Problem sheet 7

Topologie en Meetkunde, Block 3, 2022

April 27, 2022

You do not have to hand-in any solutions to the following exercises. The exercises appear according to topics and, within each topic, in order of difficulty (roughly).

1 Computing the fundamental group with easy van Kampen

Exercise 1. Compute the fundamental group of the wedge of a circle and an open disc.

Exercise 2. Compute the fundamental group of the wedge of $\mathbb{S}^k$ with $\mathbb{S}^l$, with k and l positive integers.

Exercise 3. Compute the fundamental group of the wedge of a circle and a torus.

Exercise 4. Let T^2 be the torus. Compute $\pi_1(T^2 \setminus \{p_1, p_2\}, q)$, for $p_1, p_2, q \in T^2$ distinct.

Exercise 5. Let X be $\mathbb{R}^n \setminus \{p_1, \cdots, p_m\}$, where the p_i are distinct from each other and the origin. Compute $\pi_1(X, 0)$ for all n and m (I suggest starting with n, m small).

Exercise 6. Let T^2 be the torus. Describe the homomorphism

$$i_* : \pi_1(T^2 \setminus p, q) \to \pi_1(T^2, q)$$

induced by the inclusion (i.e. describe the groups explicitly in terms of generators and relations and explain how the map acts on generators).

Exercise 7. Let X be the line with two origins. Let $A, B \cong \mathbb{R}$ and define:

$$Y := (A \coprod B)/(A \ni x \cong x \in B \mid x \neq -1, 1),$$

(i.e. a line with two pairs of double points, instead of one).

Show that X and Y are not homotopy equivalent. You may use that $[\mathbb{S}^1, X] = \mathbb{Z}$ (this was an exercise from previous sheets).

Exercise 8. Fix an integer N. Given $A, B \cong \mathbb{R}$, define:

$$X := (A \coprod B)/(A \ni x \cong x \in B \mid x \neq 1, \ldots, N).$$

Compute $\pi_1(X, 0)$.

2 Applications of general van Kampen

Exercise 9. Let X be the 2-dimensional CW-complex obtained by attaching the boundary of $\mathbb{D}^2$ to $\mathbb{S}^1$ by the map $z \to z^k$. Compute the fundamental group of X.

Exercise 10. Prove that $\mathbb{RP}^n$ is a cell complex that can be obtained from $\mathbb{RP}^{n-1}$ by adding an n-cell. Use this and van Kampen to compute the fundamental group.

Exercise 11. Let A, B be two copies of the torus $T^2 := \mathbb{S}^1 \times \mathbb{S}^1$. Compute the fundamental group of

$$C := (A \coprod B)/(A \ni (z,0) \cong (z,0) \in B).$$

Exercise 12. Let A, B be two copies of the torus $T^2 := \mathbb{S}^1 \times \mathbb{S}^1$. Fix coprime integers $p, q \in \mathbb{Z}$. Compute the fundamental group of

$$C := (A \coprod B)/(A \ni (z,0) \cong (z^p, z^q) \in B).$$

Exercise 13. We say that $\mathbb{S}^2$ is the (closed, oriented) **genus-**0 **surface** and that T^2 is the **genus-**1 **surface**. We define the **genus-**g **surface** Σ_g inductively. Take T^2 and the genus-$(g-1)$ surface Σ_{g-1} and remove little discs D_1 and D_{g-1} from each. Identify each of them with $\mathbb{D}^2$ by a homeomorphism. Then, set:

$$\Sigma_g := [(T^2 \setminus \mathring{D}_1) \coprod (\Sigma_{g-1} \setminus \mathring{D}_{g-1})]/(D_1 \cong \mathbb{D}^2 \supset \mathbb{S}^1 \ni z \cong z \in \mathbb{S}^1 \subset \mathbb{D}^2 \cong D_{g-1})$$

You may see the result, embedded as a surface in $\mathbb{R}^3$, in the figure. Compute the fundamental group of Σ_2. Can you compute the fundamental group of Σ_g?

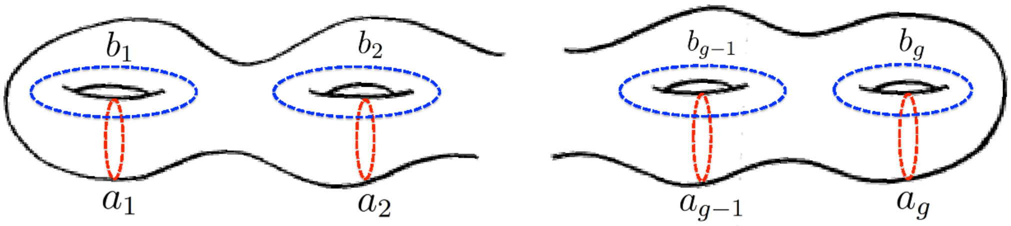

Figure 1: The genus-g surface bounding the genus-g handlebody.

