Opgaven

Inleiding Analyse in Meer Variabelen

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1 Metric spaces

1.1 Path-connectedness

Exercise 1.1. Consider a continuous map $f : X \to Y$ between metric spaces. Show that, if f is surjective and X is path-connected, then Y is path-connected.

Exercise 1.2. Let S be the unit circle in $\mathbb{R}^2$. Is S path-connected? Is $\mathbb{R}^2 \setminus S$ path-connected? Prove your statements.

Exercise 1.3. Let $A := \{x = 0\} \cup \{y = 0\}$ be the union of the axes in $\mathbb{R}^2$. Is A path-connected? Is $\mathbb{R}^2 \setminus A$ path-connected?

Exercise 1.4. We consider the open subset $U := \mathbb{R}^2 \setminus \{(x, 0) \in \mathbb{R}^2 \mid x \geq 0\}$. Show that U is open and path-connected.

Exercise 1.5. Let $U := \mathbb{R}^2 \setminus \{(0, 0)\}$ be the punctured plane. Show that U is open and path-connected.

Exercise 1.6. We consider the open subset $V := \{(x, y) \in \mathbb{R}^2 \mid |y| > |x|\}$.

(a) Show that V is not path-connected.

(b) Show that $V \cup \{(0, 0)\}$ is path-connected.

Exercise 1.7. Fix $n > 1$. Let $U \subset \mathbb{R}^n$ be the complement of finitely many points. Show that:

- U is open.
- U is path-connected.

1.2 Homotopy of curves

Exercise 1.8. Consider a metric space (X, d) and a pair of points p and q. We denote by K the set of continuous curves $c : [0, 1] \to X$ with $c(0) = p$ and $c(1) = q$. We define the relation $\sim$ on K by $c \sim d$ if c and d are homotopic relative endpoints.

(a) Show that $c \sim c$.

(b) Show that $c \sim d \implies d \sim c$.

(c) Let $c \sim d$ and $d \sim e$. Show that there exists a continuous map $H : [0, 1] \times [0, 2] \to X$ with

$$H(s, 0) = c(s) \quad \text{and} \quad H(s, 2) = e(s).$$

(d) Show that $\sim$ is an equivalence relation on K.

(e) Formulate and prove similar statements for closed curves in X.

2 (Multi)linear algebra

2.1 Linear maps

Exercise 2.1. Show that the following claims are equivalent for a covector $\alpha : \mathbb{R}^n \to \mathbb{R}$:

- $\alpha = 0$.
- α attains a maximum.
- α attains a minimum.

Exercise 2.2. Consider the linear maps $A, B, C, D : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrices

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix},$$

and

$$D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Is there a change of basis relating A and B? If so, provide it explicitly.
- Is there a change of basis relating B and C? If so, provide it explicitly.
- Can D be related to any of the other matrices by a change of basis? If so, provide it explicitly.

2.2 Kernels, images, annihilators

Exercise 2.3. Consider the covector $\alpha : \mathbb{R}^3 \to \mathbb{R}$ given by the expression $\alpha = (2 \quad 1 \quad 0)$.

- What is the dimension of the kernel of α? Find a basis for it.
- What is the dimension of the image of α? Find a basis for it.
- Give a basis for the annihilator of $\ker(\alpha)$.

Exercise 2.4. Consider the linear map $A : \mathbb{R}^3 \to \mathbb{R}^2$ given by the matrix:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 3 & 3 \end{pmatrix}.$$

- Show that A is surjective.
- Show that the kernel of A has dimension 1. Find a vector spanning it.
- Give a basis for the annihilator of $\ker(A)$.

Exercise 2.5. Consider the linear map $A : \mathbb{R}^3 \to \mathbb{R}^2$ given by the matrix:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \end{pmatrix}.$$

- What is the dimension of the kernel of A? Find a basis for it.
- What is the dimension of the image of A? Find a basis for it.
- Give a basis for the annihilator of $\ker(A)$.
- Give a basis for the annihilator of $\mathrm{im}(A)$.

Exercise 2.6. Consider the linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 5 & 3 & 1 \end{pmatrix}.$$

- What is the dimension of the kernel of A? Find a basis for it.
- What is the dimension of the image of A? Find a basis for it.
- Give a basis for the annihilator of $\ker(A)$.
- Give a basis for the annihilator of $\mathrm{im}(A)$.

Exercise 2.7. Consider the covectors $\alpha_1, \alpha_2, \alpha_3 : \mathbb{R}^3 \to \mathbb{R}$ given by $\alpha_1 = (2 \quad 1 \quad 1)$, $\alpha_2 = (3 \quad 2 \quad 1)$, and $\alpha_3 = (1 \quad 1 \quad 0)$. Consider the subspace $S \subset \mathrm{Lin}(\mathbb{R}^3, \mathbb{R})$ spanned by α_1 and α_2.

- Does S contain α_3? If so, express α_3 as a linear combination of α_1 and α_2.
- Find a basis for the annihilator $S^\perp \subset \mathbb{R}^3$.
- Extend the basis of $S^\perp$ you just found to a basis of $\ker(\alpha_3)$.

2.3 Quadratic forms

Exercise 2.8. Consider the quadratic form $Q : \mathbb{R}^2 \to \mathbb{R}$ given by the matrix:

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

That is, $Q(v) := v^t A v$.

- Write Q explicitly as a homogeneous polynomial of degree 2.
- Find a symmetric matrix B so that $Q(v) := v^t B v$.
- Is Q non-degenerate?
- If so, determine whether Q is positive definite, negative definite, or indefinite.
- Is $v = 0$ a maximum/minimum for Q?

- Find a change of basis $C : \mathbb{R}^2 \to \mathbb{R}^2$ so that $C^t AC$ is diagonal.

Exercise 2.9. Consider the quadratic form $Q : \mathbb{R}^3 \to \mathbb{R}$ given by the matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Write Q explicitly as a homogeneous polynomial of degree 2.
- Find a symmetric matrix B so that $Q(v) := v^t Bv$.
- Is Q non-degenerate?
- If so, determine whether Q is positive definite, negative definite, or indefinite.
- Is $v = 0$ a maximum/minimum for Q?
- Find a change of basis $C : \mathbb{R}^2 \to \mathbb{R}^2$ so that $C^t AC$ is diagonal.

