Overview of midterm projects

Topologie en Meetkunde, Block 3, 2021

February 23, 2021

Dear all. Instead of doing a midterm exam, you will have to work on a small project and give a short presentation. Here are the instructions:

- Form groups of two and select one of the projects (see below). Let us know about your decision by Tuesday 23rd February (just reply to the Teams post with your answer). If you do not have a partner, just let us know about your favourite two projects and we will match you with someone else.
- You will have to submit a 5-8 page document by March 28th.
- You will give a 12-15 minute presentation on your project on the week of 8th March. The presentation will allow us to give you some feedback on the current state of the project. You may submit a draft that week in order to receive further comments as well.
- Each pair will have to attend the presentation of 2-3 other groups, and fill a short form giving them feedback. The form should be sent to me and to the speakers on the day of the talk itself. You should also write a short summary of the talks you have attended and attach it to the final document.
- Feel free to depart from the aims we propose for the project if you find something interesting. However, let us know about it first, and we will meet to see if it is appropriate. The same applies if you think the goals of the project are too ambitious: we can set a more realistic scope, but this needs to be discussed.
- Do not be afraid to ask us for further references or clarification.

Guideline for the presentation:

- Both members of the group must present for approximately the same amount of time.
- You should use slides to help structure your explanations (pictures, if possible, always help!) If you opt for some other format, let us know in advance.
- The presentation is an intermediate milestone for you to get some feedback and see you are on the correct path. In particular, we do not expect you to have covered everything in the project by then (but we do expect to see enough work already!)

Guideline for the final document:

- 5-8 pages of content, written in latex, plus bibliography.

- You state clearly what the main question is (or questions). You explain why it is an interesting/natural thing to look into. (This applies to the presentation too!)
- Provide adequate references to the sources you use.
- You may blackbox difficult results, but try to give some insight on how the reader should think about them.
- Follow the standard "definition-lemma-theorem-proof" structure, but try to connect them introducing paragraphs that help the reader follow the ideas (but do not overdo it: vague analogies hurt more than they help). It is best if you write down examples that help build intuition.
- Write the document as if your audience was yourself before starting the project. How would you make the document as understandable as possible for that person?

1 Higher homotopy groups

1.1 Loop spaces and homotopy groups

General idea: In the course we focus on the fundamental group π_1 (i.e. on detecting loops that cannot be contracted because they are wrapped around a hole). One can define other π_i, $i > 1$, called higher homotopy groups, which detect how $\mathbb{S}^i$ may be wrapped in your topological space A. Up to fixing a basepoint, we are basically looking at $[\mathbb{S}^i, A]$. Part of the project consists of looking into basic properties of these π_i.

The other half consists of understanding the loop space. We mentioned in class that the space of maps $\{\mathbb{S}^1 \to A\}$ is a (really big!) topological space, and we call it the loop space $\Omega(A)$. Points in $\Omega(A)$ are loops, which means that paths in $\Omega(A)$ are homotopies of loops. From this, you can deduce that $\pi_0(\Omega(A)) \cong [\mathbb{S}^1, A]$.

The goal of the project is exploring how the homotopy groups of the loop space relate to the homotopy groups of the original topological space, generalising the expression above.

Prerequisites: None.

Aims:

- Define the loop space and its topology.
- Optional: Define the topology in any space of maps $\{X \to A\}$.
- Optional: Explain how the topology on $\{\mathbb{S}^1 \to A\}$ induces a topology on the fundamental groupoid of A.
- Discuss how taking loop space is a functor from the category of topological spaces to itself.
- Explain how taking reduced suspension and taking loop space relate to one another (Optional: look up the notion of two functors being adjoints).
- Optional: Look into what H-spaces are and show that loop spaces are an examples.
- Prove that the fundamental group of a loop space is abelian.
- Define homotopy groups, prove that the higher ones are abelian.

- Optional: Discuss how $\pi_n(\Omega(A))$ is isomorphic to $\pi_{n+1}(A)$.

References:

- Definition of homotopy groups. Hatcher (340-342)
- Lecture 2 in https://www.math.ru.nl/ gutierrez/algtop2014.htm.

Note: Apart from the project on fibrations, there is also a relationship to the project on topological groups.

