Symplectic Geometry

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Chapter 1

Linear symplectic geometry

1.1 Symplectic vector spaces and symplectic linear maps

Let V be a finite-dimensional, real vector space and $\omega : V \times V \to \mathbb{R}$ a skew-symmetric bilinear form, that is, $\omega(v, w) = -\omega(w, v)$ for all $v, w \in V$.

Definition 1.1. *The pair (V, ω) is called a symplectic vector space if ω is non-degenerate, that is:*

$$\ker(\omega) = \{v \in V \, : \, \omega(v, w) = 0 \text{ for all } w \in V\}$$

is trivial. In other words, ω induces an isomorphism $\iota_\omega : V \to V^*$, $\iota_\omega(v)(\cdot) = \omega(v, \cdot)$.

Example 1.2. Let $V = \mathbb{R}^{2n}$ and $v, w \in \mathbb{R}$. Then

$$\omega_0(v, w) = \sum_{i=1}^{n} v_{2i-1}w_{2i} - v_{2i}w_{2i-1}$$

defines the standard symplectic structure on $\mathbb{R}^{2n}$.

Example 1.3. Let V be a finite-dimensional, real vector space, and V^* its dual space. Then the canonical linear symplectic form on $V \times V^*$ is given by

$$\omega_V((v, \alpha), (v', \alpha')) = \alpha'(v) - \alpha(v').$$

Homework 1.4. Prove that in the example above the form ω is indeed skew-symmetric and non-degenerate.

Definition 1.5. *Let (V, ω) and (V', ω') be symplectic vector spaces. A linear map $\Psi : V \to V'$ is called a linear symplectic map if $\Psi^*\omega' = \omega$, i.e.,*

$$\Psi^*\omega'(v, w) = \omega'(\Psi(v), \Psi(w)) = \omega(v, w) \text{ for all } v, w \in V.$$

If Ψ is an isomorphism, we call it a linear symplectic isomorphism or linear symplectomorphism.

Example 1.6. Rotation around the origin of $\mathbb{R}^2$ by an angle θ is a linear symplectomorphism.

Homework 1.7.

1. Every linear symplectic map between symplectic vector spaces of the same dimension is a linear symplectomorphism.
2. The linear symplectomorphisms of a symplectic vector space (V, ω) form a subgroup of $GL(V)$, denoted by $\mathrm{Sp}(V, \omega)$.

One of the most important results of this section on linear symplectic structures is to prove that symplectic vector spaces are classified by their dimension, i.e., any two symplectic vector spaces of the same dimension are linearly symplectomorphic. In order to prove this, though, we will first need to learn a few things about subspaces of linear symplectic spaces.

Subspaces of a symplectic vector space

Definition 1.8. *Let (V, ω) be a symplectic vector space and $W \subset V$ a linear subspace. The space*

$$W^\omega = \{v \in V \,:\, \omega(v, w) = 0 \text{ for all } w \in W\}$$

is called the symplectic complement (or ω-complement) of W.

Notice that W and W^ω are not necessarily transverse, as is the case for a subspace and its orthogonal complement in a linear space with an inner product.

Lemma 1.9. *Let (V, ω) be a symplectic vector space. For any linear subspace $W \subset V$ we have that*

$$\dim W + \dim W^\omega = \dim V.$$

Proof. Non-degeneracy of ω implies that $\iota_\omega : V \to V^*$ is an isomorphism. The symplectic complement W^ω is the pre-image of W under ι_ω of $W^0 = \{\varphi \in V^* \,:\, \varphi(w) = 0 \text{ for all } w \in W\}$, the *annihilator* of W. The claim follows from the identity $\dim W + \dim W^0 = \dim V$. □

Corollary 1.10. *If W is a linear subspace of the symplectic vector space (V, ω), then*

$$(W^\omega)^\omega = W.$$

Definition 1.11. *Let W be a linear subspace of a symplectic vector space. We call W:*

(i) symplectic if $W \cap W^\omega = \{0\}$;

(ii) isotropic if $W \subset W^\omega$;

(iii) coisotropic if $W^\omega \subset W$;

(iv) Lagrangian if $W = W^\omega$.

Remark 1.12.

1. The linear subspace W is symplectic if and only if the restriction of ω to W is non-degenerate.

2. A linear subspace W is symplectic if and only of its symplectic complement W^ω is also a symplectic subspace.

3. A linear subspace W is isotropic if and only if its symplectic complement W^ω is coisotropic. For instance, every 1-dimensional subspace is isotropic and every codimension 1 subspace is coisotropic.

Symplectic bases

Theorem 1.13. *Two symplectic vector space are isomorphic if and only if they have the same dimension.*

Proof. We are going to prove that an arbitrary $2n$-dimensional symplectic vector space (V, ω) is isomorphic to $\mathbb{R}^{2n}$ with the standard symplectic structure. We first need to find a *symplectic basis* for V, that is, a basis $\{v_1, w_1, \dots, v_n, w_n\}$ such that

$$\omega(v_i, w_j) = \delta_{ij} \quad \text{and} \quad \omega(v_i, v_j) = 0 = \omega(w_i, w_j).$$

This can be achieved by induction over n. For $n = 1$, pick two vectors v, w such that $\omega(v, w) \neq 0$, then define $v_1 := v$ and $w_1 := \dfrac{w}{\omega(v, w)}$. Next, we assume every $2(n-1)$-dimensional symplectic vector space admits a symplectic basis and try to prove that this is also true for $2n$-dimensional vector spaces. Define v_1 and w_1 as above, then their linear span W is a symplectic subspace of V of dimension 2. Its symplectic complement W^ω is therefore also a symplectic subspace of dimension $2(n-1)$. By inductive hypothesis it admits a symplectic basis $\{v_2, w_2, \dots, v_n, w_n\}$. The collection $\{v_1, w_1, v_2, w_2, \dots, v_n, w_n\}$ is a symplectic basis of V. Define

$$\Phi : \mathbb{R}^{2n} \to V, \quad \Phi(q_1, p_1, \dots, q_n, p_n) := \sum_{i=1}^{n} (q_i v_i + p_i w_i).$$

Then Φ is a linear symplectomorphism. □

Homework 1.14. Prove that Φ defined above is indeed symplectic.

Corollary 1.15. *Let V be a $2n$-dimensional vector space. Then a skew-symmetric bilinear form on V is non-degenerate if and only if $\omega^n = \omega \wedge \ldots \wedge \omega$ does not vanish.*

Proof. If ω is degenerate, then there exists $v \neq 0$ such that $\omega(v, w) = 0$ for all $w \in V$. Extend v to a basis $\{v_1 = v, v_2, \dots, v_{2n}\}$ of V, then $\omega^n(v_1, \dots, v_{2n}) = 0$. If ω is non-degenerate, then there exists a linear symplectomorphism $\Phi : \mathbb{R}^{2n} \to V$ such that $\Phi^* \omega = \omega_0$ and hence also $\Phi^*(\omega^n) = \omega_0^n$ and the latter does not vanish. □

Homework 1.16. Prove that if Φ is an automorphism of the symplectic vector space (V, ω), then $\det \Phi = 1$.

Lagrangian subspaces

Lemma 1.17. *If W is a maximal isotropic subspace of the symplectic vector space (V,ω), i.e., it is isotropic and not contained in any isotropic subspace of strictly larger dimension, then W is Lagrangian.*

Proof. Suppose $W \neq W^\omega$, then we can choose $v \in W^\omega - W$ and produce a larger isotropic subspace, namely $W' = W + \mathbb{R}v$. □

Homework 1.18. Prove that every symplectic vector space admits a Lagrangian subspace. In particular, deduce that every symplectic vector space necessarily has even dimension.

