Problem sheet 9

Topologie en Meetkunde, Block 3, 2022

April 27, 2022

You do not have to hand-in any solutions to the following exercises. The exercises appear according to topics and, within each topic, in order of difficulty (roughly).

1 The Galois correspondence for covering spaces

Exercise 1. Let M be the (open) Möbius band.

- Prove that every path-connected covering space of M with an even number of sheets is homeomorphic to a cylinder.
- Prove that every path-connected covering space of M with an odd number of sheets is homeomorphic to M.
- Prove that the universal cover of M is homeomorphic to $\mathbb{R}^2$.
- Spell out the Galois correspondence for M.

Exercise 2. Let K be the (open) Klein bottle.

- Find a path-connected, 2-to-1 covering space of K homeomorphic to T^2.
- Find a path-connected, 2-to-1 covering space of K homeomorphic to K.
- Prove that every path-connected covering space of K with an odd number of sheets is homeomorphic to K.
- Prove that the universal cover of K is homeomorphic to $\mathbb{R}^2$.
- Compute the subgroups of $\pi_1(K, p)$ corresponding to the covering spaces above.

Exercise 3. Go over the examples in Hatcher in pages 77-78.

Exercise 4. Consider, for $m, n > 1$, the path-connected covering spaces of $\mathbb{RP}^n \vee \mathbb{RP}^m$. Enumerate all of them, up to homeomorphism. Then, for each covering space with 2-sheets:

- Endow it with a CW-structure. Compute its Euler characteristic.
- Compute its fundamental group (abstractly, not as a subgroup of the fundamental group of the base).
- Describe then the corresponding subgroup of the fundamental group of the base.

Do the same for the 3-sheeted covering spaces.

Exercise 5. Do the same for all the 2-sheeted pointed covering spaces of $\vee_m(\mathbb{S}^1, 1)$, m an integer.

Exercise 6. Describe (as explicitly as you can) the universal cover of the line with two origins. Describe all other path-connected covering spaces as quotients.

Exercise 7. We want to describe the covering spaces of the torus.

- Find two non-homeomorphic, non-compact, path-connected, covering spaces of T^2.
- Show that any 2-by-2 matrix with integer entries A and with non-zero determinant (not necessarily ± 1) defines a covering map $A : T^2 \to T^2$.
- Describe the homomorphism A_* induced by the covering map A.
- Describe the subgroups of $\pi_1(T^2, 1) \cong \mathbb{Z}^2$. Hint: every subgroup must have rank at most 2.
- Let $\tilde{X} \to T^2$ be a path-connected covering space. Show that the rank of $\pi_1(\tilde{X}, p)$ determines $\tilde{X}$ up to homeomorphism.
- Deduce that the only surface Σ_g that can be a covering space of T^2 is T^2 itself.

2 Action of the fundamental group on a covering space

Exercise 8. Find an example of a covering space $p : (\tilde{X}, \tilde{x}) \to (X, x)$ such that the endpoint map

$$\Psi_{\tilde{x}} : \pi_1(X, x) \quad \to \quad p^{-1}(x)$$

i.e. the map taking a loop to its action on the basepoint x, is not surjective.

Exercise 9. Let $p : \tilde{X} \to X$ be a covering space with X path-connected. Let $\tilde{x}_0, \tilde{x}_1 \in \tilde{X}$, possibly projecting to different points. Show that the endpoint map $\Psi_{\tilde{x}_0}$ is surjective if and only if $\Psi_{\tilde{x}_1}$ is surjective.

Exercise 10. Let $\tau : (\tilde{X}, \tilde{q}) \to (X, q)$ be a (pointed) covering space. Let $G = \tau_*(\pi_1(\tilde{X}, \tilde{q}_0))$ the corresponding subgroup. Let $G' = h^{-1}Gh \subset \pi_1(X, q)$ be a conjugate subgroup. Show that there is some other base point $\tilde{q}' \in \tilde{X}$ such that $G' = \tau_*(\pi_1(\tilde{X}, \tilde{q}'))$.

Exercise 11. Let $p : \tilde{X} \to \mathbb{S}^1$ be a covering space. Consider the homomorphism

$$\Psi : \pi_1(\mathbb{S}^1, 1) \to S_{p^{-1}(1)}$$

given by the action of the fundamental group on the fibre over 1.

- Prove that if Ψ is surjective, then p is a one or two sheeted cover.
- Find a two-sheeted example in which this occurs.
- Find a two-sheeted example in which this does not occur.

Exercise 12. Let $p : \tilde{X} \to X$ be a covering space, with $(X, q) = \vee_k(\mathbb{S}^1, 1)$. Show that if the homomorphism into the permutation group

$$\pi_1(X, q) \to S_{p^{-1}(q)}$$

is surjective, then the number of sheets is at most $k + 1$. Find an example in which the number of sheets is exactly $k + 1$. Hint: try first with $k = 2$.