1.2 Fibrations

General idea: A bundle of topological spaces is a quotient map $\pi : A \to B$ between topological spaces that is locally trivial. I.e. given any point $p \in B$, there is a neighbourhood $U \ni p$ such that $\pi^{-1}(U) \cong U \times F$, for some topological space F called the **fibre**. B is called the **base** and A is the **total space**. A typical example would be the product $A = B \times F$. The Mobius band is an example that is not a product and in which $B = \mathbb{S}^1$ and $F = \mathbb{R}$.

In topology, one studies not just bundles, but more general objects called fibrations. In a bundle, all the fibres look like F, but in a fibration they may be non-homeomorphic topological spaces.

In practice, regarding a topological space as the total space of a bundle will allow us to compute some of its properties in a tractable manner, using information about F and B. This is similar to how one may study groups by looking at subgroups and quotients. The goal of the project is to get some feeling about this.

Prerrequisites: None, but if you know already what a bundle is (of vector spaces, for instance), that may help.

Aims:

- Understand what the homotopy lifting property is.
- Understand the definition of fibre bundle, Hurewicz fibration, and Serre fibration.
- Find interesting examples of fibre bundles. Find examples of fibrations that are not fibre bundles.
- Prove that a fibre bundle satisfies the homotopy lifting property.
- Optional: learn about higher homotopy groups and understand the statement (and, if you want, the proof) of the long exact sequence of homotopy groups associated to a fibration.

References:

- Lecture 5 in https://www.math.ru.nl/ gutierrez/algtop2014.htm. Lectures 3 and 4 if you want to look into homotopy groups.
- Hatcher, page 375 onwards (for definition of fibre bundle and various examples).

2 Low-dimensional manifolds

2.1 Knots and their complements

General idea: Knots are embeddings of $\mathbb{S}^1$ into a 3-manifold (often $\mathbb{R}^3$). You can imagine them as strings in space, that may be knotted in complicated ways (like a shoelace). The central question in knot theory is to classify knots (i.e. given two embeddings of $\mathbb{S}^1$, can we find a homotopy of embeddings between them?).

A fruitful idea in knot theory is to take $\mathbb{R}^3$ and remove the knot. This leaves us with a topological space (in fact, a 3-manifold), called the knot complement. Homotopic knots will have homeomorphic complements, so one can study properties of the complement to study the knot itself. For instance, one can look at the fundamental group of the complement.

The theory of knots is one of the cornerstones of modern topology, and it is deeply intertwined with the general theory of 3-dimensional manifolds.

Prerequisites: It is probably helpful if you are familiar with smooth manifolds, but it is not strictly necessary.

Aims:

- Define what a knot and a link are.
- Optional: Look up what a wild knot is and understand the difference between a smooth and a topological embedding.
- Optional: Argue that homotopic knots have homeomorphic complements (this requires understanding the "isotopy extension theorem", which needs vector fields and flows).
- Explain and prove the Wirtinger representation theorem, which computes the fundamental group of a knot complement.
- Compute the fundamental group of the knot complement in various examples: The unknot, the Hopf link, the figure-8, torus knots...
- Optional: Look up other examples of knot invariants.

References:

- Linov - Introduction to knot theory and the knot group.
- Hatcher. Example 1.24 in page 47.
- Hatcher. Exercise 22 in page 55.

2.2 Lens spaces and Heegaard splittings

General idea: Low-dimensional topology is a (big!) area of mathematics dedicated to studying 3 and 4 dimensional manifolds (and the interactions between the two). A central question asks to classify all 3-manifolds up to homeomorphism. This is a daunting task (for instance, proving the Poincaré conjecture in dimension 3, i.e. that the only simply-connected 3-manifold is the 3-sphere, was only solved recently using tools from analysis), but many things are known.

A classic tool in the area is the use of Heegaard splittings. This consists of chopping up the 3-manifold into two equal simple pieces called "handlebodies". The manifold is recovered then from some nice combinatorial data called the Heegaard diagram (basically, a bunch of curves drawn in the surface of the handlebodies). Many deep invariants of the 3-manifold can be computed using the diagram, including the fundamental group.

Prerequisites: It may be helpful if you have seen smooth manifolds before, but it really is not needed. No tools from differential geometry are necessary for the project.

