Introduction to Contact Topology

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Contents

Contact structures in dimension 3

A three-hour lecture on Contact Topology

In this lecture:

- We will become familiar with 3-dimensional contact structures. In particular, we will prove a useful criterion to construct them (Proposition 1.5).
- We will look at curves tangent to contact structures, which are called Legendrian knots (Definition 1.7). In particular, we will provide a constructive method that produces many examples (Proposition 1.17).
- We will introduce a tool (Definition 1.18) to help us distinguish Legendrian knots.

3-dimensional contact structures

Until fairly recently (with a few exceptions) Contact and Symplectic Topology had mostly developed in dimensions 3/4. In these dimensions it is possible to have a good geometrical intuition by simply drawing what is happening.

1.1.1 Examples (25 minutes)

Let us provide some explicit examples of globally defined contact structures on 3–manifolds. They all can be shown to be contact by checking the condition $\alpha \wedge d\alpha \neq 0$.

In class activity (25 minutes): For each of the following examples (I will work out the first one myself in the board):

- Check that the given plane field is a contact structure (by looking at the condition $\alpha \wedge d\alpha \neq 0$).
- Draw the coefficients of the second vector field in the framing as a planar curve.

Try to find a pattern: What do all these curves have in common? Hint: compare their position and velocity.

Example 1.1. The structure

$$(\mathbb{R}^3, \xi_{\text{std}} = \ker(dy - zdx))$$

performs almost half a turn with respect to the line field $\langle \partial_z \rangle$. It is (globally) diffeomorphic to

$$(\mathbb{R}^3, \xi_{\text{std}'} = \ker(\cos(z)dx + \sin(z)dy)),$$

which turns infinitely many times with respect to the line field $\langle \partial_z \rangle$. It is also diffeomorphic to

$$(\mathbb{R}^3, \xi_{\text{std}''} = \ker(dz + ydx - xdy)),$$

which performs almost a $\pi/2$-turn with respect to the radial vector field $x\partial_x + y\partial_y$. This structure can be rewritten in cylindrical coordinates (r, θ, z) as

$$\xi_{\text{std}''} = \ker(dz - r^2 d\theta).$$

We simply say that all of them are the **standard contact structure** in $\mathbb{R}^3$. We invite the reader to provide explicit contactomorphisms between all of them. ♦

Example 1.2. The structure

$$(\mathbb{R}^3, \xi_{\text{OT}} = \ker(\cos(r)dx + \sin(r)rd\theta))$$

turns infinitely many times with respect to the line field $\langle \partial_r \rangle$. It is **not** diffeomorphic to the standard contact structure, and it is called the **contact structure overtwisted at infinity**. See Theorem [-] below and the subsequent discussion. ♦

Example 1.3. The structures

$$(\mathbb{T}^3, \xi_k = \ker(\cos(\pi kz)dx + \sin(\pi kz)dy)) \qquad k \in \mathbb{Z}^+,$$

turn $k/2$ times with respect to the line field $\langle \partial_z \rangle$. They are not diffeomorphic to one another. The structures are coorientable if and only if k is even. ♦

Example 1.4. Consider $\mathbb{S}^3 \subset \mathbb{C}^2$. It is defined as the level set $f^{-1}(1)$ of

$$f(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2.$$

As such, its tangent space is the kernel of the 1-form

$$df = 2(x_1 dx_1 + y_1 dy_1 + x_2 dx_2 + y_2 dy_2).$$

The complex tangencies (i.e. the vectors v such that both v and iv are in $T\mathbb{S}^3$) are simply the complex lines:

$$\begin{aligned} \xi_{\text{std}} = T\mathbb{S}^3 \cap i(T\mathbb{S}^3) = \ker(df) \cap \ker(df \circ i) = \ker(\lambda_{\text{can}}) \\ = \ker(-x_1 dy_1 + y_1 dx_1 - x_2 dy_2 + y_2 dx_2) \subset T\mathbb{S}^3. \end{aligned}$$

Which we already saw in the previous session. $(\mathbb{S}^3, \xi_{\text{std}})$ is the compactification of $(\mathbb{R}^3, \xi_{\text{std}})$. We leave this to the reader. ⧫

1.1.2 The contact condition amounts to turning (25 minutes)

The previous examples lead us thus to the following characterisation of the contact condition:

Proposition 1.5. *Fix coordinates (x, y, z) in $S \times [-1, 1]$, where S is a disc, a 2-torus, or a cylinder. Given a plane field of the form*

$$\xi = \ker(\alpha), \qquad \alpha = f dy + g dx,$$

where $f, g : S \times [-1, 1] \to \mathbb{R}$, we may look at the curves:

$$\gamma_{x_0,y_0} : [-1, 1] \to \mathbb{S}^1$$

$$\gamma_{x_0,y_0}(z) = \frac{(f(x_0, y_0, z), g(x_0, y_0, z))}{|f, g|}.$$

Then:

- *ξ is contact at the point (x_0, y_0, z_0) if and only if γ_{x_0,y_0} is an immersion at time z_0.*
- *ξ is involutive if and only if the curves γ_{x_0,y_0} are constant.*

Proof. Indeed, we check that

$$\alpha \wedge d\alpha = [f(\partial_z g) - g(\partial_z f)] dx \wedge dy \wedge dz.$$

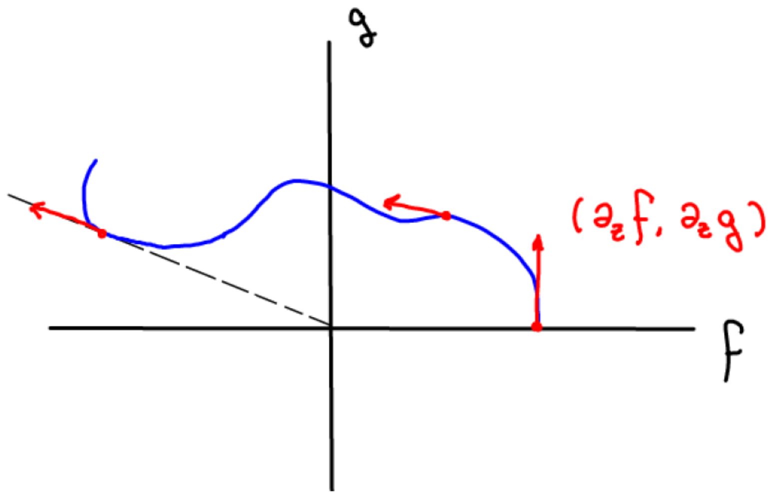

Figure 1.1: A family of curves (f, g) describing a plane field (but we only draw one of them). The points in which (f, g) is colinear with its velocity $(\partial_z f, \partial_z g)$ correspond to singular points of the normalised map $(f, g)/|f, g|$, as seen on the left hand side.

The condition $f(\partial_z g) - g(\partial_z f) \neq 0$ is equivalent to (f, g) and $(\partial_z f, \partial_z g)$ being linearly independent vectors in $\mathbb{R}^2$. This is precisely the immersion condition for $(f, g)/|f, g|$; see Figure 1.1.