2.4 Polynomials

Exercise 2.10. Consider $\mathbb{R}^3$ with coordinates (x, y, z). Consider the polynomial function $P : \mathbb{R}^3 \to \mathbb{R}$ given by

$$P(x, y, z) := x^2 z + xyz + z^2 y + x + yz + 1.$$

- What is the order of P? Is P homogeneous?
- Write out the part of P that is homogeneous of order 2.
- For each monomial appearing in P (with non-zero coefficient), indicate the multi-index α it corresponds to.

Exercise 2.11. Consider $\mathbb{R}^3$ with coordinates (x, y, z). Consider the polynomial function $P : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$P(x, y, z) = (P_1, P_2) := (x^2 z^2 + xy^2 z + z^2 y^2 + x^4 + y^3 x, y^2).$$

- What is the order of P? Is P homogeneous?
- Write out the part of P that is homogeneous of order 2.
- For each monomial appearing in P_1 (with non-zero coefficient), indicate the multi-index α it corresponds to.

3 Partial, directional, and total derivatives

3.1 Partial and directional derivatives

Exercise 3.1. Compute the partial derivatives $D_x f$, $D_y f$, and $D_z f$ of the following functions:

(a) $f(x, y, z) = \frac{z}{x^2+y^2}$, $\qquad (x, y) \neq (0, 0)$.

(b) $f(x, y, z) = x^2 + yz\sin(x)$.

(c) $f(x, y, x) = xy\sin(y)\log(x^2 + 1)$.

Exercise 3.2. Compute the partial derivatives D_r and D_φ of the functions $f, g : \mathbb{R}^2 \to \mathbb{R}$:

(a) if $f(r, \varphi) := r\cos(\varphi)$,

(b) if $g(r, \varphi) := r\sin(\varphi)$.

We can put these two functions together as $F := (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$, yielding the so-called *polar coordinates*. Show that F is partially differentiable and compute $D_1 F$ and $D_2 F$.

Exercise 3.3. We define the polynomial function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) := x^3 + x^2 y + xy^2 + y^3.$$

Show that f is directionally differentiable at $(1, 0)$ and compute the directional derivative $D_v f(1, 0)$ for every $v \in \mathbb{R}^2$.

Exercise 3.4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) := x^2 y + ye^{xy}$. Compute the following directional derivatives:

(a) $D_{(1,2)} f(0, 0)$.

(b) $D_{(0,0)} f(1, 1)$.

(c) $D_{(2,1)} f(1, 1)$.

Exercise 3.5. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by: $f(x, y) := xy^3 + e^{x\cos(y)}$. Show that f is directionally differentiable in every direction $v \in \mathbb{R}^2$ at the point $(1, 1)$. Provide a formula for $D_v f(1, 1)$.

Exercise 3.6. Consider the norm function $||x|| : \mathbb{R}^n \to \mathbb{R}$.

(a) Show that it is not directionally differentiable at $x = 0$.

(b) Prove that $D_j \|x\| = \frac{x_j}{\|x\|}$ for each $x \neq 0$.

3.2 Total differentiability

Exercise 3.7. Show that there does not exist a function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following conditions: f is differentiable at $0 \in \mathbb{R}^n$ and $D_v f(0) > 0$ for every $v \in \mathbb{R}^n$ with $v \neq 0$.

Exercise 3.8. Find all totally differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $D_v f(p) \geq 0$ for every $p \in \mathbb{R}^n$ and $v \neq 0 \in \mathbb{R}^n$.

Exercise 3.9. Show that each of the following functions is totally differentiable everywhere and compute the total differential.

(a) The function $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x, y) := (x + y, x^2 + y^2, x\,y)$.

(b) The function $g : \mathbb{R}^3 \to \mathbb{R}$ defined by $g(x,\, y,\, z) := x\,y\,z + x\,y + x$.

(c) The function $\varphi : \mathbb{R} \to \mathbb{R}^3$ defined by $\varphi(x) := (1,\, x,\, x^2)$.

Recall: You must *prove* total differentiability. To do so, the strategy is to (1) compute the partial derivatives and put them together to come up with a candidate for Df and either (2) prove that said candidate fulfills the definition of total derivative or (2') invoke the theorem telling us that continuous partial differentiability implies total differentiability. For this exercise, make sure you understand both approaches.

Exercise 3.10. The maps $g : \mathbb{R}^3 \to \mathbb{R}^2$ and $h : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by

$$g(x,\, y,\, z) := (x\,z,\, \log(y^2 + e^z)), \quad h(x, y) := (x + y,\, x\,y).$$

Compute the total derivative of $h \circ g$ in two ways:

(a) directly, i.e., by computing the partial derivatives of $h \circ g$;

(b) using the chain rule.

Exercise 3.11. The maps $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$ are defined by

$$f(x,\, y) := (x^2,\, y,\, x\,y), \qquad g(x,\, y,\, z) := (x\,y\,z,\, z\,\sin(x\,y)).$$

Compute the Jacobian matrices of $f \circ g$ and $g \circ f$. Prove that these fulfill the definition of total derivative.

Exercise 3.12. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by the expression $f(x, y) := \cos(x)\sin(y)$.

- Prove that f is totally differentiable at all points. **Note:** You can use the fact, proven in earlier courses, that the sine and the cosine are differentiable functions of one variable.
- Compute the differential $Df : \mathbb{R}^2 \to \mathrm{Lin}(\mathbb{R}^2, \mathbb{R})$.
- Compute the critical points of f.
- Draw schematically the level sets of f, indicating where the critical points are.
- For each critical point, determine whether it is a (local) maximum, a (local) minimum, or neither.

3.3 Stationary points

Exercise 3.13. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,\, y) = \left(1 - x^2\right)\, \left(1 - y^2\right)$.

a) Compute all stationary points of f.

b) Determine at all stationary points whether f has a local maximum, a local minimum, or neither. Hint: draw the zero level set of f and the regions where $f > 0$, resp. $f < 0$.

Exercise 3.14. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,\, y) = x\, y\, (x + y - 1)$.

a) Prove that f has four stationary points.

b) Prove that f has an extremum at exactly one of these points. Hint: draw the zero level set of f and the regions where $f > 0$, resp. $f < 0$.