Homework 1.19. Let (V,ω) be a symplectic vector space, $\Psi : V \to V$ a linear map. Prove that Ψ is a linear symplectomorphism if and only if

$$\Gamma_\Psi = \{(v, \Psi(v)) \,:\, v \in V\}$$

is a Lagrangian subspace of $(V \times V, (-\omega) \oplus \omega)$, where

$$(-\omega) \oplus \omega((v,w),(v',w')) = -\omega(v,v') + \omega(w,w').$$

Recall that, given a vector space (V,ω), the space $V \times V^*$ carries a canonical symplectic structure defined by

$$\omega_V((v,\alpha),(v',\alpha')) = \alpha'(v) - \alpha(v').$$

Homework 1.20. The subspaces $V \times 0_{V^*}$ and $0_V \times V^*$ are Lagrangian subspaces of $V \times V^*$.

In fact, every symplectic vector space can be written in the form above. This result can be regarded as the linear version of Weinstein's Lagrangian neighborhood theorem.

Lemma 1.21. *Given a symplectic vector space (V,ω) and a Lagrangian subspace $W \subset V$, we can find a linear symplectomorphism*

$$\Phi : (V,\omega) \to (W \times W^*, \omega_W)$$

such that $\Phi(W) = W \times \{0\}$.

Proof. We first choose U, a vector space complement to W (not necessarily Lagrangian). From U we can build, in a canonical way, a Lagrangian complement to W. Since ω is non-degenerate and W is Lagrangian, the map ι_ω induces an isomorphism $\iota'_\omega : U \to W^*$. Claim: the space $W' = \{u + Au \,:\, u \in U\}$, where A is defined by

$$-\frac{1}{2}\iota'_\omega(u) = \iota'_\omega(A(u)) \text{ for all } u \in U,$$

is a Lagrangian complement of W.
Once we have a Lagrangian complement of W, we can define

$$\Phi : V = W \oplus W' \to W \times W^*, \quad \Phi(w + w') = w + \omega(\cdot, w').$$

Obviously, $\Phi(W) = W \times \{0\}$ and one can easily verify that $\Phi^*\omega_W = \omega$. □

Homework 1.22. 1. Prove that W' defined in the above proof is indeed Lagrangian.

2. Check that $\Phi^*\omega_W = \omega$.

Remark 1.23. In the section on compatible complex structures we will see another way to construct a Lagrangian complement of a given Lagrangian subspace.

1.2 Compatible complex structures

Definition. A complex structure on a vector space V is an automorphism J of V such that $J^2 = -\mathrm{Id}$. A complex structure on a symplectic vector space (V, ω) is called ω-compatible if $g(v, w) = \omega(v, Jw)$ is a positive definite symmetric bilinear form on V, i.e., an inner product. The space of compatible complex structure is denoted by $\mathcal{J}(V, \omega)$. We call (ω, J, g) a compatible triple on V.

Example 1.24. Let $J_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be defined by

$$J_0(v_1, v_2, \dots, v_{2n-1}, v_{2n}) = (-v_2, v_1, \dots, -v_{2n}, v_{2n-1}).$$

Then J_0 is a complex structure and it is compatible with the standard symplectic form ω_0.

Homework 1.25. Prove that $g(v, w) = \omega_0(v, J_0 w)$ is the standard Euclidean inner product on $\mathbb{R}^{2n}$. Hence, in particular, J_0 is a compatible complex structure on $(\mathbb{R}^{2n}, \omega_0)$.

Remark 1.26. 1. Every compatible complex structure on the symplectic vector space (V, ω) is a symplectomorphism:

$$(J^*\omega)(v, w) = \omega(Jv, Jw) = g(Jv, w) = g(w, Jv) = -\omega(w, v) = \omega(v, w).$$

2. $(\mathbb{R}^{2n}, J_0)$ can be identified with $(\mathbb{C}^n, i)$.

3. If J is a compatible complex structure on the symplectic vector space (V, ω) and L is a Lagrangian subspace of V, then JL is a Lagrangian complement of L, that is, JL is Lagrangian and $V = L \oplus JL$. In fact, JL is the orthogonal complement of L with respect to the inner product $g(v, w) = \omega(v, Jw)$.

Homework 1.27. Prove that J is ω-compatible if and only if ω is J-invariant and $\omega(v, Jv) > 0$ for all non-zero $v \in V$.

A compatible complex structure on (V, ω) makes it into a complex inner product space, with Hermitian metric defined by

$$h(v, w) = g(v, w) + i\omega(v, w).$$

Homework 1.28. Check that h defines a Hermitian structure on V, i.e., $h(v, w) = \overline{h(w, v)}$, $h(v, v) > 0$ for $v \neq 0$, h is complex linear in the second variable ($h(v, Jw) = ih(v, w)$) and complex anti-linear in the first variable ($h(Jv, w) = -ih(v, w)$).

Lemma 1.29. *Let (ω, J, g) be a compatible triple on the vector space V. Then there exists an isomorphism $\Phi : \mathbb{R}^{2n} \to V$ such that $\Phi^*\omega = \omega_0$, $\Phi^*J = J_0$ and hence also $\Phi^*g = g_0$.*

Proof. The metric $h(v, w) = g(v, w) + i\omega(v, w)$ is Hermitian. By the Gram-Schmidt orthonormalization process there exists a basis $\{v_1, \dots, v_n\}$ of V that is unitary with respect to h. Define

$$\Phi : \mathbb{R}^n \to V, \quad \Phi(q_1, p_1, \dots, q_n, p_n) = \sum_{k=1}^{n} (q_k v_k + p_k J(v_k)).$$

Then the claim follows because $h(v_k, v_l) = \delta_{kl}$ and $h(v_k, J(v_l)) = ih(v_k, v_l)$. □

Corollary 1.30. *Let (ω, J, g) be a compatible triple on V. If (ω', J', g') is another compatible triple, then there exists an automorphism $\Psi : V \to V$ which is symplectic and satisfies $\Psi^*J' = J$ (and hence also $\Psi^*g' = g$).*

The next theorem gives a method for constructing compatible complex structures out of positive definite inner products. For a vector space V denote by $\mathrm{Met}(V)$ the space of inner products on V (Riemannian metrics).

Theorem 1.31. *Let (V, ω) be a symplectic vector space. There is a canonical continuous surjective map*

$$r : Met(V) \to \mathcal{J}(V, \omega)$$

such that $r(\omega(\cdot, J\cdot)) = J$.

Proof. Let g be a positive definite inner product on V and let $A \in GL(V)$ be defined by $\omega(v, w) = g(Av, w)$. Then A is skew-adjoint with respect to g, i.e., writing A^* for the adjoint we have that $A^* = -A$:

$$g(A^*v, w) = g(v, Aw) = g(Aw, v) = \omega(w, v) = -\omega(v, w) = g(-Av, w).$$

Hence $P = A^*A = -A^2$ is symmetric and positive definite. This implies that there exists Q (symmetric and positive definite, such that $Q^2 = P$. in fact, if we represent P by a positive definite symmetric matrix, then it can be diagonalized as $P = B\mathrm{Diag}(\lambda_1, \dots, \lambda_{2n})B^{-1}$, with positive eigenvalues λ_i, and we may define $Q := B\mathrm{Diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_{2n}})B^{-1}$ Define $r(g) = J_g = Q^{-1}A$. Then J_g is a complex structure ($J_g^2 = -\mathrm{Id}$) and it is compatible with ω. The continuity of r is proved in [MS98, Ex. 2.52]. □

In particular, this shows that any pair of structures (symplectic, complex, metric) determines a compatible triple.

Homework 1.32. 1. Prove that J_g in the proof above is a compatible complex structure on (V, ω) for any choice of metric g.