□

This is usually phrased as follows: ξ is contact if and only if it turns with respect to any line field tangent to it. This lemma will be extremely useful, because it will allow us to construct contact structures by taking any plane field and "adding to it a bit of turning". Introducing turning can be done by working locally, thanks to the model produced by the following lemma:

Lemma 1.6. *Let (M, ξ) be a 3-manifold endowed with a plane field. Fix $p \in M$. Then, there are local coordinates (x, y, z) around p in which*

$$\xi = \ker(dy + g(x, y, z)dx),$$

where g is a locally defined function.

Proof. Pick a non-vanishing vector field Z tangent to ξ, locally around p. We may then choose a locally defined surface S containing p and transverse to Z. By construction, S is transverse to ξ. Either by hand or by invoking Frobenius' theorem, we find local coordinates (x, y) in S such that the line field $\xi \cap TS$ is spanned by ∂_x. Consider now the flow ϕ_t of Z. We give coordinates (x, y, z) to the point $\phi_z(x, y)$. See Figure 1.2.

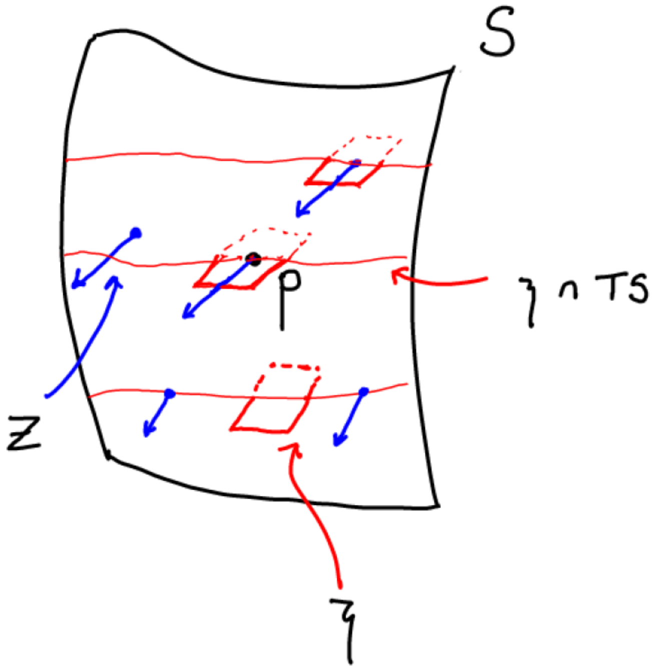

Figure 1.2: Construction of the neighbourhood of p in which ξ is in normal form. Z is a vector field transverse to ξ, S is a surface transverse to it and passing through p.

By construction ξ is tangent to ∂_z in these local coordinates. Additionally, it is tangent to ∂_x at $\{z = 0\}$. These two conditions imply the local form claimed. □

Legendrian knots I

1.2.1 Review of Smooth Knot Theory (15 minutes)

In 3-dimensional Smooth Topology, a knot is an embedding of $\mathbb{S}^1$ into a 3-manifold N. This notion is fundamental due to its role in the definition of surgery (i.e. cutting N along a knot and filling the hole in order to obtain a new manifold). One often focuses on the case in which N is $\mathbb{R}^3$ or $\mathbb{S}^3$ (and this is what we henceforth do).

The simplest knot is the **unknot**. This is the embedding, unique up to isotopy, which is the boundary of an embedded disc. Knot theory consists of determining whether

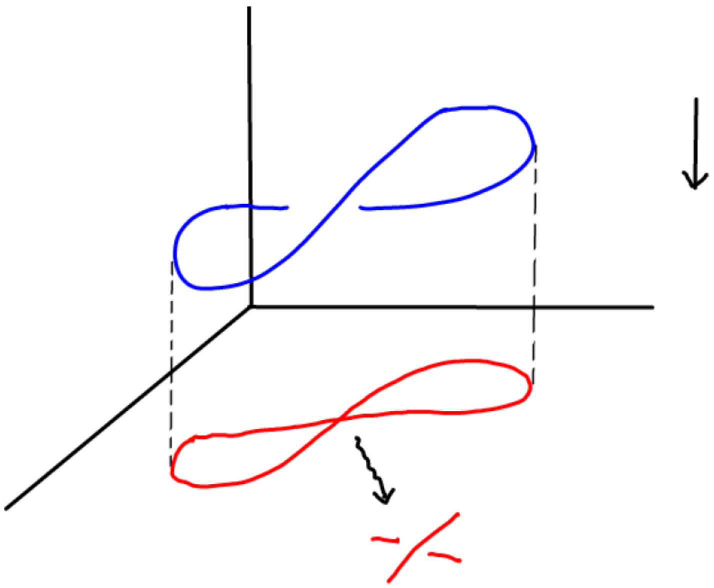

Figure 1.3: An unknot (in blue) and its projection to the horizontal plane (in red). We need to specify which is the over-pass at the point in which the projection has a self-intersection; this is depicted next to the projection with an arrow.

two knots are isotopic to one another. Even the task of determining whether a given knot is in fact the unknot is non-trivial.

The way in which one presents a knot is through a projection. That is, we pick a plane in $\mathbb{R}^3$ and we project the knot to it orthogonally. Such a projection is, generically (i.e. for most choices of plane), an immersed curve with self-intersections. These intersections are an artifact of the projection, and to distinguish them we draw the strands meeting at the intersection as an under-pass and as an over-pass. See Figure 1.3.

We are interested in classifying knots up to isotopy. As we isotope a knot, its projection varies, but (generically) it does so in a controlled way: Only three events, called the Reidemeister moves, may take place. They are depicted in Figure 1.4.

1.2.2 Legendrian Knots (5 minutes)

In a contact 3-manifold we can look at knots as well:

Definition 1.7. *Let (N^3, ξ) be a 3-dimensional contact manifold. A* ***Legendrian knot*** *is an embedding $\mathbb{S}^1 \to N$ which is everywhere tangent to ξ.*

A Legendrian knot is a Legendrian in the general sense.

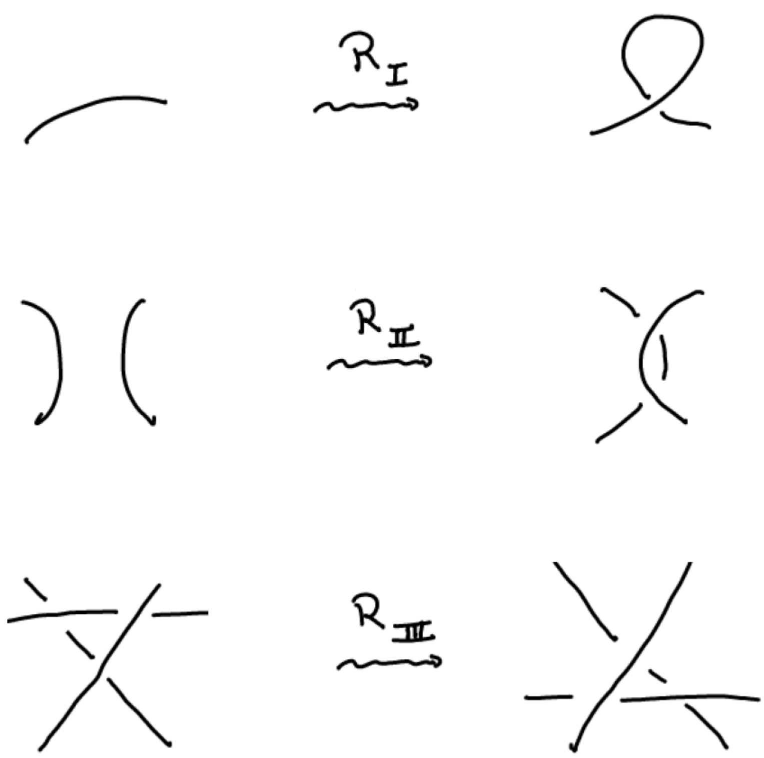

Figure 1.4: The three elementary events one might see as a knot is isotoped. They are called the first, second, and third Reidemeister moves, respectively.