Exercise 3.15. Consider the polynomial function

$$f(x,\, y) = \frac{1}{3}x^3 - x + x\, y^2$$

of two variables. Compute the partial derivatives $D_x f$ and $D_y f$ and determine the stationary points. Investigate whether f attains a maximum and/or minimum on the right half-plane

$$V_+ = \{(x,\, y) \mid x \geq 0\},$$

respectively on the left half-plane

$$V_- = \{(x,\, y) \mid x \leq 0\}.$$

If so, also compute the maximum, resp. minimum. Finally, answer the same questions with f replaced by $g(x, y) := \frac{1}{3}x^3 - x - x\, y^2$.

3.4 Total derivatives of multilinear maps

Exercise 3.16. Given are two fixed vectors b and c in $\mathbb{R}^n$. Prove that each of the maps f below is totally differentiable at every point a of $\mathbb{R}^n$ by using the definition of total differentiability. Also compute $Df(a)h$ for each point $a \in \mathbb{R}^n$ and vector $h \in \mathbb{R}^n$:

a) $f(x) = \langle b,\, x\rangle$,

b) $f(x) = \langle b,\, x\rangle\, \langle c,\, x\rangle$,

c) $f(x) = \langle x, x\rangle$,

d) $f(x) = \langle b,\, x\rangle\, c$,

e) $f(x) = \langle x,\, x\rangle\, x$.

Exercise 3.17. Suppose $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}$ are totally differentiable functions at $a \in \mathbb{R}^n$. Prove that $F : \mathbb{R}^n \to \mathbb{R}^p$, defined by

$$F(x) := g(x)f(x)$$

is also totally differentiable at a and prove that the total derivative $DF(a)$ is given by the formula:

$$DF(a)(h) = g(a)Df(a)(h) + f(a)Dg(a)(h).$$

Exercise 3.18. Suppose $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are totally differentiable functions at $a \in \mathbb{R}^n$. Prove that $F : \mathbb{R}^n \to \mathbb{R}$, defined by

$$F(x) := \langle f(x),\, g(x)\rangle$$

is also totally differentiable at a and prove that the total derivative $DF(a)$ is given by the formula:

$$DF(a)(h) = \langle Df(a)(h),\, g(a)\rangle + \langle f(a), Dg(a)(h)\rangle.$$

Exercise 3.19. Let $f : \mathbb{R}^n \to \mathbb{R}$ be the function $f(x) = ||x||^2$. Let $G := f\mathrm{Id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$.

(a) Show that G is differentiable and bijective.

(b) Prove that $DG(0) = 0$ but $\det DG(p) \neq 0$ for every $p \neq 0$.

(c) Prove that G^{-1} is not differentiable at 0.

Exercise 3.20. Let A be a symmetric n-by-n matrix. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = x^t Ax$, i.e. the associated quadratic form. Recall that we see $x \in \mathbb{R}^n$ as a column vector and x^t denotes its transpose, which is a row vector. Then:

- Write out f in terms of the coefficients A_{ij} of the matrix A. Prove that f is a polynomial of order (at most) 2 and is thus twice differentiable.
- Compute Df in terms of the coefficients A_{ij}.
- Compute the second derivatives of f.
- Show that f has a critical point at the origin.
- Show that f has other critical points if and only if A has zero determinant.

3.5 Chain rule

Exercise 3.21. The functions $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by

$$f(x, y) := x + y^2, \qquad g(x, y) := (x\,y,\, x + y).$$

a) Compute the derivatives of $f \circ g$ and of $g \circ f$.

b) Verify that the chain rule holds for the composition $f \circ g$.

Exercise 3.22. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) := x^2 + y^2 + z^2$. Consider $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\phi(u, v) := (u + v, uv, u - v)$. Let $h = f \circ \phi$.

- Compute $h(u, v)$.
- Compute the differential of h using the definition.
- Compute the differential of h using the chain rule.

Exercise 3.23. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) := \cos(x^2 + y^2)$. Let $\phi : \mathbb{R} \to \mathbb{R}^2$ be defined by $\phi(t) := (\cos t, \sin t)$. Let $h = f \circ \phi$.

- Compute $h(t)$.
- Compute the derivative of h using the definition.
- Compute the derivative of h using the chain rule.

3.6 Badly-behaved functions

Exercise 3.24. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

(a) Show that f is differentiable everywhere.

(b) Show that f' is not continuous at 0.

Exercise 3.25. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0). \end{cases}$$

has directional derivatives in every direction $v \in \mathbb{R}^2$, but is not continuous at 0.

3.7 Growth

Exercise 3.26. (a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $\sup_{x \in \mathbb{R}} |f'(x)| < \infty$, then f grows at most linearly. I.e. there is a linear function that provides an upper bound for f.

(b) Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and $\sup_{x \in \mathbb{R}^n} ||Df(x)|| < \infty$, then f grows at most linearly.

Exercise 3.27. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and there is a constant $C > 0$ such that $|f'(x)| \leq C|f(x)|$ for all $x \in \mathbb{R}$, then $f = 0$.

3.8 Functional equations

Exercise 3.28. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$f(x + y) = f(x)f(y), \qquad (x, y \in \mathbb{R}).$$

(a) Show that either $f = 0$ or $f > 0$.

(b) Show that either $f = 1$ or $f(x) = e^{ax}$ for some $a \in \mathbb{R}$.

Exercise 3.29. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and satisfies

$$f(x + y) = f(x) + f(y), \qquad (x, y \in \mathbb{R}^n).$$

(a) Show that f is linear.

(b) Show that if f is also continuous, then $f(x) = \langle a, x \rangle$ for some $a \in \mathbb{R}^n$.

Exercise 3.30. Let $U := \mathbb{R}^2 \setminus \{(0, 0)\}$ be the punctured plane. Show that there exists a unique partially differentiable $f : U \to \mathbb{R}$ satisfying $f(1, 0) = 1$ and

$$Df(x, y) = \frac{(x \quad y)}{\|(x, y)\|^2}.$$

3.9 Extra

Exercise 3.31. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and homogeneous of degree k, i.e.

$$f(\lambda x) = \lambda^k f(x), \qquad (\lambda > 0, x \in \mathbb{R}^n).$$

Show that

$$\langle \nabla f(x), x \rangle = k f(x).$$

Exercise 3.32. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial of degree 2, i.e.

$$f(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j.$$

(a) Show that $f(x) = x^t A x$ for a symmetric matrix A.