Aims:

- Introduce lens spaces as quotients of the sphere and use this to compute their fundamental groups.
- Define what a handlebody and a Heegaard splitting are.
- Compute the fundamental group of a handlebody using van Kampen. Use van Kampen to compute the fundamental group of the lens spaces (i.e. check you get the same answer as above).
- Optional: Learn a bit about Heegaard diagrams
- Discuss how lens spaces in dimension 3 are the only 3-manifolds with a Heegaard splitting of genus at most 1.

References:

- Hatcher. Example 2.43 in page 144.
- Saveliev - 4-manifolds and Kirby calculus. Chapter 1.

3 Differential topology

3.1 Pontryagin-Thom construction

General idea: In this course we look at the set $[X, A]$, particularly $[\mathbb{S}^1, A]$, in order to study the target space A. However, as one of you remarked in class, one may use it as a tool to study the source instead.

The most studied example is the so-called cohomotopy, i.e. the study of the sets $[A, \mathbb{S}^n]$. The foundational result in the theory is due to Pontryagin and Thom, who showed that, if A is a manifold, $[A, \mathbb{S}^n]$ encodes information about the submanifolds of A up to cobordism (cobordism is an equivalence relation, where two submanifolds are cobordant if there is a submanifold of dimension one more that has them as boundary).

The goal of the project is to understand the statement (and as much as possible of the proof) of the theorem of Pontryagin-Thom, which is a central result in differential topology/homotopy theory.

Prerrequisites: Differentiable manifolds. In particular: the notion of regular value, the implicit function theorem, and the tangent space.

Aims:

- Define what a framed submanifold is.
- Define the notion of cobordism between submanifolds (possibly framed).
- Explain how one assigns a framed submanifold to a map $A \to \mathbb{S}^n$. Show that this only depends on its class in $[A, \mathbb{S}^n]$.
- Explain the converse: given a framed submanifold, introduce the Thom space N of its normal bundle, and produce a map $A \to N \to \mathbb{S}^n$.
- Optional: Talk to people doing the project on knots. Learn about links in $\mathbb{S}^3$ and deduce that $[\mathbb{S}^3, \mathbb{S}^2]$ computes their cobordism classes. Provide an argument showing that any link is cobordant to the unknot. More optional: Show that $[\mathbb{S}^3, \mathbb{S}^2] \cong \mathbb{Z}$ and prove that this integer measures how twisted the framing of the link is.
- Optional: Talk to people doing the project on topological groups. Relate $[A, \mathbb{S}^1]$ to the study of hypersurfaces in A. Explain how the group structure in $[A, \mathbb{S}^1]$ is related to the connected sum in the world of hypersurfaces.

References:

- Milnor - Topology from the differentiable viewpoint. Chapter 7. Probably you should check Chapters 4 and 5 as well.

3.2 Basics of Morse theory

General idea: In order to study topological spaces, it is convenient to cut them into simpler pieces and see how they must be glued back. During the course, we will classify surfaces in this manner, cutting them into little triangles.

In the study of smooth manifolds, a fruitful idea is to use a function to cut the manifold into its level sets. In this way we regard the manifold as a "movie" of level sets of dimension one less. If the function we choose is reasonable (so-called "Morse"), most level sets will be manifolds as well, but a finite number of them will have simple singularities that encode the topology of the original manifold.

The goal of the project is to understand the basics of this theory.

Prerequisites: You should be familiar with smooth manifolds. In particular: tangent and cotangent bundles, vector fields, and flows.

Aims:

- Define what a Morse function is.
- Get an intuition for the fact that "most" functions are Morse. Optional: study the proof of this.
- Introduce the notion of gradient and prove the Morse lemma: I.e. there is a homotopy equivalence between sublevel sets as long as no critical points are crossed.
- Study the Morse model at critical points. Define what the ascending and descending manifolds are.
- Explain the crossing of critical points as attaching a cell. Optional: Deduce that a smooth manifold is homeomorphic to a CW-complex.

- Optonal: talk to people doing the project on lens spaces. Construct a Morse function on each lens space and explain the CW-complex structure that this produces.

References:

- Milnor - Morse theory. Chapters 1-3.
- My notes on Morse theory.