In the contact setting we do not project to an arbitrary 2-plane. Instead, there are two projections that are well-suited to manipulating Legendrians:

Definition 1.8. *Consider* $(\mathbb{R}^3, \xi_{\text{std}} = \ker(dy - zdx))$.

- *The map* $\pi_f : \mathbb{R}^3(x, y, z) \to \mathbb{R}^2(x, y)$ *is called the* ***front projection****.*
- *The map* $\pi_L : \mathbb{R}^3(x, y, z) \to \mathbb{R}^2(x, z)$ *is called the* ***Lagrangian projection****.*

1.2.3 Front projection (30 minutes)

One can completely recover (up to shift in the Lagrangian case) a Legendrian knot from either of its projections. Let us work this out first in the front projection:

Lemma 1.9. *Let* $\gamma : [0, 1] \to \mathbb{R}^3$ *be a Legendrian curve. Suppose that* $\pi_f \circ \gamma(t) = (x(t), y(t))$ *is immersed. Then the missing* z*–coordinate can be recovered using the expression:*

$$z(t) = \frac{dy}{dx}(t).$$

Proof. Since $\pi_f \circ \gamma(t)$ is immersed and $dy - zdx$ evaluates to zero on γ, we deduce that $\gamma^* dx = x'(t)dt$ is nonzero. Then we can solve $z(t) = dy/dx$, as claimed. □

Remark 1.10. A particular case is a curve of the form $(t, y(t), z(t))$. It must satisfy $y'(t) = z(t)$. ♦

Now, not all Legendrian curves γ in $(\mathbb{R}^3, \xi_{\text{std}})$ project to an immersed curve $\pi_f \circ \gamma$:

Lemma 1.11. *The curve* $\gamma : \mathbb{R} \to \mathbb{R}^3$ *given by*

$$\gamma(t) = (x(t) = t^2/2, y(t) = t^3/3, z(t) = t)$$

is embedded and Legendrian.

Proof. The front projection $\pi_f \circ \gamma(t) = (x(t) = t^2/2, y(t) = t^3/3)$ has a singularity (i.e. fails to be immersed) at $t = 0$, which we call the **cusp**. This is the simplest singularity a planar curve may have.

The curve γ itself is embedded, since the map $z(t)$ is a diffeomorphism of $\mathbb{R}$. For the Legendrian condition, it is sufficient to show that $\gamma^*(dy - zdx) = t^2dt - t^2dt = 0$. □

One can prove that:

Proposition 1.12. *Let γ be a Legendrian knot. After a (C^∞) small perturbation, it may be assumed that $\pi_f \circ \gamma$:*

- *Fails to be an immersion at a finite collection of points.*
- *At these points it is equivalent to the cusp or its mirror image.*

Proof. This statement requires transversality theory, which is beyond the scope of this course. The idea is roughly the following: $\pi_f \circ \gamma$ fails to be an immersion if and only if γ is tangent to the projection direction $\langle \partial_z \rangle$. When this happens, since γ itself is immersed, we have that γ must be graphical over its z–coordinate. That is, up to reparametrisation we may take $\gamma(t) = (x(t), y(t), t)$.

Transversality tells us that one can perturb γ so that these tangencies are as simple as possible. In this case, this means that they should be cuadratic so $x(t)$ should agree with $\pm t^2$ (up to reparametrisation in the domain and the target). The y coordinate is uniquely determined from x and z (by integrating), yielding $y(t) = \frac{2t^3}{3}$, i.e. the cusp. □

Apart from cusps, the planar curve $\pi_f \circ \gamma$ may fail to be embedded:

Lemma 1.13. *Let γ be a Legendrian knot. Two branches of $\pi_f \circ \gamma$ meet at an intersection point with different slopes.*

Proof. Suppose there are two distinct times t_0 and t_1 such that

$$\pi_f \circ \gamma(t_0) = (x(t_0), y(t_0)) = (x(t_1), y(t_1)) = \pi_f \circ \gamma(t_1).$$

Embeddedness of γ implies that $z(t_0) \neq z(t_1)$. This can be rewritten as:

$$\frac{dy}{dx}(t_0) = z(t_0) \neq z(t_1) = \frac{dy}{dx}(t_1)$$

i.e. the regions of $\pi_f \circ \gamma$ for times close to t_0 and for times close to t_1 have different slope, as claimed. □

In particular: At a crossing we do not need to specify whether it is an underpass or an overpass, because this is given by the slope. See Figure 1.5.

One can show (appealing again to transversality) that:

Proposition 1.14. *Let γ be a Legendrian knot. After a (C^∞) small perturbation, it may be assumed that $\pi_f \circ \gamma$:*

- *Has only finitely many self-intersections.*
- *At each intersection point only two branches meet.*

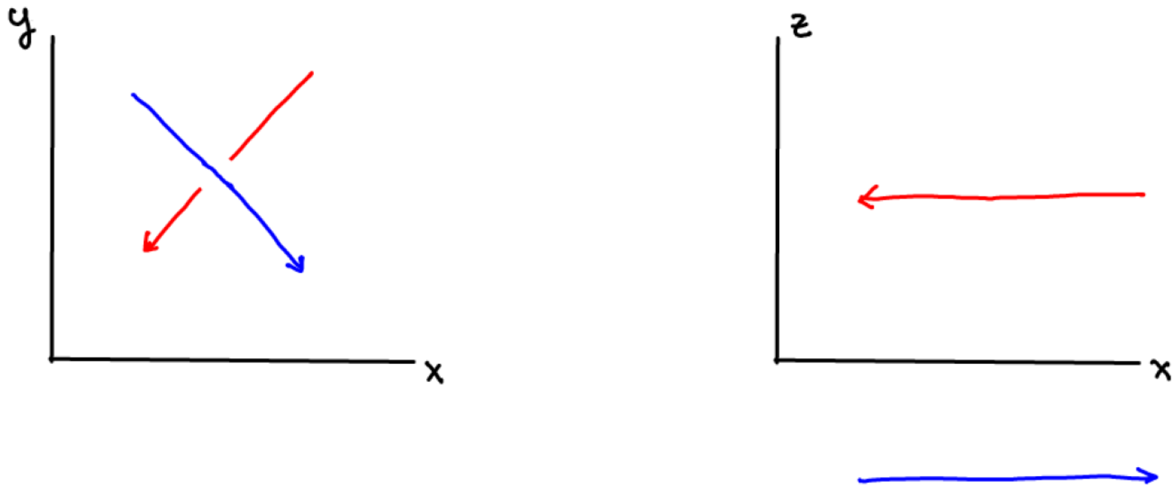

Figure 1.5: On the left we depict two curves whose front projections intersect. The slopes of the strands are constant and different, so their z–coordinates are different. In particular, the self-intersection of the front does not correspond to a self-intersection in 3-dimensional space.

