Problem sheet 8

Topologie en Meetkunde, Block 3, 2022

April 27, 2022

You do not have to hand-in any solutions to the following exercises. The exercises appear according to topics and, within each topic, in order of difficulty (roughly).

1 Constructing surfaces

Exercise 1. Show that glueing a disc with a (closed) Möbius band along their boundaries yields a projective plane. Use this to compute the fundamental group of the projective plane.

Exercise 2. Show that glueing two (closed) Möbius bands along their boundary yields a Klein bottle.

Exercise 3. Let Σ be a oriented, path-connected surface. Let $D_1, D_2 \subset \Sigma$ be disjoint discs. Endow D_1 and D_2 with the orientation inherited from Σ; endow their boundaries with the boundary orientation. Show that:

- The following surface is non-orientable:

$$\Sigma \setminus (\mathring{D}_1 \cup \mathring{D}_2)/ \cong,$$

  where $\cong$ is an equivalence relation that identifies ∂D_1 with ∂D_2 by an orientation preserving homeomorphism.

- Show that this process, applied to $\mathbb{S}^2$, yields the Klein bottle.

2 Planar representations of surfaces

Exercise 4. Let S be the one of the cell complexes shown in the Figure below.

- Is it a surface? Check that each point in S has a neighbourhood homeomorphic to a ball in the plane.
- Compute the Euler characteristic.
- Is it an orientable surface?
- Use van Kampen to write down a group presentation for the fundamental group of S. Compute the first homology.
- Determine all the g and g' such that S is homeomorphic to Σ_g or $N_{g'}$.
- Use cut-and-paste to obtain the "standard" planar representation of S.

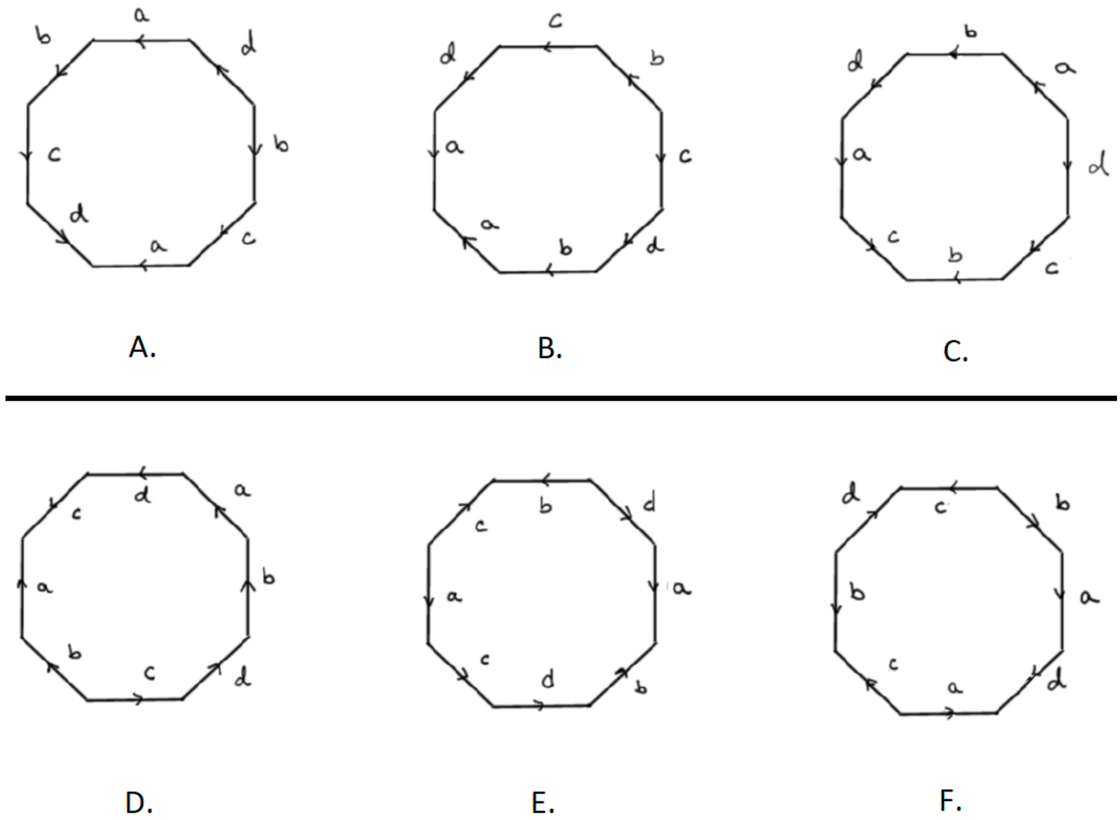

Figure 1: Planar representation for Exercise 5.

3 1-handle attachments

Exercise 5. Are there non-homeomorphic compact surfaces A and B such that $A \# T^2$ and $B \# T^2$ are homeomorphic?

Exercise 6. Let M denote the Möbius band. Prove that there are non-homeomorphic compact surfaces A and B such that $A \# M$ and $B \# M$ are homeomorphic. **Note**: You may take M to be either open or closed with boundary. In both cases you should be careful about the fact that you have an "infinity" in your surface or a boundary component.

Exercise 7. Let C denote the cylinder. Prove that are there non-homeomorphic compact surfaces A and B such that $A \# C$ and $B \# C$ are homotopy equivalent.