2. Prove that $r(\omega(\cdot, J\cdot)) = J$ for all $J \in \mathcal{J}(V, \omega)$.

Corollary 1.33. *The space $\mathcal{J}(V, \omega)$ is contractible.*

Proof. The space $\mathrm{Met}(V)$ is contractible since it is a convex subset of a vector space. Given J_0 and J_1 in $\mathcal{J}(V,\omega)$, let $g_i = \omega(\cdot, J_i\cdot)$, for $i = 0, 1$, and $g_t = (1-t)g_0 + tg_1$. For all $t \in [0,1]$, g_t is a metric and gives (by polar decomposition) an ω-compatible $\tilde{J}_t$, with $\tilde{J}_0 = J_0$ and $\tilde{J}_1 = J_1$. □

1.3 The group of linear symplectomorphisms

Recall: we denote by $\mathrm{Sp}(V,\omega)$ the group of symplectomorphisms of the symplectic vector space (V,ω). Since any two symplectic vector spaces are isomorphic, the general linear group $GL(V)$ acts transitively on the open subset of $\wedge^2 V^*$ consisting of non-degenerate skew-symmetric bilinear forms. The stabilizer of ω is exactly $\mathrm{Sp}(V,\omega)$. Hence we can compute

$$\dim \mathrm{Sp}(V,\omega) = \dim GL(V) - \dim \wedge^2 E^* = (2n)^2 - \frac{2n(2n-1)}{2} = 2n^2 + 2.$$

Example 1.34. Let W be a Lagrangian subspace of (V,ω) and identify V with $W \oplus W^*$. Given $A \in GL(W)$, let $A^* \in GL(W^*)$ be the dual map. Then $A \oplus (A^{-1})^*$ is a symplectomorphism. In other words, there is an embedding $GL(W) \to \mathrm{Sp}(V)$. This shows in particular that $\mathrm{Sp}(V,\omega)$ is not compact.

Example 1.35. Another natural subgroup of $\mathrm{Sp}(V,\omega)$ is the group $U(V)$ of automorphisms preserving the Hermitian structure for a given compatible complex structure $J \in \mathcal{J}(V,\omega)$.

Let us now fix a compatible complex structure $J \in \mathcal{J}(V,\omega)$ and let g be the corresponding inner product. Let $(\cdot)^*$ denote the transpose of an endomorphism with respect to g.

Lemma 1.36. *An automorphism $A \in GL(V)$ is symplectic if and only if $A^*JA = J$.*

Proof. An automorphism A is an element of $\mathrm{Sp}(V,\omega)$ if and only if $\omega(Av, Aw) = \omega(v,w)$ for all $v, w \in V$. Equivalently, $A \in \mathrm{Sp}(V,\omega)$ if and only if $g(JAv, Aw) = g(Jv, w)$ for all $v, w \in V$, i.e., $A^*JA = J$. □

We are now going to reorder the coordinates on $\mathbb{R}^{2n}$ in the following way: $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. This means that we get the following expression for the canonical symplectic form:

$$\omega_0(v,w) = \sum_{i=1}^{n} (v_i w_{n+i} - v_{n+i} w_i),$$

while the compatible complex structure J_0 is now defined by

$$J_0(v_1, \ldots, v_{2n}) = (-v_{n+1}, \ldots, -v_{2n}, v_1, \ldots, v_n).$$

We can now easily define the embedding

$$\Phi : GL(n,\mathbb{C}) \to GL(2n,\mathbb{R}), \quad \Phi(A + iB) = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

which allows us to consider $GL(n, \mathbb{C})$ as a subgroup of $\mathbb{R}^{2n}$. Notice that as such it consist of those matrices that commute with J_0.
Then if we denote by $\mathrm{Sp}(2n)$ the group of symplectic $2n \times 2n$ matrices, i.e., the group of symplectic automorphisms of $(\mathbb{R}^{2n}, \omega_0)$, we have the following "trefoil" identities:

$$\mathrm{Sp}(2n) \cap GL(n, \mathbb{C}) = GL(n, \mathbb{C}) \cap O(2n) = O(2n) \cap \mathrm{Sp}(2n) = U(n).$$

Here $O(2n) = \{A \in GL(2n, \mathbb{R}) \ : \ Q^*Q = QQ^* = \mathrm{Id}\}$ denotes the group of orthogonal $2n \times 2n$ matrices and should be thought of as the group of transformations of $\mathbb{R}^{2n}$ preserving the standard Euclidean metric structure, while $U(n) = \{A \in GL(n, \mathbb{C} \ : \ U^* = U^{-1}\}$ denotes the group of unitary $n \times n$ matrices. Please notice that in the last definition, U^* denotes the conjugate transpose and so we can also define unitary $n \times n$ matrices as $\{X + iY \ : \ X^*X + Y^*Y = \mathrm{Id} \text{ and } X^*Y - Y^*X = 0\}$.

Homework 1.37. Prove the trefoil identities.

1.4 Linear symplectic reduction

In this section we show that the quotient of a given co-isotropic subspace by its symplectic complement inherits a natural symplectic structure.
Suppose W is a co-isotropic subspace of the symplectic vector space (V, ω). The kernel of the restriction of ω to W is, by definition, $W \cap W^\omega$, which in the co-isotropic space is equal to W^ω. Denote by $\overline{W}$ the quotient W/W^ω and by π be the quotient map $\pi : W \to \overline{W}$. Then we can define

$$\overline{\omega}(\pi(v), \pi(w)) = \omega(v, w) \text{ for all } v, w \in W.$$

Homework 1.38.

1. Prove that the dimension of $\overline{W}$ is twice the codimension of W.
2. Prove that $\overline{\omega}$ is a well-defined symplectic structure on $\overline{W}$.

The pair $(\overline{W}, \overline{\omega})$ is called the reduced space or symplectic quotient.
An interesting fact about symplectic quotients is that Lagrangian subspaces are preserved by reduction.

Lemma 1.39. *Suppose $W \subset V$ is co-isotropic and L is a Lagrangian subspace of V.Then the image of $L \cap W$ under the reduction map $\pi : W \to \overline{W}$ is a Lagrangian subspace of $\overline{W}$.*

Proof. Since L is Lagrangian and W co-isotropic, the intersection $L \cap W$ is isotropic and so is $\pi(L \cap W)$. The result follows form a dimension count. We have namely: $\dim(\pi(L \cap W)) = \dim(L \cap W) - \dim(L \cap W^\omega) = \dim W - \dim L$, and that is exactly one half of the dimension of $\overline{W}$. □

Homework 1.40. Fill in the details of the above proof.

Chapter 2

Normal form theorems

2.1 Moser trick and Darboux theorem

We will first recall some facts about time dependent vector fields. A time dependent vector field is a family X_t of vector fields, $t \in [0,1]$ which have the form $X_t(x) = \sum a_i(t,x)\frac{\partial}{\partial x_i}$, with each a_i a smooth function of (t,x). An isotopy of M determines a time-dependent vector field by

$$\varphi_t : M \to M,\ \varphi_0 = \mathrm{id},\ \frac{d}{dt}\varphi_t = X_t \circ \varphi_t.$$

Time dependent vector fields can be viewed as vector fields on $[0,1] \times M$ by setting $\widetilde{X}(t,x) = \frac{\partial}{\partial t} \oplus X_t(x)$. In particular, a time dependent vector field X_t generates a local isotopy φ_t. If X_t is compactly supported then $\varphi_t(x)$ is defined for all $(t,x) \in [0,1] \times M$. If $X_t(x) = 0$ for all $t \in [0,1]$, then there exists an open neighborhood U of x such that $\varphi_t : U \to M$ is defined for all t.
The Lie derivative along the vector field is defined by

$$\mathcal{L}_{X_t}\omega = \frac{d}{dt}\varphi_t^*\omega|_{t=0}$$

and it satisfies

$$\frac{d}{dt}\varphi_t^*\omega = \varphi_t^*\mathcal{L}_{X_t}\omega.$$

Moreover, if $\{\omega_t\}$ is a smooth family of differential forms, one has

$$\frac{d}{dt}\varphi_t^*\omega_t = \varphi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt}\right).$$

All the normal form theorems can be proved by a technique called *Moser's trick*. Here is a first example to illustrate this technique.