3 Lens spaces

The following examples are about computing the fundamental group of a certain family of 3-manifolds known as lens spaces.

Exercise 14. Let X and Y be two copies of the solid torus $\mathbb{D}^2 \times \mathbb{S}^1$. Compute the fundamental group of

$$L_{1,0} := (X \coprod Y)/(X \supset \mathbb{S}^1 \times \mathbb{S}^1 \ni (x,y) \cong (x,y) \in \mathbb{S}^1 \times \mathbb{S}^1 \subset Y).$$

Exercise 15. Let X and Y be two copies of the solid torus $\mathbb{D}^2 \times \mathbb{S}^1$. Compute the fundamental group of

$$L_{0,-1} := (X \coprod Y)/(X \supset \mathbb{S}^1 \times \mathbb{S}^1 \ni (x,y) \cong (y,-x) \in \mathbb{S}^1 \times \mathbb{S}^1 \subset Y).$$

Exercise 16. Let X and Y be two copies of the solid torus $\mathbb{D}^2 \times \mathbb{S}^1$. Fix an integer $p \in \mathbb{Z}$. Compute the fundamental group of

$$L := (X \coprod Y)/(X \supset \mathbb{S}^1 \times \mathbb{S}^1 \ni (x,y) \cong (y + px, -x) \in \mathbb{S}^1 \times \mathbb{S}^1 \subset Y).$$

Exercise 17. Let X and Y be two copies of the solid torus $\mathbb{D}^2 \times \mathbb{S}^1$. Fix $A \in \mathrm{GL}(2,\mathbb{Z})$. Compute the fundamental group of

$$L_A := (X \coprod Y)/(X \supset \mathbb{S}^1 \times \mathbb{S}^1 \ni (x,y) \cong A(x,y) \in \mathbb{S}^1 \times \mathbb{S}^1 \subset Y).$$

You should write A explicitly and compute the fundamental group in terms of its entries.

4 Other 3-manifolds

Exercise 18. Let $\Sigma_g \subset \mathbb{R}^3$ be the genus-g surface as shown in the figure above. We say that the compact region H_g of $\mathbb{R}^3$ that it bounds is the **genus-g handlebody** (because it looks like a ball with handles). Compute the fundamental group of H_g.

Exercise 19. Let X and Y be two copies of the handlebody H_g. Compute the fundamental group of

$$M_g := (X \coprod Y)/(X \supset \Sigma_g \ni x \cong x \in \Sigma_g \subset Y).$$

Exercise 20. Fix coprime integers p, q. Let $\mathbb{S}^3 \subset \mathbb{C}^2$ and consider the equivalence relation defined by

$$(z_1, z_2) \cong (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi q i}{p}} z_2) \in \mathbb{S}^3.$$

Show that the quotient of $\mathbb{S}^3$ by this relation is a 3-manifold. Compute its fundamental group.

Exercise 21. Let $f : C \to \Sigma_g$ be a map of the cylinder into the genus-g surface such that f provides a homeomorphism of C with its image. Let $\phi : C \to C$ be a Dehn twist (see Sheet 3). Show that

$$\begin{array}{lcll} \Phi : \Sigma_g & \to & \Sigma_g & \\ \Phi(x) & = & x & \text{if } x \notin f(C) \\ \Phi(x) & = & (f \circ \phi \circ f^{-1})(x) & \text{if } x \notin f(C) \end{array}$$

defines a homeomorphism of Σ_g. The map Φ is said to be the Dehn twist defined by f.

Try to draw some simple examples and compute the induced map

$$\Phi_* : \pi_1(\Sigma_g, p) \to \pi_1(\Sigma_g, p)$$

for them (this is a rather open-ended exercise, do as much as you want).

Exercise 22. Let X and Y be two copies of the handlebody H_g. Compute the fundamental group of

$$M := (X \coprod Y)/(X \supset \Sigma_g \ni x \cong \Phi(x) \in \Sigma_g \subset Y),$$

where Φ is your favourite composition of Dehn twists (again, this is open ended, the goal is for you to see that one can construct spaces with interesting fundamental group in this manner).

Exercise 23. Construct a 3-dimensional CW-complex using a single simplex of each dimension (from 0 to 3). Describe the fundamental group you obtain. Show that the examples above built out of two copies of the handlebody H_1 can be endowed with a CW-structure of this form.