Problem sheet week 1

Topologie en Meetkunde, Block 3, 2022

May 19, 2022

You do not have to hand-in any solutions to the following exercises. The exercises appear according to topics and, within each topic, in order of difficulty (roughly).

1 Contractibility

Exercise 1. A space A is contractible if and only if $[B, A] = \{.\}$ for all topological spaces B.

Exercise 2. Let A be a topological space. Show that the following statements are equivalent:

- $x, y \in A$ are in the same path-component.
- the inclusions $i_x : \{.\} \to A$, $i_x(.) = x$, and $i_y : \{.\} \to A$, $i_y(.) = y$, are homotopic.

Exercise 3. Let A be a contractible topological space. Show that for any topological space B, there is a bijection between $[A, B]$ and $\pi_0(B)$.

Exercise 4. Let A and B be non-empty topological spaces. Define (in a natural manner) an injective map $\pi_0(B) \to [A, B]$.

Exercise 5. Show that $C = [-1, 0]^2 \cup [0, 1]^2 \subset \mathbb{R}^2$ is contractible. Show that any point $p \in C$ is a deformation retract.

Exercise 6. Prove the following statements:

- Show that the topologists comb is contractible. However, it only deformation retracts to the points not contained in the left-most tooth $\{0\} \times (0, 1]$.
- Show that the double comb is not contractible.

Exercise 7. Check the contractibility (or not) of the examples from Exercise 1 in the Sheet reviewing Inleiding Topologie. **Note:** With the tools we have at this point, you will not be able to check contractibility in some (most?) cases. However, you may want to go back to these examples as the course progresses.

2 Retracts

Exercise 8. Show that a topological space X retracts to any point $p \in X$. Find a topological space X that does not deformation retract to a point.

Exercise 9. Let $A \subset \mathbb{R}^n$ be not closed. Show that A is not a retract of $\mathbb{R}^n$.

Exercise 10. Let $A \subset \mathbb{R}$ be a discrete collection of points with cardinality $|A| > 1$. Show that A is not a retract of $\mathbb{R}$.

Exercise 11. Let X be contractible space. Let $A \subset X$ be a retract. Prove that A is also contractible.

3 Homotopy equivalences and deformation retracts

Exercise 12. Let A be the annulus $\{x \in \mathbb{R}^2 \mid 1 \leq |x| \leq 2\}$. Give a explicit formula showing that $\mathbb{S}^1_{3/2} \subset A$ is a deformation retract.

Exercise 13. Show that $\mathbb{S}^{n-1}$ is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.

Exercise 14. Let $p, q \in \mathbb{R}^2$. Show that $\mathbb{R}^2 \setminus \{p, q\}$ is homotopy equivalent to two copies of $\mathbb{S}^1$ joined at a point.

Exercise 15. Show that $\mathbb{S}^1$ is a deformation retract of $\mathbb{S}^1 \cup [0, 2] \times \{0\} \subset \mathbb{R}^2$ but they are not homeomorphic. **Note:** I strongly recommend that you read the "collapsing subspaces" lemma in page 11 of Hatcher. You should be able to do this exercise explicitly without it, but it is a extremely handy lemma that will be useful to you many times in this course.

Exercise 16. Show that there is a deformation retract of $\mathbb{T}^2 \setminus \{p\}$ (the torus minus a point) which is homeomorphic to a wedge of two circles.

Exercise 17. Let $N, S \in \mathbb{S}^2$ be the north and south poles. Show that the following spaces are homotopy equivalent to one another:

$$A := (\mathbb{S}^2 \cup [0, 1])/\{\mathbb{S}^2 \ni N \sim 0 \in [0, 1]; \mathbb{S}^2 \ni S \sim 1 \in [0, 1]\}$$
$$B := \mathbb{S}^2/\{N \sim S\}.$$

Exercise 18. Consider subsets of $\mathbb{R}^2$ with the subset topology:

- $A := \mathbb{S}^1_1((-1, 0)) \cup \mathbb{S}^1_1((1, 0))$, two circles joined at the origin.
- $B := \mathbb{S}^1 \cup (\{0\} \times [-1, 1])$, a circle with a segment connecting the poles.

Show that:

- A and B are not homeomorphic.
- A is not homeomorphic to a deformation retract of B (and viceversa).
- A and B are homotopy equivalent. Hint: they both are homeomorphic to deformation retracts of $\mathbb{D}^2 \setminus \{(\pm 1/2, 0)\}$.
- Use this to deduce that $T^2 \setminus \{p\}$ (the torus minus a point) and $\mathbb{A} \setminus \{q\}$ (the annulus minus a point in the interior) are homotopy equivalent.

In the second to last item you should be able to find explicit maps including A and B into the bigger space. You do not need to give explicit formulas for the deformation retractions (but try to be convincing).

Exercise 19. Find an example of a subset $A \subset \mathbb{R}$ which is not a deformation retract of any neighbourhood $B \supset A$.

Exercise 20. In order of difficulty: Exercises 4, 10, 11, 12, 13 5, 6, and 7 from Chapter 0 of Hatcher.

Exercise 21. Construct a topological space A that is non-Hausdorff and homotopy equivalent to $\mathbb{S}^1$.