Problem sheet 5

Topologie en Meetkunde, Block 3, 2022

April 27, 2022

You do not have to hand-in any solutions to the following exercises. The exercises appear according to topics and, within each topic, in order of difficulty (roughly).

1 Lifting criterion

Exercise 1. Let $p : \tilde{X} \to X$ be a covering space. Let $f : Y \to X$ be a map. Show that:

- If f is null-homotopic, it admits a lift.
- If Y is contractible, f admits a lift.
- If Y is simply-connected, f admits a lift.
- $[\mathbb{S}^n, \mathbb{S}^1] = \{.\}$, for all $n > 1$.
- $[\mathbb{S}^n, T^m] = \{.\}$, for all $n > 1$ and all m.

Exercise 2. Find an example of a non-contractible topological space A such that $[\mathbb{S}^k, A] = \{.\}$, for all k.

Exercise 3. Let $K \subset Y$ be a deformation retract. Let $p : \tilde{X} \to X$ be a covering space and let $f : Y \to X$ be a map. Assume that $g = f|_K$ admits a lift $\tilde{g} : K \to \tilde{X}$. Show that f admits a unique lift $\tilde{f} : Y \to \tilde{X}$ such that $\tilde{f}|_K = \tilde{g}$.

Exercise 4. Exercises 8 and 9 from page 79 of Hatcher.

2 Finding the universal cover

Exercise 5. Describe the universal cover of the wedge of a circle and an open disc. Prove that it is indeed a simply-connected covering space.

Exercise 6. Prove that the universal cover of the 2-torus $T^2 := \mathbb{S}^1 \times \mathbb{S}^1$ is $\mathbb{R}^2$.

3 CW-complexes

Exercise 7. Show that a CW-complex X is Hausdorff.

Exercise 8. Show that a CW-complex X is compact if and only if it has finitely many cells.

Exercise 9. Up to homeomorphism, how many distinct cell structures does $\mathbb{S}^1$ have? Show that, for any cell structure, it holds that:

$$\#\text{vertices} - \#\text{edges} = 0.$$

Exercise 10. Show that a CW-complex is locally contractible (i.e. every point has a contractible neighbourhood).

Exercise 11. Show that the hawaiian earring cannot be endowed with a CW-structure.

Exercise 12. Let X be a CW-complex. Let $p : \tilde{X} \to X$ be a covering space. Explain how to endow $\tilde{X}$ with a CW-complex structure.

Exercise 13. Let $X := (T^2, p) \vee (\mathbb{D}^2, q)$. Here q is a point in the interior of $\mathbb{D}^2$.

- Endow it with a CW-complex structure. State explicitly how many cells it has and how they are attached to one another.
- Compute its universal cover $\tilde{X}$.
- Describe the CW-structure that $\tilde{X}$ inherits from X through the covering map. State explicitly how many cells it has and how they are attached to one another.
- Draw (schematically) both CW-complexes, labelling the different cells.

Exercise 14. Do the same, now for $X := (\mathbb{RP}^3, p) \vee (\mathbb{D}^2, q) \vee (\mathbb{S}^2, n)$. Here q is a point in the interior of $\mathbb{D}^2$.

4 Topology of graphs

Exercise 15. Exercise 19 in page 39 of Hatcher.

Exercise 16. Exercise 17 in page 39 of Hatcher.