Theorem 2.1. *Let M be a compact manifold and $\{\omega_t\}$, $t \in [0,1]$, a smooth family of symplectic forms on M with exact time derivative:*

$$\frac{d}{dt}\omega_t = d\alpha_t.$$

Then there exists a smooth isotopy $\{\varphi_t\}$ such that

$$\varphi_t^*\omega_t = \omega_0 \quad \text{for all} \quad t \in [0,1].$$

Proof. The idea is to find the isotopy as the flow of a time-dependent vector field X_t. Suppose $\frac{d}{dt}\varphi_t = X_t \circ \varphi_t$, $\varphi_0 = \text{id}$. Then $\varphi_t^*\omega_t = \omega_0$ holds provided $\frac{d}{dt}\varphi_t^*\omega_t = 0$. Using Cartan formula and the properties of the Lie derivative we obtain:

$$\frac{d}{dt}\varphi_t^*\omega_t = \varphi_t^*\left(d\iota_{X_t}\omega_t + \frac{d\omega_t}{dt}\right)$$

and hence $\frac{d}{dt}\varphi_t^*\omega_t = 0$ if and only if

$$d\iota_{X_t}\omega_t + \frac{d\omega_t}{dt} = d(\alpha_t + \iota_{X_t}\omega_t) = 0.$$

Since ω_t is non-degenerate for all t, there exists a solution X_t of $\alpha_t + \iota_{X_t}\omega_t) = 0$. The flow $\{\varphi_t\}$ of X_t satisfies (by construction) $\varphi_t^*\omega_t = \omega_0$. □

Remark 2.2.

We can use Moser's argument to prove that if two symplectic forms agree at every point of a compact submanifold N, then they are symplectomorphic in a neighborhood of N.

Proposition 2.3 (Moser's argument)**.** *Let M be a $2n$-dimensional manifold, $N \subset M$ a compact submanifold, and ω_0 and ω_1 symplectic forms that agree at each point $x \in N$. Then there exists open neighbourhoods U_0 and U_1 of N and a diffeomorphism $\psi : U_0 \to U_1$ such that $\psi^*\omega_1 = \omega_0$ and $\psi|_N = id$.*

Proof. The first step in the proof is to show that there exists a neighbourhood of N where $\omega_1 - \omega_0$ is exact. Let $\tau = \omega_1 - \omega_0$ and let i denote the embedding of N in M. Choose a tubular neighbourhood of N, i.e., a diffeomorphism

$$\chi : i^*TM/TN \to U$$

onto an open neighborhood of N in M. For $t \in [0,1]$ define

$$\varphi_t : U \to U,\ \varphi_t(\chi(x,v)) = \chi(x,tv),$$

and let $\beta_t \in \Omega^1(U)$ be given at the point $\chi(x,v)$ by

$$\beta_t(V) = \tau\left(\frac{d}{dt}\varphi_t, (\varphi_t)_*(V)\right),$$

and $\beta = \displaystyle\int_0^1 \beta_t$. We will show that $\tau = d\beta$.

Since $\varphi_t|_N = \mathrm{id}$, α_t vanishes on N for all t. Let $t > 0$ and define $X_t = \dfrac{d}{dt}\varphi_t \circ \varphi_t^{-1}$. Then we have:

$$\frac{d}{dt}\varphi_t^* \tau = \varphi_t^* \mathcal{L}_{X_t}\tau = \varphi_t^*(\iota_{X_t} d\tau + d\iota_{X_t}\tau) = d\varphi_t^*\iota_{X_t}\tau = d\beta_t.$$

Observe now that $\varphi_0(U) \subset N$ implies that $\varphi_0^*\tau = 0$ and $\varphi_1 = \mathrm{id}$ implies $\varphi_1^*\tau = \tau$. It follows that

$$\tau = \varphi_1^*\tau = \varphi_0^*\tau = \int_0^1 (d\beta_t)\, dt = d\beta.$$

Of course, this first step would also follow from a relative version of the de Rham theorem: since τ is closed and vanishes along N, and since N and U are homotopy equivalent, there exists a 1-form β on U such that $\beta = 0$ along N and $d\beta = \tau$ on U.
Now define

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0),\ \ t \in [0,1].$$

Clearly ω_t is closed for all t. Since ω_0 and ω_1 agree along N, $\omega_t|_N = \omega_0$ and since non-degeneracy is an open condition, by restricting U if necessary, we can make sure that $\omega_t|_U$ is non-degenerate for all t. Let X_t be the unique vector field satisfying

$$\beta + \omega_t(X_t, \cdot) = 0.$$

This time we need to be more careful, because U is non-compact and the flow of X_t might not exist for all times t. Since $\beta = 0$ along N, X_t vanishes along N. Hence there exists a tubular neighborhood U_0 of Nsuch that the flow $\{\psi_t\}$ of X_t exists for all $x \in U_0$ and $t \in [0,1]$ and $\psi_t(U_0) \subset U$ for all $t \in [0,1]$. Let $U_1 = \psi_1(U_0)$: then $\psi_1 : U_0 \to U_1$ is a diffeomorphism and it satisfies:

$$\frac{d}{dt}\psi_t^*\omega_t = \psi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt}\right) = \psi_t^*(-d\beta + \omega_1 - \omega_0) = 0.$$

Since X_t vanishes along N for every t, ψ restricts to the identity on N. □

The following result, known as *Darboux's theorem* is now a simple corollary of the above proposition, but for historical reasons we will still call it a theorem. The original proof did not make use of Moser's argument.

Theorem 2.4 (Darboux's theorem)**.** *Let (M, ω) be a symplectic manifold of dimension $2n$. Then for every point $x \in M$ there exists a neighborhood U of x and a chart $\varphi : U \to \mathbb{R}^{2n}$ such that $\varphi^*\omega_0 = \omega$.*

Proof. Let ψ be any chart centered at x. By the classification result for symplectic vector spaces, we find a linear isomorphism Φ of $\mathbb{R}^{2n}$ such that ω coincides with $\Phi^*\psi^*\omega_0$ at the point x. We can now apply the above proposition, with $N = \{x\}$. □

We will consider the following equivalence notions for symplectic structures: two symplectic forms ω_1 and ω_2 on a manifold M are called:

1. *symplectomorphic* if there exists a diffeomorphism Φ of M such that $\omega_0 = \Phi^*\omega_1$;
2. *deformation equivalent* if they are related by a path of symplectic forms;
3. *isotopic* if they are related by a path of cohomologous symplectic forms;
4. *strongly isotopic* if there exists an isotopy φ_t of M such that $\varphi_1^*\omega_1 = \omega_0$.

We can immediately see that the following implications hold:

$$\text{strongly isotopic} \Rightarrow \text{symplectomorphic}$$
$$\text{isotopic} \Rightarrow \text{deformation equivalent}$$
$$\text{strongly isotopic} \Rightarrow \text{isotopic}$$

where the last implication follows from homotopy invariance of de Rham cohomology.
The following theorem shows that on closed manifold the notions of *isotopic* and *strongly isotopic* symplectic forms are in fact equivalent. In other words, on cannot change the symplectic structure within a fixed cohomology class.