In class activity (10 minutes): Given the left-hand side of Figure 1.6, draw the right hand-side. Hints:

- Look first at the cusps, what do they correspond to on the (x, z)–coordinates?
- Look now at the maxima of the upper strand and the minima of the lower one, what should they correspond to?
- Follow the slope of the upper and lower branches. When are they positive? When are they negative?

Legendrian knots II

1.3.1 Lagrangian projection (20 minutes)

Let us work in the Lagrangian projection now:

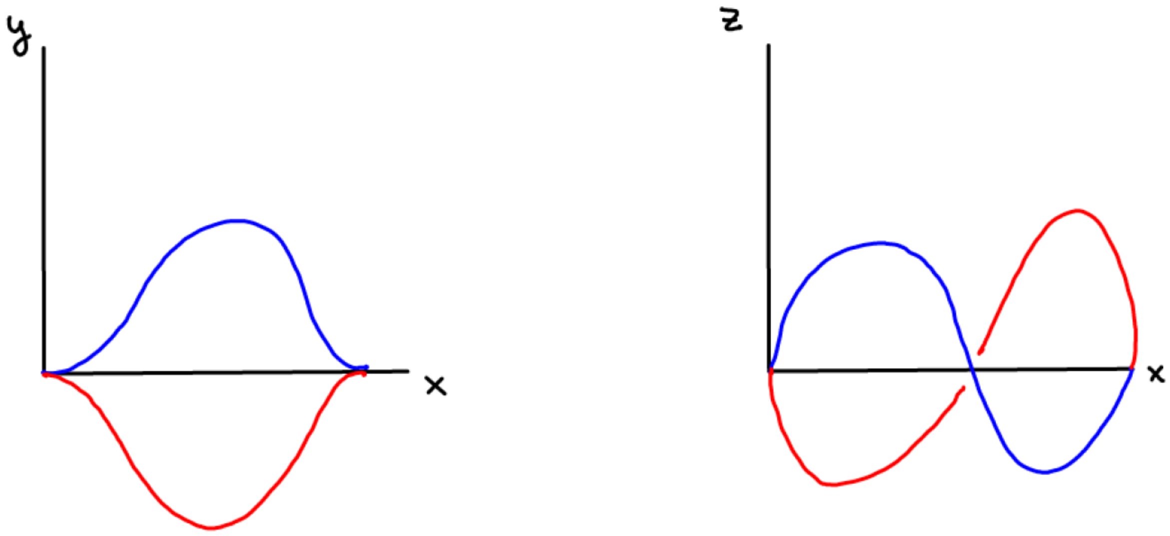

Figure 1.6: The so-called Legendrian unknot (which is in particular a smooth unknot). We draw it in both projections. In the front (left) it has two cusps. In the Lagrangian projection (right) we see a self-intersection.

Proposition 1.15. *Let $\gamma : [0,1] \to \mathbb{R}^3$ be an immersed Legendrian curve. Then:*

- *Its Lagrangian projection $\pi_L \circ \gamma(t) = (x(t), z(t))$ is an immersed planar curve.*
- *The missing coordinate can be recovered by integrating:*

$$y'(t) = z(t)x'(t), \qquad y(t) = y(0) + \int_0^t z(s)x'(s)ds.$$

Proof. The curve $\pi_L \circ \gamma$ would fail to be immersed at time t if at only if γ is tangent to ∂_y, the direction of projection. Since γ is Legendrian and immersed, this can never happen (because ∂_y is not tangent to ξ_{std}).

The second claim follows because $\gamma^*(dy - zdx) = 0$, due to the Legendrian condition. □

In particular:

Corollary 1.16. *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be an embedded Legendrian curve. Then:*

- *Its Lagrangian projection $\pi_L \circ \gamma$ bounds zero area.*
- *Let $t_0, t_1 \in \mathbb{S}^1$ be times at which $\pi_L \circ \gamma(t_0) = \pi_L \circ \gamma(t_1)$. Then the curve $\pi_L \circ \gamma|_{[t_0,t_1]}$ does not bound zero area.*

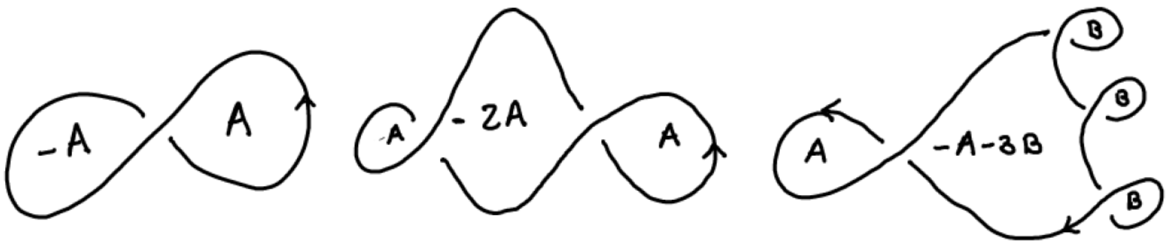

Figure 1.7: The Lagrangian projection of 3 Legendrians. In every region we specify its area, the total sum of which must add to zero. The area of each region additionally allows us to compute which are the over-passes.

Proof. See Figure 1.7.

For the first statement: Since γ is a closed curve, it bounds a (possibly not embedded) disc D. Then we may apply Stokes to show that:

$$0 = \int_\gamma dy = \int_\gamma zdx = \int_{\pi_L \circ \gamma} zdx = \int_{\pi_L(D)} dzdx,$$

where the first equality follows from the fact that γ is closed.

For the second statement we have that $\pi_L \circ \gamma|_{[t_0,t_1]}$ is a closed planar curve, so it bounds a (possibly non-immersed) disc D. Then we have:

$$0 \neq y(t_1) - y(t_0) = \int_{\gamma|_{[t_0,t_1]}} dy = \int_{\gamma|_{[t_0,t_1]}} zdx = \int_D dzdx,$$

where the first inequality is due to the embeddedness condition. □

1.3.2 Construction of Legendrians (20 minutes)

The first meaningful question one might pose is: how rich is Legendrian Knot Theory? For instance, can any smooth knot be represented by a Legendrian knot? The answer is yes:

Proposition 1.17. *Let γ be a smooth knot in $\mathbb{R}^3$. Then, it is smoothly isotopic to a Legendrian knot $\widetilde{\gamma}$ (which is not unique!).*

Proof. This is a proof by picture. It can be done in either projection. See Figures 1.8 and 1.9.

