Problem sheet 4

Topologie en Meetkunde, Block 3, 2022

April 27, 2022

You do not have to hand-in any solutions to the following exercises. The exercises appear according to topics and, within each topic, in order of difficulty (roughly).

1 General properties of covering spaces

Exercise 1. Find an example of a covering space $p : \tilde{X} \to X$ and an open $U \subset X$ that is not evenly-covered.

Exercise 2. Find an example of a covering space $p : \tilde{X} \to X$, $\tilde{X} \ncong X$, such that every open set of X is evenly-covered.

Exercise 3. Show that if an open $U \subset X$ is evenly-covered, the same is true for any other open $V \subset U$.

Exercise 4. Let $p : \tilde{X} \to X$ be a covering space and $A \subset X$ a subspace. Show that $p : p^{-1}(A) \to A$ is a covering space.

Exercise 5. Show that if X is not compact, neither is any of its covering spaces. Find an example where X is compact but some covering space is not.

Exercise 6. Let M be a manifold. Show that every covering space of M is a manifold as well. You may ignore second countability.

Exercise 7. Find a space X such that no covering space of X is simply-connected.

2 The fundamental groupoid as a space

Exercise 8. Show that $\Pi_1(X)$ is homeomorphic to $X \times X$ if and only if X is simply-connected.

Exercise 9. Consider $\Pi_1(X)$ with X simply-connected. Show that the subspace of composable arrows $\Pi_1(X) \times_{s,t} \Pi_1(X)$ is homeomorphic to $X \times X \times X$.

Exercise 10. Show that $\Pi_1(X)$ is contractible if and only if X is contractible.

Exercise 11. Consider $\Pi_1(\mathbb{S}^1)$. Show that the subspace of composable arrows $\Pi_1(\mathbb{S}^1) \times_{s,t} \Pi_1(\mathbb{S}^1)$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$.

Exercise 12. Show that $\Pi_1(T^n)$ is homeomorphic to $T^n \times \mathbb{R}^n$. Describe the structure maps in terms of this identification.

Exercise 13. Let $f : X \to Y$ be a map. Show that $f_* : \Pi_1(X) \to \Pi(Y)$ is a continuous map. Show that f_* is a homeomorphism if and only if f is a homeomorphism.

Exercise 14. Suppose X is locally simply-connected. This exercise requires you to know how to topologise $\mathrm{Maps}([0,1], X)$ first (this was an exercise in Sheet 2). Show that the open sets

$$\mathcal{U}_V := \{[\nu] \in \Pi_1(X) \mid \nu \in V\},$$

where $V \in \mathrm{Maps}([0,1], X)$ ranges over all opens, defines a topology in $\Pi_1(X)$. Show that this agrees with the usual topology defined in class.

3 Applications of $[\mathbb{S}^1, \mathbb{S}^1] \cong \mathbb{Z}$

Exercise 15. Exercises 8 and 9 from page 39 of Hatcher.

Exercise 16. Let L be the line with two origins (we introduced it in the first sheet). Show that:

- For every A connected and Hausdorff, $[L, A] = \{.\}$. Deduce that L is not homotopy equivalent to $\mathbb{S}^1$.
- Find a connected space A such that $[L, A] \neq \{.\}$.
- $[\mathbb{S}^1, L] = \mathbb{Z}$. **Hint:** Check the proof of $[\mathbb{S}^1, \mathbb{S}^1] = \mathbb{Z}$ and see that it can be adapted to this case.