Theorem 2.5 (Moser's stability). *Let M be a closed manifold. Suppose, ω_t, $t \in [0,1]$ is a smooth family of symplectic forms on M such that the cohomology class $[\omega_t]$ is independent of t. Then there exists a smooth isotopy $\varphi_t : M \to M$ such that $\varphi_t^*\omega_t = \omega_0$ for all $t \in [0,1]$.*

Proof. The main ingredient of this proof is the existence of a smooth family of 1-forms σ_t such that $\dfrac{d}{dt}\omega_t = d\sigma_t$. Then the claim follows immediately from Moser's trick. □

Example 2.6 (Closed symplectic surfaces). Let Σ be a closed orientable surface. Then symplectic structures on Σ are just area forms and they are classified (up to isomorphism) by their total area.
In fact, if ω_0 and ω_1 are symplectomorphic, then their total area is equal by the change of variables forum for integrals.
Conversely, assume that $\operatorname{Area}(\Sigma, \omega_0) = \operatorname{Area}(\Sigma, \omega_1)$ and that the orientations induced by ω_o and ω_1 agree. Then $[\omega_0] = [\omega_1]$ and hence $\omega_1 - \omega_0 = d\alpha$ for some $\alpha \in \Omega^1(\Sigma)$. Define $\omega_t = (1-t)\omega_0 + t\omega_1$. Then ω_t is a symplectic form for all t, and $[\omega_t] = [\omega_0]$ (i.e., ω_0 and ω_1 are isotopic). By Moser's stability there exists a smooth isotopy φ_t of Σ such that $\varphi_t^*\omega_t = \omega_0$ for all t (i.e., ω_0 and ω_1 are strongly isotopic). In particular, $\varphi_1^*\omega_1 = \omega_0$, i.e., ω_1 and ω_0 are isomorphic. If the orientations do not agree, let Ψ be an orientation-reversing diffeomorphism of Σ and apply the previous argument to ω_0 and $\Psi^*\omega_1$. The required symplectomorphism is $\Psi \circ \varphi_1$.

2.2 Symplectic, (co-)isotropic and Lagrangian submanifolds

Definition. A submanifold N of a symplectic manifold (M, ω) is called symplectic (isotropic, co-isotropic, Lagrangian) if $T_xN \subset T_xM$ is a symplectic (isotropic, co-isotropic, Lagrangian) subspace for every $x \in N$.

Example 2.7.
1. Let (M_i, ω_i), $i = 1, 2$, be symplectic manifolds. Let π_i, $i = 1, 2$, denote the projection $M_1 \times M_2 \to M_i$. Then $\pi_1^*\omega_1 - \pi_2^*\omega_2$ is a symplectic form on $M_1 \times M_2$. For every $x \in M_2$, $M_1 \times \{x\}$ is a symplectic submanifold of $M_1 \times M_2$.

2. The graph of a diffeomorphism $f : M_1 \to M_2$ is a Lagrangian submanifold of $M_1 \times M_2$ if and only if f is symplectic.

3. For any smooth manifold L, the zero section and the fibers of the cotangent bundle T^*L are Lagrangian submanifolds (with respect to the canonical symplectic form).

Weinstein's celebrated Lagrangian tubular neighborhood theorem states that a neighborhood of a Lagrangian submanifold $L \subset (M, \omega)$ depends symplectically only on the differential topology of L.

Theorem 2.8 (Weinstein's Lagrangian neighbourhood theorem)**.** *Let L be a compact Lagrangian submanifold of the symplectic manifold (M, ω). Then there exist tubular neighborhoods U of the zero section in T^*L and U' of L in M and a diffeomorphism $\varphi : U \to U'$ such that $\varphi|_L$ is the inclusion and $\varphi^*\omega = \omega_{can}$.*

Proof. We are first going to show that we can "match" ω and ω_{can} along the zero section of T^*L. Then we will apply Moser's argument.
Choose an ω-compatible almost complex structure J on TM, i.e., $J : TM \to TM$ and $J^2 = -\text{id}_{TM}$. Such a compatible almost structure always exists. Let $g = \omega(\cdot, J\cdot)$ be the associated metric. Then for every $q \in M$, JT_qL is the orthogonal complement (w.r.t. g) of T_qL in T_qM. The exponential map $\exp : (T_qL)^\perp \to M$ is a diffeomorphism onto its image when restricted to a small ϵ-neighbourhood of the zero section. Let $k : T^*M \to TM$ be the isomorphism induced by g, so that $g(k(p), v) = p(v)$. Then on a suitable small neighborhood V of the zero section of T^*L we can define

$$\Phi : V \to M, \quad (q, p) \mapsto \exp_q(Jk(p)).$$

Under the identification $T_{(q,0)}(T^*L) = T_qL \oplus T_q^*L$ we get:

$$d\Phi_{(q,0)}(v, p) = v + Jk_q(p),$$

where the sum on the right-hand side is the sum of vectors in T_qM. Now we compute:

$$\begin{aligned}(\Phi^*\omega)_{(q,0)}((v,p),(v',p')) &= \omega_q(v+Jk(p), v'+Jk(p'))\\ &= \omega_q(v, Jk(p')) - \omega(v', Jk(p))\\ &= g_q(v,k(p')) - g_q(v',k(p))\\ &= p'(v) - p(v')\\ &= (d\lambda_{\mathrm{can}})_{(q,0)}((v,p),(v',p')).\end{aligned}$$

Notice that in the above calculations we have used that ω is J-invariant and that $\omega_q(v,v') = 0 = \omega_q(k(p),k(p'))$ because T_qL is Legendrian. We have thus shown that $d\lambda_{\mathrm{can}}$ and ω agree over the zero section of T^*L, and we can apply Moser's argument to finish the proof. □

We end this section with a more general normal form theorem for compact submanifolds of a symplectic manifold (M,ω). For this we will first need to introduce some definitions. Let $\pi : E \to X$ be a (smooth) real vector bundle of rank $2n$. A symplectic structure on E is a collection $(\omega_x)_{x\in X}$, where ω_x is a symplectic bilinear form on E_x, which varies smoothly with x (i.e., a smooth section of $\Lambda^2E^* \to X$ such that each $\omega_x \in \Lambda^2 E_x^*$ is a linear symplectic form. We call (E,ω) a *symplectic vector bundle.*
A complex structure J on the symplectic vector bundle E is called ω-compatible if for each $x\in X$, J_x is ω_x-compatible. The space of compatible complex structures on (E,ω) is denoted by $\mathcal{J}(E,\omega)$.

Theorem 2.9 (Space of compatible complex structures on a symplectic vector bundle)**.** *For any symplectic vector bundle (E,ω), the space of compatible complex structures $\mathcal{J}(E,\omega)$ is non-empty and contractible*

An *isomorphism of symplectic vector bundles* (E,ω) and (E',ω') is a vector bundle isomorphism $\Phi : E \to E'$ (covering the identity map) such that $\Phi^*\omega' = \omega$. Given a sub bundle of a symplectic vector bundle (E,ω) over X we define

$$W^\omega = \{(x,v) \,:\, x\in X \text{ and } v \in W_x^{\omega_x}\}.$$

Example 2.10.