□

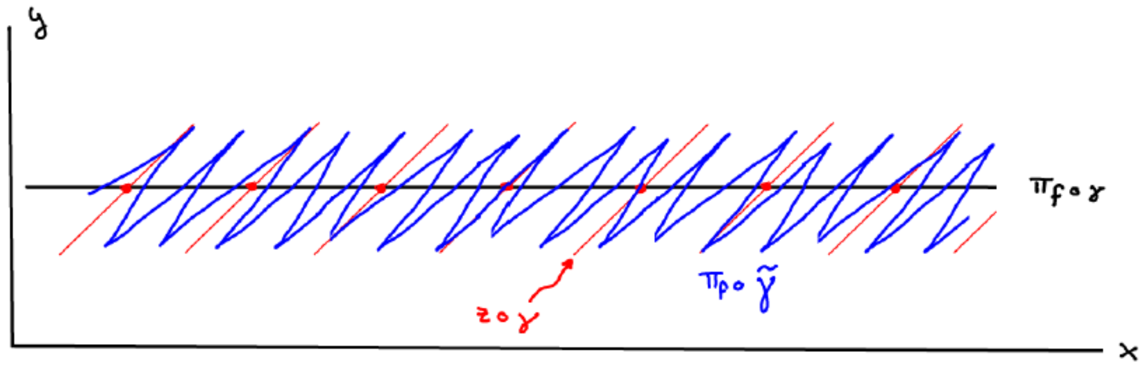

Figure 1.8: We describe a non-Legendrian curve γ in 3-space by looking at its front projection (in black) and specifying its missing z–coordinate by drawing a slope (in red). We claim that we can approximate it by a Legendrian curve. Indeed, we draw a curve (in blue) whose front projection is close to the black curve and whose slope is very close to the red slopes. At the turning points it has cusps. Its unique lift is the desired Legendrian $\tilde{\gamma}$ approximating γ.

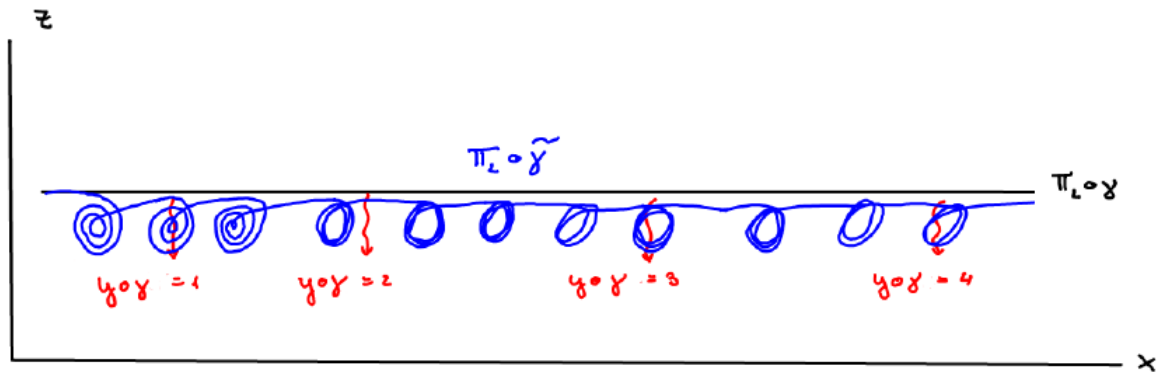

Figure 1.9: We describe a non-Legendrian curve in 3-space by looking at its Lagrangian projection (in black); we must keep track of the missing z–coordinate (which is a number at each point of the curve, recorded in red). We draw a blue curve which is very close to the black curve and that has many loops. The area of these loops accounts exactly for the desired displacement in z. In this manner, its unique lift is the Legendrian that we desired. The self-intersections that appear when we introduce loops do not lift to actual self-intersection (because the loops bound positive area), so the Legendrian constructed is embedded.

1.3.3 Rotation number (10 minutes)

Legendrian Knot Theory studies the question: when are two given Legendrian knots homotopic to one another (as Legendrian knots)? It is a necessary condition that they are smoothly isotopic, but this is not sufficient. The first additional invariant we can define is:

Definition 1.18. *Let $\gamma : \mathbb{S}^1 \to (\mathbb{R}^3, \xi_{\text{std}} = \ker(dy - zdx))$ be an immersed Legendrian. Its Lagrangian projection $\pi_L \circ \gamma$ is a closed and immersed planar curve in $\mathbb{R}^2(x,z)$. As such, we may look at the Gauss map:*

$$\rho(\gamma) : \mathbb{S}^1 \to \mathbb{S}^1 \subset \mathbb{R}^2(x,z)$$

$$\rho(\gamma)(t) = \frac{(\pi_L \circ \gamma)'(t)}{|(\pi_L \circ \gamma)'(t)|}.$$

Then, the ***rotation number*** *of γ is the degree of $\rho(\gamma)$, which is an integer.*

Remark 1.19. Looking at this definition, you should convince yourself that it depends on the orientation we put in $\mathbb{R}^2(x,z)$,. Here we are assuming that it is the standard one given by the basis $\{\partial_x, \partial_z\}$.

Lemma 1.20. *The rotation number is invariant under homotopies of immersed Legendrians.*

Proof. Let $(\gamma_s)_{s\in[0,1]}$ be a family of immersed Legendrians. The corresponding projections $(\pi_L \circ \gamma_s)_{s\in[0,1]}$ are also immersed. As such, we obtain a homotopy of maps:

$$\rho(\gamma_s)(t) = \frac{(\pi_L \circ \gamma_s)'(t)}{|(\pi_L \circ \gamma_s)'(t)|}$$

We conclude by recalling that the degree is a homotopy invariant of maps $\mathbb{S}^1 \to \mathbb{S}^1$. □

We will use this Lemma in the exercises to distinguish Legendrian knots that are smoothly isotopic.

Remark 1.21. To a Legendrian knot one can assign another invariant called the *Thurston-Bennequin number*, which measures the twisting of ξ_{std} with respect to the knot. We will not look into this any further; I invite you to read the book by Geiges. ♦

Exercises

Make sure you are comfortable with the definitions and statements marked as important (in blue) in the notes. Then, take a look at the first exercise of each block below (we will spend the first hour of Session 16 discussing them).

1.4.1 Contact forms, Reeb fields

Exercise 1.1. Prove that the following 1-forms are contact forms in $\mathbb{R}^3$ (in either standard coordinates (x, y, z) or polar coordinates (r, θ, z)). Compute their Reeb vector fields. Describe their closed Reeb orbits (i.e. the orbits of the Reeb vector field which are periodic), computing their periods.

- $\alpha_1 = dy - zdx$,
- $\alpha_2 = \cos(z)dx + \sin(z)dy$,
- $\alpha_3 = dz - ydx + xdy$,
- $\alpha_4 = \cos(r)dx + \sin(r)rd\theta$.

Exercise 1.2. Prove that the following plane fields are contact structures:

$$(\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3, \xi_k = \ker(\cos(\pi kz)dx + \sin(\pi kz)dy)) \qquad k \in \mathbb{Z}^+.$$

Compute the Reeb vector field of the given contact forms. Describe their closed Reeb orbits (with their periods).

Proof. First observe that, for k odd, the forms given are in fact not well-defined at $z = 0, 1$. This tells us that the corresponding plane fields are not coorientable (this showed up in the last exercise of the previous sheet already). This is not a problem when we check the contact condition, which is just a local computation. We write:

$$\alpha_k = \cos(\pi kz)dx + \sin(\pi kz)dy, \qquad d\alpha_k = \pi k(-\sin(\pi kz)dzdx + \cos(\pi kz)dzdy)$$

$$\alpha_k \wedge d\alpha_k = -\pi k dx \wedge dy \wedge dz$$

which is a volume form, so ξ_k is contact.

