Problem sheet 3

Topologie en Meetkunde, Block 3, 2022

April 27, 2022

You do not have to hand-in any solutions to the following exercises. The exercises appear according to topics and, within each topic, in order of difficulty (roughly).

1 Pointed topological spaces

Exercise 1. Find an example of a pair of pointed topological spaces (X_1, p_1) and (X_2, p_2) that are non-homeomorphic (as pointed spaces) but X_1 is homeomorphic to X_2. Can you find an example where X_1 and X_2 are path-connected?

Exercise 2. Show that the following pairs are homeomorphic (as pointed topological spaces):

- $(\mathbb{R}^n, p)$ and $(\mathbb{R}^n, q)$, for all $p, q \in \mathbb{R}^n$.
- $(\mathbb{S}^n, p)$ and $(\mathbb{S}^n, q)$, for all $p, q \in \mathbb{S}^n$.

2 Fundamental group(oid)

Exercise 3. Find an example of a topological space X and two points $p, q \in X$ such that $\pi_1(X, p)$ is not isomorphic to $\pi_1(X, q)$.

Exercise 4. Let X be a space. Fix a subspace Y with $i : Y \to X$ the inclusion. Describe the natural groupoid structure that

$$G := \coprod_{p,q \in Y} \pi_1(X, p, q) \rightrightarrows Y$$

inherits from $\Pi_1(X) \rightrightarrows X$.

Show that:

- The inclusion $G \to \Pi_1(X)$ is a groupoid homomorphism.
- The inclusion $G_p \to \pi_1(X, p)$ is a group isomorphism for each $p \in Y$.
- There is a sequence of groupoid homomorphisms $\Pi_1(Y) \to G \to \Pi_1(X)$ that composes to i_*.

Exercise 5. Let $\Pi_1(A)$ be the fundamental groupoid. Assume that A is path-connected. Let $p = q_0, q_1, \cdots, q_{n-1}, q_n = p \in A$. Fix a group isomorphism

$$\beta : \pi_1(A, p) \to \pi_1(A, p).$$

Show that the following conditions are equivalent:

- β is the conjugation map of a loop γ based at p.
- β is a composition $\beta_{\nu_n} \circ \cdots \circ \beta_{\nu_1}$, where

$$\beta_{\nu_i} : \pi_1(A, q_{i-1}) \to \pi_1(A, q_i)$$

  is the conjugation map of a path ν_i connecting q_{i-1} with q_i.

Exercise 6. Let A be a topological space and $q_1, q_2 \in A$ points. Consider the map

$$\begin{array}{ccc} \pi_1(A,p) & \to & [\mathbb{S}^1, A] \\ [\gamma] & \to & [\gamma] \end{array}$$

which forgets the basepoint p.

Show that two classes

$$[\gamma] \in \pi_1(A, q_1), \qquad [\nu] \in \pi_1(A, q_2)$$

have the same image in $[\mathbb{S}^1, A]$ if and only if they are conjugate to one another by some $[\alpha] \in \pi_1(A, q_1, q_2)$.

3 Pushforward of a map

Exercise 7. Find an example where $f : A \to B$ induces isomorphisms

$$f_* : \pi_1(A,p) \quad \to \quad \pi_1(B, f(p))$$

for all $p \in A$, but

$$f_* : \Pi_1(A) \quad \to \quad \Pi_1(B)$$

is not surjective.

Exercise 8. Let $X \subset Y$ and write f for the inclusion. Show that $f_* : \Pi_1(X) \to \Pi(Y)$ may not be injective itself.

Exercise 9. Let $f : Y \to X$ surjective. Show that $f_* : \Pi_1(Y) \to \Pi(X)$ may not be surjective itself.

Exercise 10. Exercises 11, 13 from page 39 of Hatcher.

Exercise 11. Let $\gamma_k : \mathbb{S}^1 \to \mathbb{S}^1$ be $\gamma_k(z) = z^k$. Show that the map

$$(\gamma_k)_* : [\mathbb{S}^1, \mathbb{S}^1] \to [\mathbb{S}^1, \mathbb{S}^1]$$

is surjective if and only if $k = \pm 1$. Compute its image for all k.