1. If (M,ω) is a symplectic manifold, then (TM,ω) is symplectic vector bundle.
2. If $E\to X$ is any vector bundle, there is a canonical symplectic structure on the Whitney sum $E\oplus E^*$, defined by

$$(\omega_E)_x((v,\eta),(v',\eta')) = \eta(v') - \eta'(v) \quad \text{for all} \quad v,v'\in E_x,\ \eta,\eta'\in E_x^*.$$

3. if N is a symplectic submanifold of M, then TN is a symplectic subbundle of $TM|_N$ (i.e., T_xN is a symplectic subspace of T_xM for every $x\in N$), and we have a symplectic direct sum decomposition

$$TM|_N = TN \oplus TN^\omega.$$

Proposition 2.11 (Normal form for subbundles of symplectic vector bundles)**.** *Let* (E, ω) *be a rank* $2n$ *symplectic vector bundle and* $W \subset E$ *a rank* $2k + l$ *subbundle, such that* $V = W \cap W^\omega$ *has constant rank* l*. Then*

$$(E, \omega) \cong (W/V, \omega) \oplus (W^\omega/V, \omega) \oplus (V \oplus V^*, \omega_V).$$

Proof. Pick a compatible almost complex J on (E, ω). Then

$$U_1 = W \cap JW, \quad U_2 = W^\omega \cap JW^\omega, \quad \text{and} \quad U_3 = JV$$

are smooth subbundles of E. Then the decomposition

$$E = U_1 \oplus V \oplus U_2 \oplus U_3$$

induces a symplectic vector bundle isomorphism

$$\begin{gathered}(E, \omega) \cong (W/V, \omega) \oplus (W^\omega/V, \omega) \oplus (V \oplus V^*, \omega_V), \\ u_1 + v + u_2 + u_3 \mapsto \big(u_1, u_2, (v, -i_{u_3}\omega)\big)\end{gathered}$$

□

Now we can state the more general normal form result for submanifolds of a symplectic manifold.

Theorem 2.12 (Symplectic normal forms)**.** *Let* ω_0 *and* ω_1 *be symplectic forms on a manifold* M*, and* $N \subset M$ *a compact submanifold such that* $\omega_0|_N = \omega_1|N$*. Suppose moreover that* $K := \ker(\omega_0|_N) = \ker(\omega_1|N)$ *has constant rank, and the bundles* $(TN^{\omega_0}/K, \omega_0)$ *and* $(TN^{\omega_1}/K, \omega_1)$ *are isomorphic as symplectic vector bundles. Then there exist tubular neighborhoods* U_0 *and* U_1 *of* N *and a diffeomorphism* $\varphi : U_0 \to U_1$ *such that* $\varphi|_N = id$ *and* $\varphi^*\omega_1 = \omega_0$*.*

Proof. By the previous proposition, we have that

$$(TM|_N, \omega_0) \cong (TN/K, \omega_0) \oplus (TN^{\omega_0}/K, \omega_0) \oplus (K \oplus K^*, \omega_K),$$

and similarly for ω_1.
By hypothesis, the right hand side is isomorphic for ω_0 and ω_1. Hence the left hand side is also isomorphic, i.e, there exists a symplectic vector bundle isomorphism $\Phi : TM|_N \to TM|_N$ with such that $\Phi|_{TN} = \mathbb{1}$. Then by the Whitney Extension theorem, Φ extends to a an embedding $\varphi : U \to M$ of a tubular neighborhood of N such that $\varphi|N = \text{id}$ and $d\varphi_x = \Phi_x$ for all $x \in N$. It follows that $\varphi^*\omega_1 = \omega_0$ along N. We can now apply Moser's argument to the submanifold N and the symplectic forms ω_0 and $\varphi^*\omega_1$. □

All the normal forms for submanifolds of a symplectic manifold are corollaries of this result. For instance, if the submanifold N is symplectic with respect to both ω_0 and ω_1, then the bundle K is trivial and we have the vector bundle isomorphism

$$(TM|_N, \omega_0) \cong (TN, \omega_0) \oplus (TN^{\omega_0}, \omega_0),$$

and similarly for ω_1. If we assume the two symplectic forms to coincide along N and the symplectic normal bundles $(TN^{\omega_0}, \omega_0)$ and $(TN^{\omega_1}, \omega_1)$ to be isomorphic (as symplectic vector bundles), then we satisfy the conditions of the symplectic normal forms theorem and we get a symplectomorphism of neighborhoods U_0 and U_1 of N, equipped with ω_0 and ω_1, respectively, that fixes N.

Corollary 2.13 (Symplectic neighbourhood theorem)**.** *Let ω_0 and ω_1 be symplectic forms on a manifold M, and $N \subset M$ a compact submanifold such that $\omega_0|_N = \omega_1|_N$ is symplectic. Suppose moreover that the bundles $(TN^{\omega_0}, \omega_0)$ and $(TN^{\omega_1}, \omega_1)$ are isomorphic as symplectic vector bundles. Then there exist tubular neighborhoods U_0 and U_1 of N and a diffeomorphism $\varphi : U_0 \to U_1$ such that $\varphi|_N = id$ and $\varphi^*\omega_1 = \omega_0$.*

2.3 Some applications of normal forms

Fixed points of a symplectomorphism

Let (M, ω) be a symplectic manifold, φ a symplectomorphism of M. We can reformulate the problem of finding fixed points of φ in terms of intersections of Lagrangian submanifold. There is in fact a one-to-one correspondence between

$$\{\text{fixed points of } \varphi\} \xleftrightarrow{1-1} \{\text{intersections of } \Gamma_\varphi \text{ with the diagonal } \Delta \subset M \times M\},$$

where Γ_φ denotes the graph of φ.

Corollary 2.14. *Suppose (M, ω) is compact and $H^1_{dR} = 0$. Then every $\varphi \in Symp(M, \omega)$ C^1-close to the identity has at least 2 fixed points.*

Proof. Recall that by Weinstein's Lagrangian neighborhood theorem, a neighborhood of a Lagrangian submanifold can always be symplectically identified with a neighborhood of the zero section in the cotangent bundle of the manifold. If φ is C^1-close to the identity, $\Gamma_\varphi \subset T^*M$ is close to the zero section M_0 and can be written as the graph of a 1-form σ. But $\sigma = dh$ as $H^1_{\mathrm{dR}} = 0$, and

$$p \in \Gamma_\varphi \cap M_0 \Leftrightarrow dh_p = 0 \Leftrightarrow p \in \mathrm{Crit}(h).$$

Since M is compact, h has at least one maximum and one minimum. □

Remark 2.15.

1. This result fails if $H^1_{\mathrm{dR}} \neq 0$; think of translation on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

2. If $H^1_{\mathrm{dR}} \neq 0$, the result holds for Hamiltonian symplectomorphisms.

Gompf's connected sum construction

Suppose we have two symplectic embeddings $i_j : (N, \sigma) \to (M_j, \omega_j)$, $j = 1, 2$, of a compact codimension 2 submanifold in M_1 and M_2. Suppose also that both embeddings have trivial normal bundle. Then from the symplectic neighborhood theorem we get symplectic embeddings $\varphi_j : N \times D_\epsilon \to M_j$, such that $\varphi_j(N \times \{0\}) = i_j(N)$ and where the symplectic form on the trivial disc bundle is $\sigma \oplus dx \wedge dy$. Remove the origin from each D_ϵ to get an annulus $A = D_\epsilon \setminus \{0\}$. The annulus admits a symplectic automorphism that interchanges the boundary components:

$$\psi : A \to A; \quad \psi(r, \theta) = (\sqrt{\epsilon^2 - r^2}, -\theta).$$

Hence we can construct the *fibre connected sum* of M_1 and M_2 along N:

$$M_1 \sharp_N M_2 = M_1 \setminus i_1(N) \cup_{\mathrm{id} \times \psi} M_2 \setminus i_2(N)$$

where $\mathrm{id} \times \psi : \varphi_1(N \times D_\epsilon) \to \varphi_2(N \times D_\epsilon)$. This manifold carries a natural symplectic structure, which coincides with ω_1 and ω_2 outside of a tubular neighborhood of N. This means that away from N, nothing changes, so for instance symplectic/Lagrangian submanifolds of M_j which are disjoint from N stay symplectic/Lagrangian.
The above construction was used by Gompf to prove:

Theorem 2.16. *Any finitely generated group G appears ads the fundamental group of some compact, 4-dimensional symplectic manifold.*

Gompf's contraction is an example of how to construct new symplectic manifolds from old ones by *surgery*. In the next section we will see how to construct new symplectic manifolds from *symmetry*.

