Review from Inleiding Topologie

Topologie en Meetkunde, Block 3, 2022

February 1, 2022

You do not have to hand-in any solutions to the following exercises.

Exercise 1. For each of the following topological spaces, check whether they are:

- Connected (if not, how many components are there?) and locally connected.
- Path-connected (if not, how many components?) and locally path-connected.
- Haussdorff (and other separation properties if you want).
- Compact and locally compact.

For the latter ones, I recommend drawing them to get intuition. Which of them are homeomorphic to one another? (For many pairs, you will not be able to answer this question)

1. $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^n$ with their usual Euclidean topology.
2. The open hypercube $(0,1)^n \subset \mathbb{R}^n$ with the subset topology.
3. The sphere $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ with the subset topology.
4. Projective space $\mathbb{RP}^n := \{x \in \mathbb{S}^n\}/\{x \cong -x\}$ with the quotient topology.
5. A set A with the discrete topology (i.e. every subset is open).
6. The torus $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$ with the product topology.
7. The closed, orientable surface Σ_g with g holes with the subset topology from any inclusion into $\mathbb{R}^3$. Remark: we will see these later in the course.
8. The hypercube $[0,1]^n \subset \mathbb{R}^n$ with the subset topology.
9. The closed unit ball $\mathbb{D}^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ with the subset topology.
10. The closed upper half-space $\mathbb{H}^n := \{x \in \mathbb{R}^n \mid x_1 \geq 0\}$ with the subset topology.
11. The union of two circles at a point $\{x \in \mathbb{R}^2 \mid |x - (1,0)| = 1\} \cup \{x \in \mathbb{R}^2 \mid |x + (1,0)| = 1\}$.
12. The line with two origins: Set $A, B = \mathbb{R}$. Then consider $(A \coprod B)/\{A \ni x \cong x \neq 0 \in B\}$ with the quotient topology induced from the disjoint union of A with B.
13. The hawaiian earring $\bigcup_{n\in\mathbb{Z}^+} \mathbb{S}^1_{1/n}((1/n, 0))$ with the topology induced from $\mathbb{R}^2$.

14. The topologists comb $([0,1]\times\{0\})\cup(\{0\}\times[0,1])\cup(\cup_{n\in\mathbb{Z}^+}\{1/n\}\times[0,1])$ with the induced topology from $\mathbb{R}^2$.

15. The double comb $([-1,0]\times\{-1\})\cup([0,1]\times\{1\})\cup(\{0\}\times[-1,1])\cup(\cup_{n\in\mathbb{Z}^+}\{1/n\}\times[0,1])\cup(\cup_{n\in\mathbb{Z}^+}\{-1/n\}\times[-1,0])$.

Exercise 2. Prove the **pasting lemma**: Let $f: X\to Y$ be a function (not necessarily continuous) between topological spaces. Let $A,B\subset X$ be open subsets satisfying $A\cup B = X$. Then, f is continuous if and only if $f|_A$ and $f|_B$ are continuous.

Prove the same claim assuming instead that A and B are both closed. Prove the same claim where, instead of two subsets, you have finitely many covering X.

Exercise 3. Show that $\mathbb{S}^n$ is homemorphic to $\mathbb{D}^n/\mathbb{S}^{n-1}$. Hint: use identifications of $\mathbb{R}^n$ with $\mathbb{S}^n\setminus\{p\}$ (the stereographic projection) and with $\mathbb{D}^n\setminus\mathbb{S}^{n-1}$.

Exercise 4. Show that:

- $\mathbb{S}^{n-1}$ is homemorphic to $(\mathbb{R}^n\setminus\{0\})/(x\cong\lambda x \text{ for } \lambda>0)$.
- $\mathbb{RP}^{n-1}$ is homemorphic to $\mathbb{S}^{n-1}/(x\cong -x)$.

Exercise 5. Let $\phi:\mathbb{S}^{n-1}\to\mathbb{S}^{n-1}$ be a homeomorphism. Show that:

- The map $\tilde\phi:\mathbb{D}^n\to\mathbb{D}^n$ defined in spherical coordinates by $\tilde\phi(r,p) := (r,\phi(p))$ is a homeomorphism.
- The quotient space $\mathbb{D}^n\coprod_{\mathbb{S}^{n-1}\cong\phi(\mathbb{S}^{n-1})}\mathbb{D}^n$ is homeomorphic to $\mathbb{S}^n$.

Exercise 6. Let A be the hawaiian ring and B the disjoint union of $\mathbb{N}$ copies of $\mathbb{S}^1$ with their points $1\in\mathbb{S}^1$ identified (so B has the quotient topology). Prove that A and B are not homeomorphic by:

- Proving that the points $0\in A$ and $1\in B$ (where all the circles meet) must be identified by any homeomorphism between A and B.
- Proving that there is no neighbourhood of $0\in A$ homeomorphic to a neighbourhood of $1\in B$.

We do not have the technology yet but, later in the course, prove that these two spaces are not homotopy equivalent.