We now compute the Reeb field. **Important remark**: for k odd, the Reeb field is not well-defined! If a contact structure is not coorientable, it does not make sense to talk about its Reeb field, because the Reeb field is defined in terms of a contact *form*. Now, for k even, the kernel of $d\alpha_k$ is spanned by

$$R_k = \cos(\pi kz)\partial_x + \sin(\pi kz)\partial_y$$

which satisfies $\alpha_k(R_k) = 1$, so it is the Reeb vector field.

A torus $\{z = z_0\}$ is foliated by closed orbits of R_k if and only if $\cos(\pi k z_0)$ and $\sin(\pi k z_0)$ are linearly dependent over the rationals, i.e. either $\cos(\pi k z_0) = 0$ or $\tan(\pi k z_0)$ is a rational number. Otherwise the torus is foliated by orbits which are copies of $\mathbb{R}$. If $\cos(\pi k z_0) = 0$ or $\sin(\pi k z_0) = 0$ the period of the orbits is 1. Otherwise, if $\tan(\pi k z_0) = p/q$, with p, q coprime integers, the corresponding orbits close up for the first time when you move q in the direction of x and p in the direction of y. Since the speed in x is $\cos(\pi k z_0)$, this tells us that the period is $q/\cos(\pi k z_0) = p/\sin(\pi k z_0)$. □

1.4.2 Classification of contact structures

Exercise 1.3. Show that any two plane fields in $\mathbb{R}^3$ are homotopic to one another. Show that the space of plane fields in $\mathbb{S}^3$ has $\mathbb{Z}$ components.

Proof. The cotangent bundle of $\mathbb{R}^3$ is trivial $T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$. In particular, its projectivisation is trivial too $\mathbb{P}T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{RP}^2$. Giving a plane field in $\mathbb{R}^3$ amounts to giving a section $s : \mathbb{R}^3 \to \mathbb{P}T^*\mathbb{R}^3$ (indeed, such an s has a well-defined kernel at each point, which is the corresponding plane field). Thus, plane fields in $\mathbb{R}^3$ up to homotopy are the same as sections of $\mathbb{P}T^*\mathbb{R}^3$ up to homotopy, i.e. the same as maps $\mathbb{R}^3 \to \mathbb{RP}^2$ up to homotopy. Since $\mathbb{R}^3$ is contractible, all of them are homotopic to one another. This also shows that the space of plane fields in $\mathbb{R}^3$ is contractible.

Any closed 3-manifold is parallelisable (this is a non-trivial theorem!) As such, $\mathbb{P}T^*\mathbb{S}^3 \cong \mathbb{S}^3 \times \mathbb{RP}^2$. Plane fields in the 3-sphere are thus described by maps $\mathbb{S}^3 \to \mathbb{RP}^2$. The possible homotopy classes are then given by $\pi_3(\mathbb{RP}^2) = \pi_3(\mathbb{S}^2) = \mathbb{Z}$, where we use that the universal cover of $\mathbb{RP}^2$ is $\mathbb{S}^2$. □

Exercise 1.4. Consider $(\mathbb{R}^3, \xi_{\mathrm{std}} = \ker(dy - z dx))$. Show any arbitrarily big (but compact) domain of $\mathbb{R}^3$ can be mapped to an arbitrarily small one by a contactomorphism of ξ_{std}.

Proof. Consider the family of maps $f_\lambda(x, y, z) = (\lambda x, \lambda^2 y, \lambda z)$. Since

$$f_\lambda^*(dy - z dx) = \lambda^2(dy - z dx)$$

we conclude that they are contactomorphisms for every $\lambda \neq 0$. By taking λ sufficiently small, we may map any arbitrarily big compact set in $\mathbb{R}^3$ to a small one. □

The previous exercise can be used to prove the following statement:

Proposition 1.22. *$(\mathbb{R}^3, \xi_{\mathrm{std}} = \ker(dy - z dx))$ is contactomorphic to an arbitrarily small ball (also endowed with ξ_{std}).*

Proof. The rough idea is that one can construct a contactomorphism $\mathbb{D}^3 \to \mathbb{R}^3$ as the limit of a family of embeddings $\mathbb{D}^3 \to \mathbb{R}^3$ that preserve the contact structure and whose images are progressively bigger. Formalising this statement requires the use of contact Hamiltonians, which we did not cover in the course. □

> *Exercise* 1.5. Let ξ_0 and ξ_1 be contact structures in $\mathbb{R}^3$. Show that they are homotopic (as contact structures) to one another if and only if they induce the same orientation. Hint: use Darboux and think about the space of embeddings $\mathbb{R}^3 \to \mathbb{R}^3$.

Proof. Suppose that ξ_0 induces the standard orientation. It is sufficient to show that it is homotopic to $\xi_{\mathrm{std}} = \ker(dy + zdx)$, which also induces the standard orientation. Use Darboux' theorem to find a open ball U_0 with coordinates (x', y', z') in which $\xi_0 = \ker(dy' + z'dx')$. We can assume that U_0 is the image on an orientation preserving embedding $\psi_0 : \mathbb{R}^3 \to \mathbb{R}^3$ which is additionally a contactomorphism (by using the Proposition preceding this exercise).

We claim that the space of embeddings $\mathbb{R}^3 \to \mathbb{R}^3$ preserving the orientation is connected. The intuitive idea is that one can precompose any embedding f with a homotopy of embeddings $(\rho_r)_{r\in(0,\infty]} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\rho_r(\mathbb{R}^3) = \mathbb{D}^3_r$. As ε goes to zero, the map $f \circ \rho_\varepsilon$ sees progressively less and less of f and remembers only the differential of f at the origin. This effectively provides a retraction of the space of embeddings onto $\mathrm{GL}(\mathbb{R}^3)$, which has two components, corresponding to the two orientations.

Assuming this, find a path ψ_t between ψ_0 and the identity $\psi_1 = \mathrm{Id}_{\mathbb{R}^3}$. Thus, the family $(\psi_t)^*\xi_0$ is a homotopy between $(\psi_0)^*\xi_0 = \xi_{\mathrm{std}}$ and $(\psi_1)^*\xi_0 = \xi_0$. □

1.4.3 Legendrians and their front projection

> *Exercise* 1.6. Check that any legendrian $[0,1] \to (\mathbb{R}^3, \ker(dy - zdx))$ which is graphical over the x–coordinate can be reparametrised to be of the form $(x, y(x), y'(x))$, with y a function of x.

> *Exercise* 1.7. As shown in the previous exercise, $(\mathbb{R}^3, \alpha_{\mathrm{can}} = dy - zdx)$ is the space of 1-jets of functions from $\mathbb{R}$ to $\mathbb{R}$. I.e, we think of y as a function of x and z as its derivative.
>
> Consider the maps:
>
> - $f_1(t) = (x(t) = t^2, y(t) = t^3)$.
> - $f_2(t) = (x(t) = t^l, y(t) = t^k)$, with $k > l$ positive integers.