Chapter 3

Contact geometry

3.1 Contact structures and Reeb dynamics

Integrability and the theorem of Frobenius

We will start by recalling some facts about integrable distributions. Let M be a smooth n-dimensional manifold, $1 \leq d \leq n$. A d-dimensional distribution $\mathcal{D}$ on M is a choice of a d-dimensional subspace $\mathcal{D}_p$ of T_pM for each $p \in M$. A distribution $\mathcal{D}$ is smooth if for every $p \in M$ there exists a neighborhood U of p and d smooth vector fields $X_1, \ldots, X_d$ on U which span $\mathcal{D}$ at each point of U. A vector field X on M is said to lie in the distribution $\mathcal{D}$ if $X_p \in \mathcal{D}_p$ for all $p \in M$.

An immersed submanifold $i : N \to M$ is an integral submanifold of a distribution $\mathcal{D}$ if $i_*(T_pN) = \mathcal{D}_{i(p)}$ for all $p \in N$. A distribution $\mathcal{D}$ is called integrable if through each point of M there exists an integral manifold of $\mathcal{D}$. The theorem of Frobenius states that a distribution is integrable if and only if it is involutive, i.e., $[X, Y] \in \mathcal{D}$ whenever X and Y are smooth vector fields lying in $\mathcal{D}$.

Theorem 3.1 (Frobenius)**.** *Let $\mathcal{D}$ be a d-dimensional, smooth distribution on M. Then $\mathcal{D}$ is integrable if and only if it is involutive*

Remark 3.2. (i) A 1-dimensional distribution is the same as a non-vanishing vector field, the image of an integral curve is an integral manifold. So a 1-dimensional distribution is always integrable by local existence of integral curves.

(ii) The smooth distribution $\mathcal{D}$ defined on $\mathbb{R}^3$ by the vector fields

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \quad \text{and} \quad Y = \frac{\partial}{\partial y}$$

is not integrable: there is no integral manifold through the origin of $\mathbb{R}^3$. Equivalently, it is not involutive: in fact, $[X, Y] = -\dfrac{\partial}{\partial z}$, which does not lie in $\mathcal{D}$.

Distributions can also be described in terms of smooth 1-forms.

Lemma 3.3. *A d-dimensional distribution $\mathcal{D}$ on M is smooth if and only if each point p has a neighborhood U on which there exist smooth 1-forms $\omega^1, \ldots, \omega^{n-d}$ such that*

$$\mathcal{D}_q = \ker \omega^1_q \cap \ldots \cap \ker \omega^{n-d}_q$$

for all $q \in U$.

We say that a k-form ω annihilates $\mathcal{D}$ if $\omega(X_1, \ldots, X_k) = 0$ whenever $X_1, \ldots, X_k$ are local sections of $\mathcal{D}$. The following proposition gives a characterization of integrable (involute) distributions.

Proposition 3.4. *Suppose $\mathcal{D}$ is a smooth distribution on M. Then $\mathcal{D}$ is integrable if and only if the following condition is satisfied: if η is any smooth 1-form that annihilates $\mathcal{D}$ on an open subset $U \subset M$, then $d\eta$ also annihilates $\mathcal{D}$ on U.*

We now turn to hyperplane distributions (i.e., the case $d = n - 1$). Let ξ be a hyperplane distribution, then it can always be described locally as the kernel of a 1-form α and therefore integrability in this context is equivalent to the following: $d\alpha = 0$ when restricted to vectors in ξ, or equivalently, $\alpha \wedge d\alpha = 0$. The contact condition is as far from this as possible: it requires $d\alpha$ to restrict to a non-degenerate form on ξ.

Contact structures and Reeb vector fields

Let M be a smooth manifold, $\xi \subset TM$ a hyperplane field (i.e., a smooth codimension 1 sub-bundle). Write $TM = \xi \oplus \xi^\perp$ with respect to an auxiliary Riemannian metric g. Locally, ξ can always be described as the kernel of a 1-form, by trivializing $\xi^\perp$ around a point. This can be done globally if and only if ξ is co-orientable (i.e., $\xi^\perp$ is orientable).

Definition 3.5. *Let ξ be a co-orientable hyperplane field on a smooth manifold M of dimension $2n+1$ and α a 1-form such that $\xi = \ker \alpha$. Then $d\alpha$ is non-degenerate on ξ is and only if*

$$\alpha \wedge (d\alpha)^n \neq 0.$$

If this is the case, ξ is called a contact structure and α is called a contact form.

Definition 3.6. *Associated to any contact form there is a uniquely defined vector field R_α, called the Reeb vector field of α. It is defined by the conditions*

$$\iota_{R_\alpha} d\alpha = 0 \quad \textit{and} \quad \iota_{R_\alpha} \alpha = 1.$$

Definition 3.7. *Let (M, ξ) be a contact manifold with contact form α. A contactomorphism of (M, ξ) is a is a diffeomorphism of M which preserves the contact structure, i.e., $f_*(\xi) = \xi$. A contactomorphism is called strict if it also preserves the contact form.*

Remark 3.8. (i) Even though the definition of a contact structure ξ is independent of the choice of contact form, choosing different contact forms leads to different vector fields and flows with different properties.

(ii) If α is a contact form for the contact structure ξ and R_α its Reeb vector field, the flow of R_α preserves α and hence ξ.

Example 3.9. (i) On $\mathbb{R}^{2n+1}$ with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$, the 1-form

$$\alpha_1 = dz + \sum_{i=1}^{n} x_i dy_i$$

is a contact form. The corresponding Reeb vector field is $R_1 = \dfrac{\partial}{\partial z}$.

(ii) The restriction of the form

$$\alpha_0 = \sum_{j=1}^{n+1} (x_j dy_j - y_j dx_j)$$

to $S^{2n+1} \subset \mathbb{R}^{2n+2}$ defines a contact structure.

(iii) If B is a manifold, the restriction of the Liouville form λ_{can} on T^*B to the unit cotangent bundle ST^*B is a contact form.

The last two examples actually fit in a larger class which we will describe in the next section.

Hypersurfaces of contact type

One comes across contact strucures naturally when considering hypersurfaces in a symplectic manifold that are everywhere transverse to a given Liouville vector field.
Recall that a Liouville vector field Y on a symplectic manifold (W, ω) is a smooth vector field on W such that $\mathcal{L}_Y \omega = \omega$.

Lemma 3.10. *Let Y be a Liouville vector field on the symplectic manifold (W, ω). Then $\alpha := \iota_Y \omega$ restricts to a contact form on each hypersurface $M \subset W$ which is everywhere transverse to Y.*

Proof. It follows from Cartan's formula that $d\alpha = \omega$. Now suppose $\dim W = 2n$ and $M \subset W$ is a smooth codimension one submanifold which is everywhere transverse to Y. Then

$$\alpha \wedge (d\alpha)^{n-1} = \iota_Y \omega \wedge \omega^{n-1} = \frac{1}{n} \iota_Y (\omega^n).$$

Since $\omega_n \neq 0$ on W, the last term must be non-zero along M provided $Y_p \notin T_p M$ for all $p \in M$. □

If M is as in the above lemma, it is called a hypersurface of *contact type*.

Moser's argument and Darboux's theorem

Just as in the case of symplectic manifolds, there is a local normal form theorem for contact manifolds.

Theorem 3.11 (Darboux). *Let α be a contact form on the $(2n+1)$-dimensional manifold M and $p \in M$. Then there exists coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$ in a neighborhood U of p such that*

$$\alpha|_U = dz + \sum_{i=1}^{n} x_i dy_i.$$

Theorem 3.12 (Gray Stability). *Let ξ_t, $t \in [0,1]$ be a smooth family of contact structures on a closed manifold M. Then there exists an isotopy $(\varphi_t)_{t\in[0,1]}$ of M such that*

$$d\varphi_t(\xi_0) = \xi_t \quad \textit{for all} \quad t \in [0,1].$$

Remark 3.13. (i) In contrast to the symplectic case, there is no cohomological condition.