(b) Show that $\nabla f(x) = 2Ax$.

(c) Show that $\nabla^2 f(x) = 2A$.

Exercise 3.33. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \|x\|$.

(a) Show that f is differentiable at every $x \neq 0$ and compute $\nabla f(x)$.

(b) Show that f is not differentiable at 0.

Exercise 3.34. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \sqrt{|xy|}$.

(a) Show that f is differentiable at every (x, y) with $xy \neq 0$.

(b) Show that f is not differentiable at $(0, 0)$.

4 Derivatives of higher order

4.1 Computing higher partial derivatives

Exercise 4.1. We consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(0,0) := 0$ and by

$$f(x,y) := \frac{|x|xy}{\sqrt{x^2+y^2}} \qquad \text{if} \quad (x,y) \neq (0,0).$$

(a) Show that $D_1 f(0,y)$ exists for all $y \in \mathbb{R}$ and determine the function $y \mapsto D_1 f(0,y)$.

(b) Show that $D_2 f(x,0)$ exists for all $x \in \mathbb{R}$ and determine the function $x \mapsto D_2 f(x,0)$.

(c) Show that $D_2 D_1 f(0,0)$ and $D_1 D_2 f(0,0)$ exist but are not equal to each other.

How can (c) be reconciled with the theorem on switching of partial derivatives?

4.2 Stationary points and extrema

Exercise 4.2. Determine all local extrema of the functions $f : \mathbb{R}^2 \to \mathbb{R}$ defined below.

(a) $f(x,y) = e^{x^2-y^2} - 2x^2 - y^2$;

(b) $f(x,y) = x^2 - xy + y^2 + e^{xy}$;

(c) $f(x,y) = 2x^2 + xy + y^2 + e^{xy}$;

(d) $f(x,y) = e^{-x^2-y^2}(x^2 + 2y^2)$.

Determine, for each extremum, whether it is a maximum or minimum and whether it is local or global. Does the function have any other critical points?

Exercise 4.3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by $f(x,y) := (1-x^2)(1-y^2)$.

- Show that f is smooth.
- Determine all stationary points of f.
- Determine, for each stationary point, whether it is a minimum, a maximum, or neither.
- Determine, for each maximum/minimum, whether it is local or global.

Exercise 4.4. We consider a smooth function $g : \mathbb{R} \to \mathbb{R}$. The functions $f, F : \mathbb{R}^2 \to \mathbb{R}$ are defined by $f(x,y) := x^2 + y^2$ and $F := g \circ f$.

(a) Prove that F is smooth.

(b) Compute the first and second order derivatives of F in terms of the partial derivatives of g.

(c) Determine the stationary points of F in terms of those of g.

(d) Determine, for each critical point, whether it is a minimum, a maximum, or neither. Determine, for each maximum/minimum, whether it is local or global.

Exercise 4.5. Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions. We consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, y) = f(x + g(y))$.

(a) Give the first and second order partial derivatives of F in terms of the partial derivatives of f and g.

(b) Show that F is a C^2-function.

(c) Determine the critical points of F in terms of f and g.

(d) Determine, for each critical point, whether it is a minimum, a maximum, or neither. Determine, for each maximum/minimum, whether it is local or global.

4.3 Hessian

Exercise 4.6. Let $U \subset \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}$ a C^2-function.

(a) Show that $g := \mathrm{grad} f$ is a C^1-function $U \to \mathbb{R}^n$.

(b) Show that for all $x \in U$ we have:

$$Dg(x) = H_f(x).$$

Exercise 4.7. (a) Show that every polynomial function $f : \mathbb{R}^n \to \mathbb{R}$ of degree at most 2 can be uniquely written as

$$f(x) = a + \langle b, x\rangle + \frac{1}{2}\langle Cx, x\rangle \qquad (x \in \mathbb{R}^n).$$

Here $a \in \mathbb{R}$ is a constant, $b \in \mathbb{R}^n$ a constant vector and $C \in M_n(\mathbb{R})$ a *symmetric* $n \times n$-matrix.

(b) Show that for every $x \in \mathbb{R}^n$ it holds that $\mathrm{grad} f(x) = b + Cx$. Define now $v : \mathbb{R}^n \to \mathbb{R}^n$ by $v(x) = \mathrm{grad} f(x)$.

(c) Show that v is totally differentiable at every $x \in \mathbb{R}^n$ with total derivative $Dv(x) : \mathbb{R}^n \to \mathbb{R}^n,\ h \mapsto Ch$.

(d) Show that $H_f(x) = C$ for all $x \in \mathbb{R}^n$.

Exercise 4.8. Given is a polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ of degree at most 2. Let $a \in \mathbb{R}^n$. Show that the following statements are equivalent:

(1) $D^\alpha p(a) = 0$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2$.

(2) $\lim_{x \to a} \|x - a\|^{-2} p(x) = 0$.

(3) $p = 0$.

Exercise 4.9. Let C be a symmetric 2×2 matrix. We assume that $\det C \neq 0$. From linear algebra it is known that there exist an orthonormal basis $\{v_1, v_2\}$ of $\mathbb{R}^2$ and $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ such that

$$Cv_j = \lambda_j v_j, \qquad (j = 1, 2).$$

We assume that $\det C < 0$. Then $\lambda_1\lambda_2 < 0$. After possibly reordering, we may therefore assume that $\lambda_1 > 0$ and $\lambda_2 < 0$.

(a) Show that there exist vectors $u, v \in \mathbb{R}^2$ such that

$$\langle Cu, u\rangle > 0 \quad \text{and} \quad \langle Cv, v\rangle < 0$$

Now let $U \subset \mathbb{R}^2$ be open and $f : U \to \mathbb{R}$ a C^2-function with a stationary point $a \in U$. Assume further that $\det H_f(a) < 0$.

(b) Show that there exist two vectors $u, v \in \mathbb{R}^2$ of length 1 and a $\delta > 0$ such that for all $x \in B(a; \delta)$ it holds that

$$\langle H_f(x)u, u\rangle > 0, \quad \text{and} \quad \langle H_f(x)v, v\rangle < 0.$$

(c) Let u satisfy (b). Show that for $t \in \mathbb{R}$ we have:

$$0 < |t| < \delta \quad \Longrightarrow \quad f(a + tu) > f(a).$$

(d) Show that f has no local extremum at a.

