Problem sheet on category theory

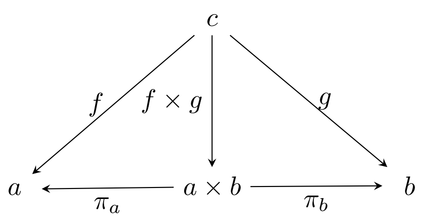
Topologie en Meetkunde, Block 3, 2022

February 1, 2022

This problem sheet may help you put some of the concepts of the course within a larger context. Categorical notions will keep appearing in most subsequent courses in Math that you may take. Whenever you encounter a new concrete mathematical construction, it is good to think about how it fits into this language.

Exercise 1. Provide proofs for each of the Lemmas in the notes on "The category of topological spaces".

Exercise 2. Let C be a category, and let $a, b \in C$ be objects. The product of a and b is an element $a \times b \in C$, together with morphisms $\pi_a : a \times b \to a$ and $\pi_b : a \times b \to b$, satisfying the **universal property of the product:** For any element $c \in C$ and pair of morphisms $f : c \to a$ and $g : c \to b$, there exists a unique morphism $f \times g : c \to a \times b$ such that:

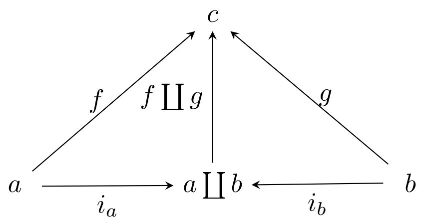


- Prove that $a \times b$ is unique up to isomorphism. I.e if there is another object $d \in C$ with morphisms $d \to a$ and $d \to b$, satisfying the universal property, then there is a unique isomorphism $\phi : d \to a \times b$.
- Prove that the usual products in sets, topological spaces, and groups satisfy this definition.

Exercise 3. Let C be a category, and let $a, b \in C$ be objects. The coproduct of a and b is an element $a \coprod b \in C$, together with morphisms $i_a : a \to a \coprod b$ and $i_b : b \to a \coprod b$, satisfying the **universal property of the coproduct:** For any element $c \in C$ and pair of morphisms $f : a \to c$ and $g : b \to c$, there exists a unique morphism $f \coprod g : a \coprod b \to c$ such that:

- Prove that $a \coprod b$ is unique. I.e if there is another object $d \in C$ with morphisms $a \to d$ and $b \to d$, satisfying the universal property, then there is a unique isomorphism $\phi : d \to a \coprod b$.
- Prove that the coproduct in sets and topological spaces is the disjoint union.
- Prove that the coproduct in abelian groups and vector spaces is the same as the product.

Exercise 4. Let G and H be groups. We define their **free product** $G * H$ as:



- The set of words in G and H:

$$\{a_1 a_2 \dots a_n \mid a_i \in G \text{ or } H,\ n \text{ any non-zero integer}\}$$

  up to the equivalence relation

$$a_1 a_2 \dots a_i a_{i+1} \dots a_n \cong a_1 a_2 \dots (a_i a_{i+1}) \dots a_n$$

  where $a_i, a_{i+1} \in G$ or $a_i, a_{i+1} \in H$ (i.e. if they are in the same group, we compose them).

- With group operation given by concatenation of words (i.e. just put them one after the other).

A word is **reduced** if you cannot compose pairs any further (i.e. elements in the word alternate as being in G or H). The word with no letters is said to be the **empty word**. Show that:

- $G * H$ is a group.
- The coproduct $\mathbb{Z} * \mathbb{Z}$ is not abelian.
- $G * H$ is the coproduct in the category of groups.

Exercise 5. Let A be a topological space. Consider the category whose objects are the opens in A and whose morphisms are the inclusions between them. Show that the product in this category is the intersection and the coproduct is the union.

Show that the same is true if you take A to be a set and then you form the category of subsets.

Exercise 6. Given a topological space, we can just regard it as a set, forgetting its topology. This defines a functor (called the **forgetful functor**) from the category of topological spaces to the category of sets. Show that:

- This is indeed a functor (check what it does on morphisms).
- It maps the product to the product and the coproduct to the coproduct.

Exercise 7. Let G be a group. Its commutator subgroup $[G, G]$ is the subgroup generated by the commutators (i.e. elements of the form $aba^{-1}b^{-1}$). The abelianisation of G is the quotient $\text{Ab}(G) := G/[G, G]$. Prove that:

- $\text{Ab}(G)$ is abelian.
- The abelianisation process defines a functor from the category of groups to the category of abelian groups. Make sure you explain what the functor does on morphisms.
- The abelianisation maps the product to the product and the coproduct to the coproduct.