(ii) Contact forms do not satisfy stability.

Hamiltonian and Reeb dynamics: the Weinstein Conjecture

Let (W, ω) be a symplectic manifold and $H : W \to \mathbb{R}$ a smooth Hamiltonian function. If Σ is a regular level set of H (i.e., $dH(x) \neq 0$ for all $x \in \Sigma$), existence of periodic solutions of the Hamilton's equations $\dot{x}(t) = X_H(x(t))$ on Σ depends only on the hypersurface and the symplectic structure, not on the Hamiltonian function H. This is because, up to parametrization, these solutions are closed characteristics of the line bundle

$$\mathcal{L}_\Sigma = \{(x, v) \in T\Sigma) \,:\, \omega_x(v, w) = 0 \text{ for all } w \in T_x\Sigma\}.$$

Because of this, it makes sense to ask the question of existence of periodic orbits without reference to a specific Hamiltonian function: when does a hypersurface Σ in the symplectic manifold (W, ω) admit a Hamiltonian periodic orbit? In 1986, Viterbo proved the following theorem:

Theorem 3.14. *A compact hypersurface Σ in the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$ always admits periodic Hamiltonian orbits if it is of contact type.*

Viterbo's result prompted Weinstein to formulate his famous conjecture: if $Sigma$ is a compact hypersurface of contact type in a symplectic manifold (W, ω), then it always carries periodic orbits.
Another point of view on this conjecture is the one coming from contact structures an Reeb dynamics. If $\Sigma \subset (W, \omega)$ is a hypersurface of contact type, in fact, with transverse Liouville vector field Y, then $(\Sigma, \alpha := \iota_Y \omega|_\Sigma)$ is a contact manifold and the Reeb vector field is also a section of the characteristic line bundle. Hence the hyeprsurface of contact type $\Sigma \subset (W, \omega)$ admits periodic Hamitlonian orbits if and only if the contact manifold

$(\Sigma, \alpha := \iota_Y \omega|_\Sigma)$ admits periodic Reeb orbits. We can therefore generalize the Weinstein conjecture and ask: "When does the closed, strict contact manifold (Σ, α) admit a periodic Reeb orbit?"
A seminal result in this direction is Hofer's proof off the Weinstein Conjecture for S^3:

Theorem 3.15. *Every contact form on the three-sphere admits a periodic Reeb orbit.*

The importance of this theorem lies not only in the result, but also in the method of proof, which introduced a whole new technique in the study of contact geometry and Reeb dynamics, namely that of *pseudo-holomoprhic curves in symplectizations*. Another interesting feature of this proof is that it is a proof by cases: it distinguishes between forms defining a *tight* contact structure and forms defining an *overtwisted* contact structure.

3.2 Tight vs. overtwisted and symplectic fillability in dimension 3

Characteristic foliations

Let (M, ξ) be a contact 3-manifold and $S \subset M$ an embedded oriented surface. For each $x \in S$ we can consider the space $l_x = \xi_x \cap T_x S$. This subspace of $T_x M$ will be a line in $T_x S$ at most points, but at some points, which we call *singular*, we will have $l_x = T_x S$. The *characteristic foliation* is the induced (1-dimensional) singular foliation on S.

Example 3.16. If $S = S^2 \subset (\mathbb{R}^3, \xi = \ker(dz + xdy - ydx)$, so that $\xi = \mathrm{span}\{x\partial_z - \partial_y, y\partial_z + \partial_x\}$, the poles $(x = y = 0)$ will be singular points, whereas at all other points l will be one-dimensional.

Remark 3.17. Singular points cannot form an open subset of S because of the contact condition.

Example 3.18. Consider now $\mathbb{R}^3$ but with a different contact structure, namely $\xi = \ker(\cos r\, dz + r \sin r\, d\varphi)$, and let S be the closed disk with radius φ in the plane $z = 0$. Then ξ is horizontal in $r = 0$ and $r = \pi$, i.e., at the center and along the boundary of the disk. Between 0 and φ, the contact planes make one turn, so their intersection with TS is the span of ∂_r. We call this the *standard overtwisted disk*. If we push up the interior of S very slightly, the points at htte boundary are not singular any more, but l is spanned by ∂_φ at these points, so the boundary becomes a closed leaf in the characteristic foliation. the other leaves spiral out form the center point and converge towards the boundary.

Charactersitic foliations completely determine the contact structure in a neighborhood of the surface S:

Theorem 3.19. *Let (M_i, ξ_i), $i = 0, 1$, be contact manifolds, S_i embedded surfaces. If there exists a diffeomorphism $f : S_0 \to S_1$ which preserves the characteristic foliation $(Tf(l_0) = l_1)$, then f may be extended to a contactomorphism from a neighborhood of S_0 to a neighborhood of S_1.*

Definition 3.20. *A contact structure ξ on a 3-manifol M is called overtwisted (OT) if there exists an overtwisted disk $D_{OT} \subset M$. It is called tight otherwise.*

The following theorem is interesting because (combined with Darboux's theorem) it shows that every contact structure is locally tight, so the existence of an overtwisted disk really is a global question.

Theorem 3.21 (Bennequin)**.** $(\mathbb{R}^3, \xi = \ker(dz - ydx))$ *does not have an overtwisted disk.*

It follows from Eliashberg overtwisted classification that any 3-manifold has an overtwisted contact structure, On the other hand, there exist closed contact 3-manifolds that do not support any tight contact structure. It thus seems natural to ask how to detect "tightness" of a contact structure. One possible way is by studying the existence of *symplectic fillings.*

Definition 3.22. *Let $(M, \xi = \ker\alpha)$ be a contact manifold, so that $(\xi, d\alpha|_\xi)$ has a symplectic vector bundle structure, Let (W, ω) be a symplectic manifold. We call W a weak symplectic filling of M if $\partial W = M$ as oriented manifolds and $\omega|_\xi^{n-1} > 0$. WE call W a strong symplectic filling of M if $\partial W = M$ an there exists a Liouville vector field, defined near ∂W, pointing outwards along ∂W and satisfying $\xi = \ker(\iota_Y \omega|_{TM})$ as co-oriented contact structures. In this case we also say that M is the ω-convex boundary of W. If the Liouville vector filed points into W, then one calls M the ω-concave boundary.*

Remark 3.23. (i) The boundary of a strong filling is of contact type.

(ii) If ξ_0 and ξ_1 are contact structures on M filled (in the strong sense) by the same symplectic manifold (W, ω), then they are isotopic. In other words, a strong symplectic filling pretty much determines the contact structure on its boundary.

(iii) Convex and concave are different concepts in this context than in the usual geometric situation. One can find, for instance, Liouville vector fields Y_+ and Y_- on $(\mathbb{R}^2, \omega = r\,dr \wedge d\varphi)$ defined near S^1, everywhere transverse to S^1, with Y_+ pointing out of the unit disk and Y_- pointing into the unit disk.

We can now finally state the theorem relating the tightness of the contact structure on a 3-manifold to the existence of a symplectic filling:

Theorem 3.24 (Eliashberg-Gromov)**.** *If a contact manifold (M, ξ) is weakly symplectically fillable, then the contact structure is tight.*

Example 3.25. $(S^3, \xi = \ker(x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2))$ can be filled by $(B^4, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$, hence it is tight..

Remark 3.26. This theorem was one of the main reference points for everyone attempting to generalize the definition of overtwisted contact structure to higher dimensions: overtwistedness should represent an obstruction to symplectic fillability.

Bibliography

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