to \mathbb{R}$ be a function with a Morse critical point at 0 (i.e. a point such that $f'' \neq 0$).

a. Show that this critical point is isolated.
b. Try to convince yourself of the following fact: if we perturb f slightly, the resulting function will still have a unique Morse critical point somewhere close to zero.

Item (b) says that having a Morse critical point is an open condition for a function. Now we make this claim rigorous:

c. The set of all smooth functions $[-1,1] \to \mathbb{R}$ is an (infinite dimensional) vector space. Observe that $|f|_{C^0} = \sup_{x\in[-1,1]} |f(x)|$ is a norm and as such it induces a topology called the C^0-*topology*.
d. Observe that $|f|_{C^1} = |f|_{C^0} + \sup_{x\in[-1,1]} |f'(x)|$ is another norm; it induces the so-called C^1*–topology*.
e. Observe that a sequence of functions f_n converges to f in the C^0-topology if they converge to f pointwise. They converge to f in the C^1-topology if their derivatives converge to f' pointwise as well.
f. Find a sequence converging to zero in the C^0 topology but not in the C^1-topology.
g. Define inductively the norm $|f|_{C^i} = |f|_{C^{i-1}} + \sup_{x\in[-1,1]} |f^(i)(x)|$. Find sequences converging to zero in the C^{i-1}-topology but not in C^i.

Now we let f be a function with Morse critical point at 0.

h. Show that if there is a sequence f_n converging to f in the C^2-topology, there is a (unique!) sequence of points such that x_n is a Morse critical point for f_n and $x_n \to 0$.

Exercise 7. A function $\mathbb{R} \to \mathbb{R}$ is said to be Morse if all of its critical points are Morse. We have just seen that small perturbations of Morse functions are still Morse. What happens when the perturbation is not small? Then critical points might appear and disappear. Consider the family

$$f_s : \mathbb{R} \to \mathbb{R}, \qquad s \in [-1,1]$$
$$f_s(x) = x^3 - sx.$$

a. Describe the locus of points in the plane (x, s) such that f_s has a critical point at x. Draw in the $(f_s(x), s)$ plane the locus of critical values.
b. What is happening to the functions f_s as we cross $s = 0$? Make a qualitative picture.
c. Compare this with Exercise 1: Take a Morse function in $\mathbb{S}^1$ and then modify it as in item (b) to add a new maximum and a new minimum. Show that all Morse functions in $\mathbb{S}^1$ are related to one another by this process.
d. Suppose $f_s : \mathbb{S}^1 \to \mathbb{R}$ is now a family of functions on the circle such that f_s is Morse except at a discrete collection of times $s_0, \cdots, s_n$. At each of these times, a pair of critical points either appears or disappears. Try to draw (qualitatively) the shape of the set of critical points in the cylinder (x, s). Draw in the $(f_s(x), s)$-plane the locus of critical values. (This exercise is very open. Try to come up with families f_s and see how their behaviour can be understood by looking at the set of critical points drawn in (x, s). This is the simplest example in *Cerf theory* and the theory of *generating functions*.)

Consider the family

$$f_s : \mathbb{R}^2 \to \mathbb{R}, \qquad s \in [-1,1]$$
$$f_s(x, y) = x^2 + (y^3 - sy).$$

e. Describe the critical locus of f_s. What happens to f_s when we cross $s = 0$?

The phenomenon at $s = 0$ is called a *birth/death*.

2. Second exercise session

Exercise 8. Let us compute the Morse homology of $\mathbb{S}^1$.

a. Consider the height function on $\mathbb{S}^1$. Choose a pseudogradient for it. Compute the Morse homology.
b. Compute the Morse homology of all the Morse functions on $\mathbb{S}^1$. (You should always get the same result!)

Exercise 9. Let us compute the Morse homology of $\mathbb{S}^2$.

a. Consider the height function on $\mathbb{S}^1$. Choose a pseudogradient for it. Compute the Morse homology.
b. Choose a pseudogradient in the bean and compute its Morse homology.

Exercise 10. Let us compute the Morse homology of T^2. Consider the height function. Choose a pseudogradient (carefully!) to do it.