4.4 Taylor polynomials

Exercise 4.10. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := \sin(xy)$. Compute its third order Taylor polynomial at $(0, 0)$. Indicate clearly the homogeneous terms.

Hint: you could try to do this directly by computing all derivatives up to order three. A quicker approach is to: (1) compute the third order Taylor polynomial of the sine, (2) compute the third order Taylor polynomial of xy, (3) compose them, (4) discard terms of order greater than 3.

Exercise 4.11. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := \sin(xy)$, from the previous exercise. Let $R(x)$ be the remainder of its second order Taylor polynomial at $(0, 0)$. Provide a finite upper bound for $|R(x)|$ over the unit ball.

Exercise 4.12. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := e^{x \sin(y)}$.

- Compute its third order Taylor polynomial at $(1, 0)$, indicating clearly the homogeneous terms.
- Let $R(x)$ be the remainder of its second order Taylor polynomial at $(0, 0)$. Provide a finite upper bound for $|R(x)|$ over the ball of radius 1 centered at $(1, 0)$.

4.5 Extra

Exercise 4.13. Given is a C^2 function $g : \mathbb{R}^2 \to \mathbb{R}$ and a twice differentiable function $f : \mathbb{R} \to \mathbb{R}$. Show that

$$\frac{\partial^2 f(g(x,\, y))}{\partial x\, \partial y} = \frac{\partial^2 f(g(x,\, y))}{\partial y\, \partial x}.$$

Warning: we have not assumed that the second derivative f'' of f is continuous.

Exercise 4.14. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^p$ be open subsets, $f \in C^k(U, \mathbb{R}^p)$ and $g \in C^k(V, \mathbb{R}^q)$ such that $f(U) \subset V$. Show that $g \circ f \in C^k(U, \mathbb{R}^q)$. Hint: reduce to $q = 1$ and use induction on k.

Exercise 4.15. Let $\mathcal{F}$ be the space of functions $f : \mathbb{R}^2 \to \mathbb{R}$. For $h_1, h_2 \in \mathbb{R} \setminus \{0\}$ we define the linear maps $\Delta_j : \mathcal{F} \to \mathcal{F}$, for $j = 1, 2$, by

$$\Delta_j f(x) = \frac{f(x + h_j e_j) - f(x)}{h_j}, \quad (x \in \mathbb{R}^2),$$

for $f \in \mathcal{F}$. Show that for all $f \in \mathcal{F}$ it holds that:

$$\Delta_1(\Delta_2 f) = \Delta_2(\Delta_1 f).$$

Exercise 4.16. Let U be an open subset of $\mathbb{R}^n$. A vector field on U is a map $U \to \mathbb{R}^n$. If $f : U \to \mathbb{R}$ is a differentiable real-valued function on U, then the gradient $\mathrm{grad} f : U \to \mathbb{R}^n$ of f is a vector field on U.

If $v : U \to \mathbb{R}^n$ is a differentiable vector field, then the *divergence* $\mathrm{div}(v) : U \to \mathbb{R}$ of v is defined as

$$(\mathrm{div}(v))(x) := \sum_{i=1}^{n} \frac{\partial v_i(x)}{\partial x_i}.$$

Now let $n = 3$. Then we define the vector field $\mathrm{rot}(v) : U \to \mathbb{R}^3$ by

$$(\mathrm{rot}(v))_1 = D_2 v_3 - D_3 v_2, \quad (\mathrm{rot}(v))_2 = D_3 v_1 - D_1 v_3, \quad (\mathrm{rot}(v))_3 = D_1 v_2 - D_2 v_1.$$

The vector field $\mathrm{rot}(v)$ is called the *curl* of v.

a) Show that for every $\mathbb{C}^2$ function f on U we have: $\mathrm{rot}(\mathrm{grad} f) = 0$.

b) Show that for every C^2 vector field v on U we have: $\mathrm{div}(\mathrm{rot} v) = 0$.

5 Multivariate Riemann integral

Preliminary instructions: improper integrals

In the exercises below, you will encounter "improper" integrals of the type

$$\int_a^\infty f(x)dx$$

Here, $a \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$ is a function that is locally Riemann-integrable, i.e., Riemann-integrable over $[a, \beta]$ for every $\beta \geq a$. The integral is called *convergent* if the limit $\int_a^\beta f(x)dx$ exists as $\beta \to \infty$. In that case we define

$$\int_a^\infty f(x)dx := \lim_{\beta\to\infty} \int_a^\beta f(x)dx.$$

A non-convergent integral is called *divergent*.

We can similarly define convergence of integrals over unbounded intervals of the form $(-\infty, b]$, over open intervals (a, b), or over the whole real line $(-\infty, \infty)$. Write I for such an open and assume that $f : I \to \mathbb{R}$ is locally Riemann-integrable, i.e., integrable over any closed interval of the form $[\alpha, \beta] \subset I$. Then we can say that $\int_I f(x)dx$ is convergent if the integrals $\int_\alpha^\beta f(x)dx$ have a limit as α and β go to the ends of I (possibly $\pm\infty$).

5.1 Continuity

Exercise 5.1. Let $f(x, y) := e^{-xy}y : \mathbb{R}^2 \to \mathbb{R}$. Prove that:

- f is continuous,
- for every $y \geq 0$ the improper integral

  $$F(y) := \int_0^\infty e^{-xy}ydx$$

  exists,
- but the function $F : [0, \infty) \to \mathbb{R}$ is not continuous at 0.

5.2 Computing integrals

Exercise 5.2. Compute $\int_0^A (x^2 + t)^{-2}dx$ for $t > 0$ by differentiating with respect to t the integral $\int_0^A (x^2 + t)^{-1}dx$.

Exercise 5.3. Compute the improper integral $\int_0^\infty (x^2 + t)^{-2}dx$ and verify that it equals minus the derivative with respect to t of the improper integral $\int_0^\infty (x^2 + t)^{-1}dx$.

Exercise 5.4. Compute $\int_0^\infty (x^2 + 1)^{-2}dx$ and $\int_0^\infty (x^2 + 1)^{-3}dx$.