Exercise 12. Fix $z \in \mathbb{S}^1$. Show that the map:

$$\Pi_1(\mathbb{S}^1) \to \Pi_1(\mathbb{S}^1)$$

$$[\gamma] \to [z.\gamma]$$

given by the usual product as complex numbers is a groupoid isomorphism. Describe explicitly how the sets $\pi_1(\mathbb{S}^1, p, q)$ are mapped to one another.

Exercise 13. Consider the map

$$\gamma_k(z) := z^k : \mathbb{S}^1 \to \mathbb{S}^1.$$

Compute the induced fundamental groupoid homomorphism

$$(\gamma_k)_* : \Pi_1(\mathbb{S}^1) \to \Pi_1(\mathbb{S}^1)$$

and show that it is a continuous map. For which k is it a groupoid isomorphism? For which k is it a homeomorphism?

Exercise 14. How many homotopy classes of homotopy equivalences $f : \mathbb{S}^1 \to \mathbb{S}^1$ are there?

4 Some more difficult exercises

Exercise 15. Fix a matrix $A \in \mathrm{GL}(2, \mathbb{Z})$ (note that this means that the determinant of A has to be an invertible element in $\mathbb{Z}$, i.e. ± 1).

- Thinking of T^2 as a quotient of the plane $\mathbb{R}^2$, show that A defines a homeomorphism of T^2 to itself.
- Describe the induced group homomorphism

  $$A_* : \pi_1(T^2, p) \to \pi_1(T^2, p)$$

  in terms of generators.

Exercise 16. Let $C := [0, 1] \times \mathbb{S}^1$ be the cylinder. Fix points $p = (0, 1)$ and $q = (1, 1)$. Prove that $\pi_1(C, p, q) \cong \mathbb{Z}$.

Then, given an integer $n \in \mathbb{Z}$, show that there is a homeomorphism $\phi_n : C \to C$ such that:

- ϕ_n restricted to the boundary $\partial C = \{0, 1\} \times \mathbb{S}^1$ is the identity.
- $(\phi_n)_* : \pi_1(C, p, q) \to \pi_1(C, p, q)$ corresponds to adding n.

Any such map ϕ_1 is called a **Dehn twist**.

Prove that ϕ_n can be defined as the composition of ϕ_1 with itself n times.

Exercise 17. We write $\mathrm{Maps}([0, 1], A)$ for the set of all maps $[0, 1] \to A$. We can endow it with a topology (the **compact-open topology**) by defining a subbasis:

$$\mathcal{U}_{K,V} := \{f \in \mathrm{Maps}([0, 1], A) \mid f(K) \subset V\}$$

for every compact $K \subset [0, 1]$ and any open $V \subset A$.

- Show that this does define a topology.
- Show that if $A = \mathbb{R}^n$, this is the **C^0-topology**, i.e. the topology defined by the subbasis:

  $$\mathcal{V}_{f,\varepsilon} := \{g : [0, 1] \to \mathbb{R}^n \mid \sup_{x \in [0,1]} |f(x) - g(x)| < \varepsilon\}.$$

- Show that there is a natural inclusion $A \subset \mathrm{Maps}([0, 1], A)$.
- Show that $\mathrm{Maps}([0, 1], A)$ is homotopy equivalent to A (and, in fact, A is a deformation retract using the previous inclusion).
- Show that the **evaluation map** at $s \in [0, 1]$

  $$\begin{aligned} \mathrm{ev}_s : \mathrm{Maps}([0, 1], A) &\to A \\ \mathrm{ev}_s(\gamma) &:= \gamma(s), \end{aligned}$$

  is continuous.
- Realise the set $\mathrm{Maps}(\mathbb{S}^1, A)$ of all loops $\mathbb{S}^1 \to A$ as a subset of $\mathrm{Maps}([0, 1], A)$. As such, it inherits a topology.
- Show that a function $F : [0, 1] \to \mathrm{Maps}(\mathbb{S}^1, A)$ is continuous if and only if $\gamma_t := F(t)$ is a homotopy of loops.
- Construct a bijection between $\pi_0(\mathrm{Maps}(\mathbb{S}^1, A))$ and $[\mathbb{S}^1, A]$.