Lift them to Legendrians in $\mathbb{R}^3$ (i.e. find expressions $z(t)$ such that $(x(t), y(t), z(t))$ is a parametrised curve tangent to $\ker(\alpha_{\text{can}})$). Which of the resulting Legendrians are immersed?

Proof. We need the expression $y'(t) - z(t)x'(t)$ to hold. For f_2 this means that:

$$kt^{k-1} - lz(t)t^{l-1} = 0, \qquad z(t) = \frac{k}{l}t^{k-l}.$$

As soon as $k \geq l$, this is a well-defined expression. Now, the tangent vector to f_2 is:

$$f_2'(t) = (x'(t), y'(t), z'(t)) = (lt^{l-1}, kt^{k-1}, \frac{k(k-l)}{l}t^{k-l-1})$$

which vanishes at $t = 0$ if and only if $l > 1$ and $k > l + 1$. Otherwise f_2 is immersed (for instance, if $k = l + 1$, as is the case for f_1). □

Exercise 1.8. This is a follow-up of the previous exercise. Lift the following maps to Legendrians in $\mathbb{R}^3$:

- $f_\varepsilon(t) = (x(t) = \int_0^t (s^2 - \varepsilon)ds, y(t) = \int_0^t s(s^2 - \varepsilon)ds)$, where $\varepsilon \in \mathbb{R}$ is a parameter.
- $g_\varepsilon(t) = (x(t) = \int_0^t (s^2 - \varepsilon)ds, y(t) = \int_0^t (s^2 - \varepsilon)^2 ds)$, where $\varepsilon \in \mathbb{R}$ is a parameter.

For which values of the parameter are the resulting Legendrians embedded? Draw their front and Lagrangian projections schematically as ε varies.

The first family is called the **first Reidemeister move**. The second one is called the **stabilisation**.

Proof. We compute as before. For f_ε the expression $t(t^2 - \varepsilon) + z(t)(t^2 - \varepsilon)$ implies that $z(t) = t$. In particular, the curves are immersed for all times. Since $z(t)$ is strictly increasing, it follows that they are embedded too. This implies that the family constructed is a homotopy of embedded legendrians.

For g_ε we solve $(t^2 - \varepsilon)^2 + z(t)(t^2 - \varepsilon)$, yielding $z(t) = t^2 - \varepsilon$. In particular, the curve $g_0 = (t^3/3, t^5/5, t^2)$ has a singular point at $t = 0$. All other curves are immersed because the only critical point of $z(t)$ takes place at $t = 0$, which is not critical for $x(t)$, whose critical points are at $t = \pm\sqrt{\varepsilon}$. Additionally, they are embedded: this follows because $y(t)$ is strictly increasing outside of the origin but $z(t)$ is decreasing for $t < 0$ and increasing for $t > 0$. Thus, the family g_ε is not a homotopy of immersed/embedded legendrians, because a singularity appears at $\varepsilon = 0$.

Use Wolfram Alpha (or something else) to plot these! In each of the projections, determine which is the over-crossing. □

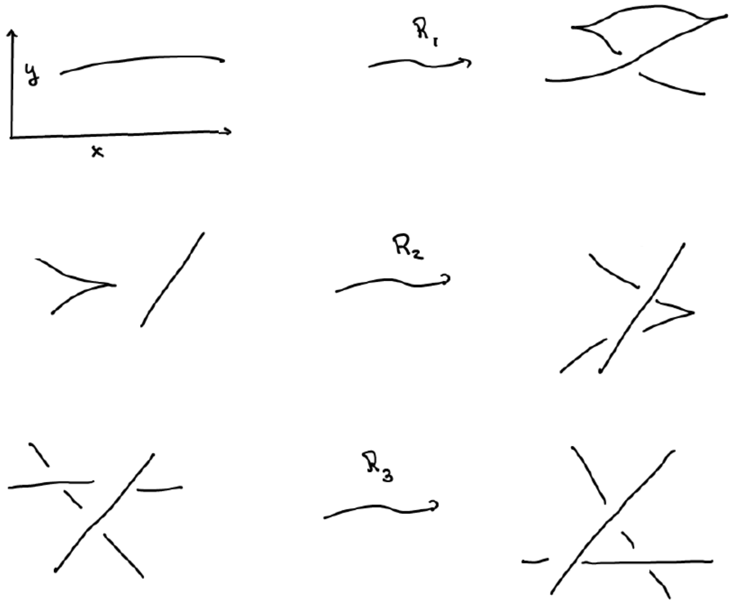

Figure 1.10: The three Legendrian Reidemeister moves in the front projection. The over-crossings represent strands with greater slope and therefore greater z-value.

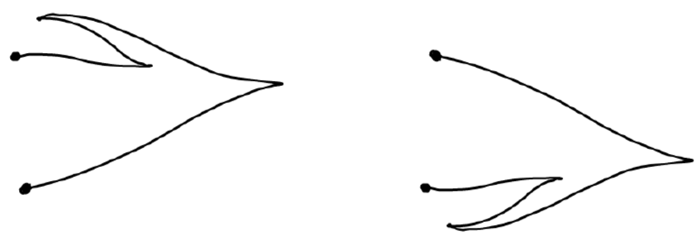

Figure 1.11: Two pieces of Legendrian knot, shown in the front projection.

Exercise 1.9. Using the three Reidemeister moves (Figure 1.10) show that there is a homotopy of Legendrian embeddings connecting the following two local configurations shown in Figure 1.11.

Proof. See Figure 1.12. □

1.4.4 The rotation number

Exercise 1.10. Let $\gamma : \mathbb{S}^1 \to (\mathbb{R}^3, \ker(dy - zdx))$ be a Legendrian knot. Show that the rotation number of $\gamma(-t)$ is minus the rotation number of $\gamma(t)$.

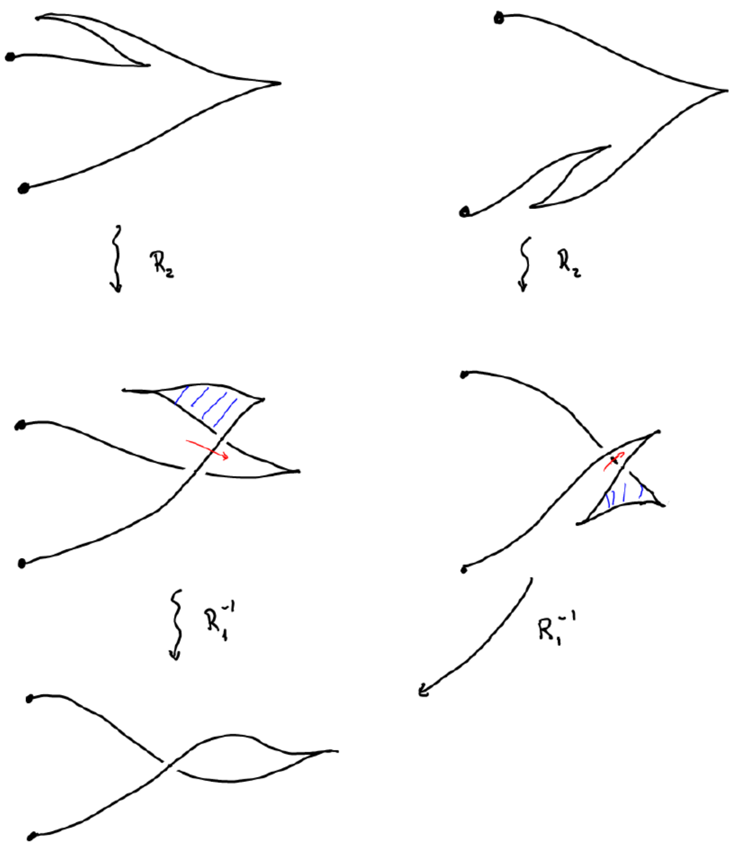

Figure 1.12: The embedded Legendrian homotopy between the two configurations, expressed in terms of Reidemeister moves. The blue areas correspond to Reidemeister I moves.