Exercise 5.5. Define $f : \mathbb{R} \to \mathbb{R}$ by:

$$f(a) = \int_0^1 \frac{e^{-a^2(1+t^2)/2}}{1 + t^2} dt.$$

a) Prove that $f(0) = \pi/4$. Prove by differentiating with respect to a, followed by a substitution, that

$$f'(a) = -e^{-a^2/2} \int_0^a e^{-x^2/2}\, dx, \quad a > 0.$$

Define $g : \mathbb{R} \to \mathbb{R}$ by:

$$g(a) = f(a) + \left(\int_0^a e^{-x^2/2}\, dx \right)^2 /2.$$

b) Prove that $g' = 0$ on $\mathbb{R}$. Conclude that $g(a) = g(0) = \pi/4$ for all $a \in \mathbb{R}$.

c) Prove that for every $a \in \mathbb{R}$ we have $0 \le f(a) \le e^{-a^2/2}$. Prove that $f(a) \to 0$ as $a \to \infty$. Finally, use this to prove that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}.$$

This is a well-known formula of Gauss.

Exercise 5.6. Let

$$F(x) := \int_0^{\pi/2} \log(1 + x\cos^2\theta)\, d\theta, \quad x > -1.$$

Prove, by differentiating with respect to x and using the substitution $t = \sin\theta/\cos\theta$, that

$$F'(x) = \frac{1}{x} \int_0^{\infty} \left(\frac{1}{t^2+1} - \frac{1}{t^2+1+x} \right) dt = \frac{\pi}{2} \frac{\sqrt{1+x} - 1}{x\sqrt{1+x}}.$$

Compute $F(x)$. *Hint*: write $G(u) = F(u^2 - 1)$ and study $G'(u)$. What is $F(0)$?

Exercise 5.7. Consider the function $f : (0,\infty) \times (0,1) \to \mathbb{R}$ given by $f(x,t) := \dfrac{t^x - 1}{\ln(t)}$. Define $F(x) := \int_0^1 f(x,t)dt : (0,\infty) \to \mathbb{R}$. The goal of this exercise is to show that $F(x) = \ln(x+1)$. However, a priori it is not even clear that $F(x)$ is well-defined. Indeed, we must pay attention to the fact that $f(x,0)$ and $f(x,1)$ are not defined.

To address this, we introduce the auxiliary function $G(x,s) := \int_s^{1-s} f(x,t)dt : (0,\infty) \times (0,1/2) \to \mathbb{R}$.

- Show that f is C^1. **Hint**: Don't forget to study the function $(x,t) \mapsto t^x$ first.
- Show that G is continuous **Hint**: Consider the function $h(s,z) := z(1-2s) + s$. Use this to make a change of variable that replaces $t \in [s, 1-s]$ with $z \in [0,1]$.
- Prove that G is C^1.
- Compute $D_x G(x,s)$.
- Show that there is a unique continuous function $H : (0,\infty) \times [0,1/2) \to \mathbb{R}$ such that $H(x,s) = D_x G(x,s)$ for each $s \neq 0$.
- Use H to show that $F(x) := \lim_{s\to 0} G(x,s)$ is indeed $\ln(x+1)$.

5.3 Differentiation under the integral sign

Exercise 5.8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x,y) = \sin(x)^5 y + \cos(y)x$. Define the function $F(y) := \int_0^1 f(x,y)dx$.

- Show that $F : \mathbb{R} \to \mathbb{R}$ is smooth.
- Compute DF.
- Show that $|F(y)| \leq |y| + \frac{1}{2}$.

Exercise 5.9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x,y) := x^2 + \sin(y)^5 y$. Define the function $F(x,y) := \int_0^y f(x,a)da$.

- Show that $F : \mathbb{R}^2 \to \mathbb{R}$ is a C^2-function.
- Compute the total derivative of F.
- Determine the critical points of F.

Exercise 5.10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be C^1. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$F(x,y) := \int_0^{x^2} f(x^2 + y, x + t)dt.$$

- Prove that F is continuous.
- Prove that F is C^1.
- Compute its total differential DF, written in terms of the derivatives of f.

Hint: Whenever you are asked to show that a function is C^k, it is often helpful to write it as a composition/product/sum of simpler functions. In this case, you may want to consider:

$$\begin{aligned} h : \mathbb{R}^3 \to \mathbb{R}^2, &\quad h(x,y,t) := (x^2 + y, x + t) \\ g : \mathbb{R}^3 \to \mathbb{R}, &\quad g := f \circ h \\ G : \mathbb{R}^3 \to \mathbb{R}, &\quad G(x,y,s) := \int_0^s g(x,y,t)dt \end{aligned}$$

Exercise 5.11. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Suppose that f is differentiable with respect to the first variable and that $D_1 f : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Define

$$F(x) := \int_a^x f(x,y)dy.$$

a) Prove that F is continuously differentiable and that

$$F'(x) = f(x,x) + \int_a^x D_x f(x,y)dy.$$

b) Prove that

$$\int_a^c f(c,y)dy = \int_a^c f(x,x)dx + \int_a^c \int_a^x D_x f(x,y)dydx.$$

Exercise 5.12. Prove successively

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(e^{-x\sin t}\cos(x\cos t)\right) &= \frac{1}{x}\frac{\partial}{\partial t}\left(e^{-x\sin t}\sin(x\cos t)\right), \quad (x\neq 0),\\
\frac{d}{dx}\int_0^{\pi/2} e^{-x\sin t}\cos(x\cos t)dt &= -\frac{\sin x}{x}, \quad (x\neq 0),\\
\int_0^y \frac{\sin x}{x}dx &= \frac{\pi}{2} - \int_0^{\pi/2} e^{-y\sin t}\cos(y\cos t)dt, \quad (y>0),\\
\left|\int_0^{\pi/2} e^{-y\sin t}\cos(y\cos t)dt\right| &\leq \int_0^{\epsilon} e^{-y\sin t}dt + \int_{\epsilon}^{\pi/2} e^{-y\sin t}dt\\
&\leq \epsilon + \frac{\pi}{2}e^{-y\sin\epsilon}, \quad (0<\epsilon<\pi/2),\\
\lim_{y\to\infty}\int_0^y \frac{\sin x}{x}dx &= \frac{\pi}{2}.
\end{aligned}
$$

Note: in the course "Functions and Series," the limit is computed in a very different way.