4 Categorical ideas

4.1 A convenient category of topological spaces

General idea: The category of topological spaces is wild. Many times, there is some weird counterexample to some property that you would like to be true. By getting rid of those pathological examples, doing topology becomes easier. In this project, we explore the space of maps $C(X, Y)$ between topological spaces X and Y, and in particular its relation with the product $X \times Y$.

Aims:

- Define the topology on $C(X, Y)$
- Get a feeling of what cartesian closedness should mean. In particular, show that $Z^{X \times Y} = (Z^Y)^X$ for sets X, Y, Z.
- Prove that there is a well-defined continuous map $C(X \times Y, Z) \to C(X, C(Y, Z))$.
- Show that if X, Y are locally compact and Z is hausdorff, then the natural map $C(X \times Y, Z) \to C(X, C(Y, Z))$ is a homeomorphism. Interpret this result!
- Don't forget to give examples!
- Optional: Define what compactly generated spaces (also called k-spaces) are.
- Optional: Explain how from every topological space X one can associate a compactly generated space $k(X)$ (i.e. explain the k-functor), and use this to define a topology on the product
- Optional: Prove the main theorem in Steenrod's paper: The canonical identification $C(X \times Y, Z) = C(X, C(Y, Z))$ is a homeomorphism.
- Optional: Explore a bit what other options there are to still have cartesian closedness (i.e. what other categories of topological spaces one could consider).

References:

- Hatcher: appendix on compact-open topology.
- Steenrod: A convenient category of topological spaces. https://projecteuclid.org/euclid.mmj/1028999711
- https://ncatlab.org/nlab/show/convenient+category+of+topological+spaces

4.2 The nerve theorem

General idea: Consider the following real world application: You have a cloud of data points. These points are not distributed uniformly over the space of all possible values, but instead concentrate along some smaller topological space. Your goal is to recover properties of the topological space (for instance, the fundamental group), from the data cloud. This is the key motivation behind topological data analysis.

What we do is we thicken each point to a tiny ball, and we imagine that this produces, out of the data cloud, a covering of our topological space. It turns out that, out of this covering, we can produce a new topological space, called the nerve. This is constructed in a combinatorial way, by looking at how the opens intersect one another. One can prove that, for reasonable covers, the nerve is homotopy equivalent to the original space. This means that the cover produced from the data cloud knows everything about the topological subspace of interest.

Prerequisites: None.

Aims:

- Consider a Hausdorff topological space endowed with a covering. Define what the nerve is.
- Optional: Read a bit on topological data analysis and how to construct covers out of data clouds.
- Prove Weil's theorem: The nerve is homotopically equivalent to the original space.

References:

- van de Griend - Thesis. Chapters 1 and 3.1.
- K. Heal - Variations on the nerve theorem.
- Edelsbrunner, Harer - Computational topology. An introduction. This is an introduction to topological data analysis. Chapter III explains how to associate a combinatorial complex to a point cloud (there are many ways!).

4.3 Topological groups

General idea: Consider the matrix groups (for instance $\mathrm{GL}_n(\mathbb{R})$). These are both groups and topological spaces (in fact, they are smooth manifolds), and the two structures are compatible, in the sense that the group operations are continuous. A topological group generalises this idea.

The goal of the project is learning about various examples of topological groups and showing that their fundamental groups are always abelian.

Prerequisites: None, but it may be helpful if you have seen Lie groups before.

Aims:

- Define what a topological group is. Check that complex multiplication turns $\mathbb{S}^1$ into a topological group.

- Show that if X is a topological group, the space of maps $C(A, X)$ is also a topological group, for any A.
- Deduce that $[A, X]$ is then a group. Show that $[-, X]$ defines a functor from the category of topological spaces into the category of groups.
- Explain the group structure on $[\mathbb{S}^1, \mathbb{S}^1]$ induced in this manner.
- Prove that the fundamental group of a topological group is abelian.
- Show that the group of homeomorphisms of a compact Hausdorff topological space is a topological group. For this item, you may want to talk to someone working on the convenient category project.
- Optional how much to do: Discuss some concrete examples of topological groups (for instance, Lie groups?), and explain how to compute their fundamental groups.

References:

- Start by doing Exercise 5 in Sheet 2.
- Bredon - Topology and Geometry. Section I.15.
- Munkres - Topology. Exercises starting on p. 145. Also Exercise 7 in p. 335.

Note: This project is most closely related to the project on loop spaces.