Exercise 11. Let us look at the genus-g surface Σ_g.

a. Find a Morse function on Σ_g, pick a pseudogradient and compute its Morse homology.
b. Find $2g$ curves $\{\alpha_i\}$ in Σ_g such that every other curve is homologous to a combination of the α_i. (This is probably impossible to prove if you do not know some tricks in homology. However, try to play around with T^2 and Σ_2 a bit to see what is going on. It is convenient for you to think of Σ_2 as two tori from which we have removed corresponding discs and glued together).

Exercise 12. Let $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ be two Morse functions on two different manifolds.

a. Show that $f + g$ is a Morse function in the product $M \times N$.
b. What are the indices of the critical points of f and g?
c. Let p by a critical point of $f + g$. When is it in the kernel of the corresponding homomorphism ∂? When is it in the image?
d. Can you compute the Morse homology of $M \times N$ in terms of the homology of M and N?

3. More exercises

Exercise 13. Let M be a connected 3-dimensional manifold endowed with a proper Morse function $f : M \to \mathbb{R}$.

a. Suppose f has no critical points. Describe M.
b. Suppose f has just a minimum, describe M.
c. Suppose f has an index 1 point, describe M.
d. Suppose f has an index 2 point, describe M.

For (c) and (d) particularly, draw the local model around the index 1 or 2 point and see how it can be glued to the examples obtained from (a).

Exercise 14. Consider the unit sphere in $\mathbb{R}^4$:

$$\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 + z^2 + w^2 = 1\}.$$

a. Show that $\mathbb{S}^3$ can be covered by two charts, each of which is diffeomorphic to $\mathbb{D}^3$. Hint: use the stereographic projection. Convince yourself that then we can think of $\mathbb{S}^3$ as $\mathbb{R}^3$ plus the point at infinity.
b. Look at the height function $f(x, y, z, w) = w$ in $\mathbb{S}^3$. Describe the critical points, level sets, and sublevel sets.
c. Find a Morse function on $\mathbb{S}^3$ having a critical point of each index. Hint: use the models from the previous exercise.

d. Compute the Morse homology of $\mathbb{S}^3$ using the height function and the function from (c).
e. Looking at (c), reason that $\mathbb{S}^3$ can be obtained by gluing together two solid tori $\mathbb{S}^1 \times \mathbb{D}^2$. Describe how this gluing is done.

Exercise 15. Let M be a closed, connected, 3-dimensional manifold endowed with a Morse function f.

a. Show that f cannot have exactly 3 critical points.
b. Suppose f has 4 critical points. What are their possible indices?
c. Suppose f has 4 critical points. When is M just $\mathbb{S}^3$?

Exercise 16. Let M be a closed, connected, 3-dimensional manifold endowed with a Morse function f. Suppose f has 4 critical points of different indices.

a. Construct M piece by piece by looking at f. Look at what happens when we cross a critical point. How do the sublevel sets change?
b. Show that M can be obtained by gluing together two solid tori $\mathbb{S}^1 \times \mathbb{D}^2$ along their common boundary T^2. Try to describe the different ways in which one can glue.
c. Compute the Morse homology of M in terms of the gluing.

These manifolds are called *lens spaces*. They are the simplest 3-manifolds apart from the sphere.

Exercise 17. The alternated sum $\chi(M) = \sum_i (-1)^i b_i$ of the Betti numbers is called the *Euler characteristic*. Show that $\chi(M) = 0$ if M is a closed 3-dimensional manifold. Step by step:

a. Show that $\chi(M)$ equals the alternated sum
$$\sum_i (-1)^i \dim(C_i(\nabla f))$$
for any pseudogradient f. Here dim denotes the number of $\mathbb{Z}$ factors in the abelian group $C_i(\nabla f)$.
b. Study how $\sum_i (-1)^i \dim(C_i(\nabla f))$ changes when f is replaced by $-f$.
c. Check that the same is true if M is $2n+1$ dimensional.

Exercise 18. Let M be a closed, connected, 3-dimensional manifold whose fundamental group is not abelian. Show that any Morse function $f : M \to \mathbb{R}$ must have at least 6 critical points. Note: skip this one if you do not know what the fundamental group is!

Exercise 19. Let S be a closed surface endowed with a triangulation T. Show that the Euler characteristic (defined in terms of the Betti numbers) agrees with the expression:
$$\#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{faces}\}.$$
Hint: use T to construct a function f whose critical points are in correspondence with vertices/edges/faces.