5.4 Switching in the order of integration

Exercise 5.13. a) Show that the function $(x,y)\mapsto y/(x^2+y^2)$ is continuous on $[0,1]\times[1,2]$.

b) Verify by direct calculation that

$$\int_1^2\int_0^1 \frac{y}{x^2+y^2}\,dx\,dy = \int_0^1\int_1^2 \frac{y}{x^2+y^2}\,dy\,dx.$$

Exercise 5.14. Let $a,b,c,d\in\mathbb{R}$ with $0\leq a<b$ and $0<c<d$.

a) Show that the function $f:(x,y)\mapsto 1/(x+y)$ is continuous on $[a,b]\times[c,d]$.

b) Verify by direct calculation that

$$\int_a^b\int_c^d \frac{1}{x+y}\,dy\,dx = \int_c^d\int_a^b \frac{1}{x+y}\,dx\,dy.$$

Exercise 5.15. Consider the function $f:[0,1]^2\to\mathbb{R}$ given by $f(x,y):=e^{xy}y$.

a) Show that the function $F:[0,1]\to\mathbb{R}$ given by $F(x):=\int_0^1 f(x,s)ds$ is smooth.

b) Show that the function $G:[0,1]^2\to\mathbb{R}$ given by $G(x,y):=\int_0^{yx^2} f(x,s)ds$ is smooth.

c) Compute $\int_0^1 F(t)dt$.

6 The inverse function theorem and its applications

6.1 Diffeomorphisms

Exercise 6.1. Recall that the spherical coordinates

$$\Phi : U := (0, \infty) \times (-\pi, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}^3$$

are defined as

$$\Phi(r, \phi, \theta) := r(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta).$$

Show that

$$\det D\Phi(r, \phi, \theta) = r^2\cos\theta, \qquad ((r, \phi, \theta) \in U).$$

Exercise 6.2. Recall that the hyperbolic functions $\cosh, \sinh : \mathbb{R} \to \mathbb{R}$ are defined by

$$\cosh t := \frac{e^t + e^{-t}}{2}, \qquad \sinh t := \frac{e^t - e^{-t}}{2}.$$

This allows us to define coordinates $F : U := (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ using the expression:

$$F(\rho, t) = \rho(\cosh t, \sinh t).$$

(a) Compute $\det DF(\rho, t)$.

(b) Show that F is a diffeomorphism from U onto an open subset V of $\mathbb{R}^2$. Determine V.

Exercise 6.3. Consider the map $f : U := (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(r, t) = r(\tanh t, \frac{1}{\cosh t}).$$

(a) Show that f is a C^1 map and that

$$\det Df(r, t) = -\frac{r}{\cosh t}.$$

(b) Show that f is a diffeomorphism from U onto the upper half-plane $V := \{x \in \mathbb{R}^2 \mid x_2 > 0\}$.

Exercise 6.4. Consider the set U of points $x \in \mathbb{R}^2$ with $n(x) := 1 + x_1 + x_2 \neq 0$ and the function $f : U \to \mathbb{R}^2$ defined by

$$f(x) = \frac{x}{n(x)}.$$

(a) Show that U is an open subset of $\mathbb{R}^2$.

(b) Show that f is injective.

(c) Show that

$$\det Df(x) = \frac{1}{n(x)^3} \quad (x \in U).$$

(d) According to the inverse function theorem, f is a diffeomorphism from U onto an open subset V of $\mathbb{R}^2$. Verify this by explicitly determining V and $f^{-1} : V \to U$.

Exercise 6.5. Consider the map $f : U = \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ given by $f(x) = \|x\|^{-2}x$.

(a) Show that $U = \mathbb{R}^n \setminus \{0\}$ is an open subset of $\mathbb{R}^n$.

(b) Show that f is a diffeomorphism from U onto itself.

(c) Assume $n = 2$ and compute the Jacobian $\det Df(x)$ for all $x \in \mathbb{R}^2$.

We now indicate a method to compute the Jacobian for general $n \geq 2$.

(d) Show that $\det Df(r, 0, \dots, 0) = -r^{-2n}$ for all $r > 0$.

(e) Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal linear map. Show that for all $x \in U$

$$A \circ Df(x) = Df(Ax) \circ A.$$

Hint: consider $A \circ f$.

(f) Find a formula expressing $\det Df(x)$ in terms of the norm $\|x\|$, for $x \in U$. *Hint*: from linear algebra it is known that for each $x \in U$ there exists an orthogonal map A such that $Ae_1 = \|x\|^{-1}x$.

Notation 6.6. For the following exercises, we introduce the following auxiliary notation: Given a function $g : \mathbb{R}^n \to \mathbb{R}^n$, we say that a subset $U \subset \mathbb{R}^n$ is *good* for g if:

- U is open and path-connected.
- $g|_U : U \to g(U)$ is a C^1-diffeomorphism.
- U is maximal: i.e. there is no $V \supset U$, strictly larger than U, satisfying the previous two properties.

Exercise 6.7. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $f(x, y) := (x(1 - x), y^2)$. Let $p = (1, 1) \in \mathbb{R}^2$. Find an open $U \subset \mathbb{R}^2$ that is good for f and contains p.

Exercise 6.8. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $f(x, y) := (\cos(x), x\sin(y))$. For each $p \in \mathbb{R}^2$, either: (1) find an open $U \subset \mathbb{R}^2$ that is good for f and contains p or (2) show that such a U cannot exist.

Exercise 6.9. Given $x \in \mathbb{R}$, consider the function $f_x(t) := t^3 + tx : \mathbb{R} \to \mathbb{R}$. Then, for each $x \in \mathbb{R}$:

- Draw the function f_x. Since this depends on x, draw a handful of cases (it is enough if you draw it for $x = 0$, for some x positive, and for some x negative).
- Compute the critical points of f_x.
- Show that if t is critical, there is no $U \subset \mathbb{R}$ containing it that is good for f_x.
- Find all the $U \subset \mathbb{R}$ that are good for f_x. Describe $f_x(U)$ explicitly.
- Show that the list of good U you provide is complete. I.e. there are no other open subsets that are good for f_x.