Exercise 1.11. For each integer $k \in \mathbb{Z}$, find a Legendrian knot in $(\mathbb{R}^3, \ker(dy - zdx))$ with k as its rotation number. To describe the knots, draw schematically their front and Lagrangian projections, stating what convention you use to draw the crossings. It is sufficient that you provide the picture for $k = 0, 1, 2$ and you briefly explain how the general case goes. Remark: you should explain why the two projections you draw indeed correspond to the same knot and you should explain how the rotation is computed from them.

Proof. See Figure 1.13. The main idea is to simply draw curves γ_k in the (x, z)-plane (the Lagrangian projection) bounding zero area and such that γ_k' has degree k. Any such curve will lift to a closed Legendrian, thanks to the formula

$$y(t) = y(0) + \int_0^t zdx,$$

to fix this lift we pick $y(0)$ arbitrarily. According to the previous exercise, it is enough to construct γ_k for $k \geq 0$.

Now, on the left hand side is the unknot, as seen in class. Its Lagrangian projection is a figure eight, which bounds zero area and has rotation zero. The idea now is to add to this Lagrangian projection additional loops: Adding a turn either clockwise or counterclockwise substracts or adds 1 to the rotation number, respectively. This is depicted in pictures two and three. One can make the curve γ_k describe one big lobe in clockwise direction and $k+1$ lobes in counterclockwise direction (the cases depicted are $k = 1, 2$). It is important to make sure that the $k + 1$ lobes bound together the same (unsigned) area as the big lobe, in order to yield a closed curve (this is badly depicted in the picture!).

In order to produce the front projection, we look at the points in which the Lagrangian projection is tangent to the z-direction. These points (marked in the figure) correspond to the cusps of the front. Each strand in-between these points is graphical over the x direction, so we can draw it by recalling that z recovers the slope in the front. In particular: each time we transverse one of the right-most z-tangencies, we are increasing in z, so the corresponding cusp in the front is transversed downwards (because the slope is increasing). Similarly, every time we cross one of the tangencies in-between the small lobes, we are decreasing in slope; thus, the corresponding cusp is also transversed downwards. This tells us that we keep making zig-zags in the front projection.

□

Exercise 1.12. Given $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ Legendrian immersion and a trivialisation of $\xi_{\text{std}} = \ker(dy - zdx)$, there is a map:

$$\rho(\gamma) : \mathbb{S}^1 \to \mathbb{S}^1$$

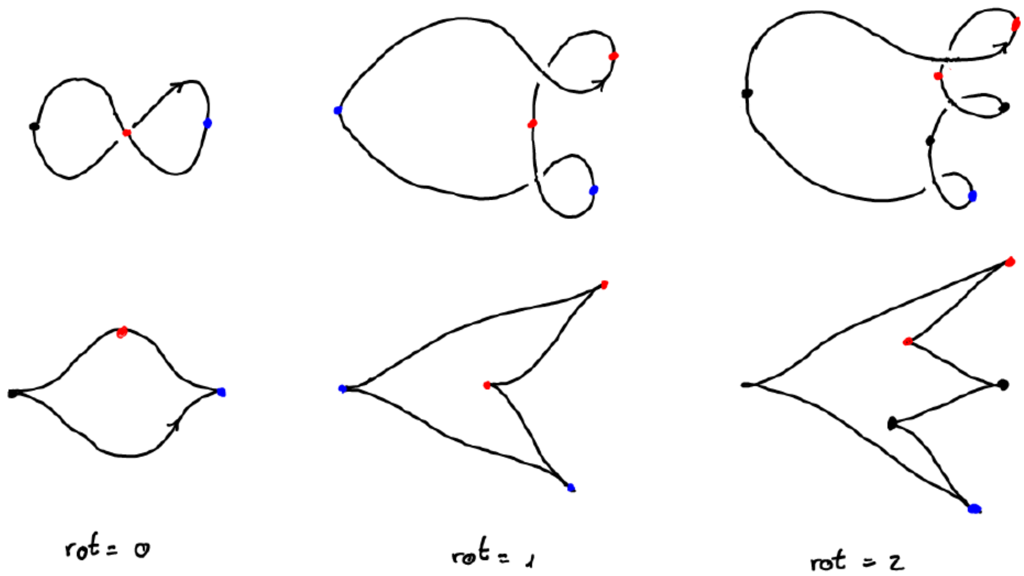

Figure 1.13: Three unknots. The Lagrangian projection on top and the front projection at the bottom. Some of the points are coloured to identify them between the two projections. All the pictures are supposed to be symmetric with respect to the x-axis (the horizontal one in both cases). The over-crossings correspond to greater z and thus to greater slope.

$$\rho(\gamma)(t) = \frac{\gamma'(t)}{|\gamma'(t)|} \in \xi_{\gamma(t)} \cong \mathbb{R}^2$$

where the identification $\xi_{\gamma(t)} \cong \mathbb{R}^2$ depends on the choice of trivialisation. Show that the absolute value of the degree of $\rho(\gamma)$ is independent of the trivialisation. How is this related to the rotation number of γ?

Proof. Trivialisations of the plane field ξ_{std} correspond to framings $\{X, Y\}$ of ξ_{std} (indeed, we map X to ∂_x in $\mathbb{R}^2$ and Y to ∂_y). Such a trivialisation $\xi_{\text{std}} \cong \mathbb{R}^3 \times \mathbb{R}^2$ in particular provides an orientation of ξ_{std} by taking the standard orientation in $\mathbb{R}^2$ on each fibre.

Now, since $\mathbb{R}^3$ is contractible, the space of sections of ξ_{std} is contractible. As such, any two choices of framing $\{X_0, Y_0\}$ and $\{X_1, Y_1\}$ inducing the same orientation are homotopic to one another by a family $\{X_s, Y_s\}_{s\in[0,1]}$. For each s, the corresponding map $\rho_s(\gamma)$ is a planar curve (defined as in the statement, where we indicate the dependence with respect to the trivialisation using the subscript s). The degree of a planar curve is constant in its homotopy class, i.e. since $s \to \deg(\rho_s(\gamma))$ is continuous and takes values in the integers, it is constant. When we consider $\{Y, X\}$ instead of $\{X, Y\}$, the degree changes signs.

The rotation number is defined as the degree computed in the standard trivialisation $X = \partial_x + z\partial_y$ and $Y = \partial_z$.

What you should take from this exercise is that the rotation number can be defined on any contact 3-manifold (M, ξ), in an analogous manner, once we fix a trivialisation of ξ (which is not always possible, because ξ might not be trivial as a bundle). □