Problem sheet week 2

Topologie en Meetkunde, Block 3, 2022

May 19, 2022

You do not have to hand-in any solutions to the following exercises. The exercises appear according to topics and, within each topic, in order of difficulty (roughly).

1 Maps to auxiliary spaces

In class we are looking at $[C, A]$ as an invariant of the space of interest A, where C is auxiliary. Something we may also do is look at $[A, C]$ as an invariant of A instead. The following exercises develop this idea.

Exercise 1. Let $f : A \to B$ be a map. Let C be some auxiliary topological space. Show that the **pullback**:

$$\begin{aligned} f^* : [B, C] &\to [A, C], \\ f^*([g]) &:= [g \circ f] \end{aligned}$$

is a well-defined map. Check that $f_1^* \circ f_2^* = (f_2 \circ f_1)^*$ (i.e. the morphisms compose in the opposite direction as we pass from topological spaces to sets).

Remark: Observe that $[-, C]$ is a contravariant functor from the category of topological spaces to the category of sets. I.e. the orientation of the morphisms gets reversed when we apply the functor.

Exercise 2. Check that f^* is a bijection whenever f is homotopic to a homeomorphism.

Exercise 3. Let $A \subset B$ be a retract; denote the inclusion by i and the retraction by r. Show that i^* is surjective and r^* is injective.

Exercise 4. Check that f^* is a bijection whenever f is a homotopy equivalence.

Exercise 5. The circle $\mathbb{S}^1$ is a group under multiplication as complex numbers. You may want to prove that:

- The product $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ is a continuous map.
- The inverse $\mathbb{S}^1 \to \mathbb{S}^1$ is a continuous map.

We say that $\mathbb{S}^1$ is a **topological group** (because the structures as group and topological space are compatible).

Using the previous facts, prove that:

- The set of all maps $\{A \to \mathbb{S}^1\}$ is an abelian group, for all A.
- $[A, \mathbb{S}^1]$ is an abelian group, for all A.
- This group structure yields a group isomorphism $[\mathbb{S}^1, \mathbb{S}^1] \cong \mathbb{Z}$.

- Let $p \in A$ and $q \in B$; define $D = (A \coprod B)/(p \cong q)$. Show that $[D, \mathbb{S}^1] \cong [A, \mathbb{S}^1] \oplus [B, \mathbb{S}^1]$ as groups.

More generally: Observe that if B is any topological group, then $\{A \to B\}$ and $[A, B]$ are groups as well.

2 Homotopy classes of paths/loops

Exercise 6. Show that $\pi_1(A, p, q) = \{.\}$ for every $A \subset \mathbb{R}^n$ convex and every $p, q \in A$.

Exercise 7. Let A be a path-connected topological space; fix a point $p \in A$. Show that any map $\gamma_0 : \mathbb{S}^1 \to A$ is homotopic to a map γ_1 such that $\gamma_1(1) = p$.

Exercise 8. Exercises 4 and 5 from page 38 of Hatcher.

Exercise 9. Let U_1 and U_2 be two open subsets covering a topological space X. Let $\gamma : [0, 1] \to X$ be a path. Show that there is a finite collection of points

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

such that $\gamma([t_i, t_{i+1}])$ is contained completely in U_1 or U_2.

3 Homotopy classes of maps

Exercise 10. Show that $\mathbb{S}^1$ is not a retract of $\mathbb{R}^2$.

Exercise 11. Let A be a non-empty topological space. Show that there is a map $\gamma : \mathbb{S}^1 \to A \times \mathbb{S}^1$ which is not homotopic to a constant map.

Exercise 12. Let $T^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (n times). Show that for all $n, m \in \mathbb{Z}^+$, the set $[T^n, T^m] \neq \{.\}$.

Exercise 13. Show that $\mathbb{R}^2$ and the closed upper-half plane $\mathbb{H}^2$ are not homeomorphic. **Hint:** Assume otherwise and reach a contradiction by removing a point.

Exercise 14. Exercise 16, items a, b, c, from page 39 of Hatcher.