Consider then the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x,t) := (x, t^3 + tx) = (x, f_x(t)).$$

Then:

- Compute the points (x,t) where the determinant of $DF(x,t)$ vanishes. Recall that such a point is said to be *critical*.
- Show that if (x,t) is critical, there is no U containing it that is good for F.
- Find two different subsets $U, U' \subset \mathbb{R}^2$ that are good for F.

Hint: For the last item, look at the good subsets of each f_x and see whether you can put some of them together to produce U and U'.

6.2 Submanifolds

Exercise 6.10. Show that the *unit sphere* $S = \{x \in \mathbb{R}^n \mid \|x\|^2 = 1\}$ is a submanifold.

Exercise 6.11. Let $A : \mathbb{R}^n \to \mathbb{R}^p$ be a surjective linear map. Show that the linear subspace $N = \{x \in \mathbb{R}^n \mid A(x) = 0\}$, the kernel, is a submanifold.

Exercise 6.12. Let $A : \mathbb{R}^n \to \mathbb{R}^p$ be a linear map that is not surjective. Show that the linear subspace $N = \{x \in \mathbb{R}^n \mid A(x) = 0\}$, the kernel is not a regular level set of A. Prove that you can find some other linear map $B : \mathbb{R}^n \to \mathbb{R}^q$, with $q < p$, such that N is a regular level set of B.

Exercise 6.13. Show that the *ellipsoid* $M_c = \{x \in \mathbb{R}^3 \mid x^2 + 2y^2 + 3z^2 = c\}$ is a submanifold for every $c \neq 0$. Draw it for various c, noting that it is empty for $c < 0$.

Exercise 6.14. Show that the *hyperboloid* $M_c = \{x \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = c\}$ is a submanifold for every $c \neq 0$. Draw M_c for various values of c. Prove that:

- For $c > 0$, the submanifold M_c is path-connected.
- For $c < 0$, the submanifold M_c is not path-connected.
- M_0 is path-connected but becomes disconnected if you remove the points in which it fails to be a submanifold.

Exercise 6.15. Show that the *paraboloid* $M_c = \{x \in \mathbb{R}^3 \mid x^2 + 2y^2 = c + z\}$ is a submanifold for every c. Draw it for various values of c.

Exercise 6.16. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Show that the *graph*

$$\text{graph}(f) = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = f(x_1, \cdots, x_n)\}$$

is a submanifold.

6.3 Lagrange Multipliers

Exercise 6.17. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x, y, z) := 2x^2 + y^2 + 3z^2$. Consider the function $g : \mathbb{R}^3 \to \mathbb{R}$ given by $g(x, y, z) := x^2 - y^2 - z^2$. Define $M_a := g^{-1}(a)$ for each $a \in \mathbb{R}$.

- Show that M_a, for each $a \neq 0$, is a submanifold.
- Find the maxima and minima of $f|_{M_a}$, for each a. Determine whether they are local or global.
- Compute the Lagrange multipliers for each critical point of $f|_{M_a}$.

Exercise 6.18. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x, y, z) := \sin(x)$. Consider the function $g : \mathbb{R}^3 \to \mathbb{R}$ given by $g(x, y, z) := x^2 + y^2 + z^2$. Define $M_a := g^{-1}(a)$, $a \in \mathbb{R}$.

- Determine for each $a \neq 0$ whether M_a is a submanifold.
- Determine the critical points of $f|_{M_a}$ using the Lagrange multiplier method.
- For each a, find the maxima and minima of $f|_{M_a}$. Determine whether they are local or global.

Exercise 6.19. Consider the following data in $\mathbb{R}^3$:

- The function $f(x, y, z) := x^2 + y^2 - z : \mathbb{R}^3 \to \mathbb{R}$.
- The function $g(x, y, z) := z - 1 : \mathbb{R}^3 \to \mathbb{R}$.
- The function $(f, g) : \mathbb{R}^3 \to \mathbb{R}^2$ that they form together.
- The open $U := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < 1\}$, its closure $\bar{U} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}$, and its boundary $\partial U = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.

Then:

a. Show that $A := U \cap f^{-1}(0)$ is a submanifold of U. What is its dimension?

b. Show that $B := U \cap g^{-1}(0)$ is a submanifold of U. What is its dimension?

c. Show that $C := (f, g)^{-1}(0)$ is a submanifold of $\mathbb{R}^3$. What is its dimension? Show that it is contained in ∂U.

Consider now the function $h : \mathbb{R}^3 \to \mathbb{R}$ given by $h(x, y, z) := x^2 - y^2$.

d. Draw A, B, and C. In a separate picture, draw some of the level sets of h (say, a positive level set, a negative level set, and the zero level set). For the next two items it may be convenient to compare these two pictures (perhaps by making further pictures in which you show how different level sets of h interact with A, B, and C).

e. Using Lagrange multipliers, determine the critical points of $h|_A$, $h|_B$, and $h|_C$.

f. Determine the minima/maxima of $h|_{A \cup B \cup C}$. State whether they are local or global.

Exercise 6.20. Let A be a symmetric $n \times n$ matrix with real coefficients. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = \langle Ax, x \rangle.$$

Let $S = \{x \in \mathbb{R}^n \mid \|x\|^2 = 1\}$.

(a) Show that there exists $a \in S$ such that $f(x) \leq f(a)$ for all $x \in S$.

(b) Show that there exists $\lambda \in \mathbb{R}$ such that $Aa = \lambda a$. ***Hint***: use the Lagrange method with $g(x) = \|x\|^2 - 1$.

Exercise 6.21 (Continuation of Opgave 6.20)**.** Now consider several elements $a_1, \ldots, a_k \in S$ such that $a_1, \ldots, a_k$ are linearly independent. Let V be the linear subspace $a_1^\perp \cap \cdots \cap a_k^\perp$ of $\mathbb{R}^n$ and assume that $A(V) \subset V$.

(a) Show that there exists $b \in S \cap V$ such that $f(x) \leq f(b)$ for all $x \in V$.

(b) Let b be as in (a). Show that there exists $\lambda \in \mathbb{R}$ such that $Ab = \lambda b$. ***Hint***: use the Lagrange method with suitable functions $g, g_1, \ldots, g_k$ and use that $A(V) \subset V$.

(c) Prove that there exists an orthonormal set of vectors $a_1, \ldots, a_n \in \mathbb{R}^n$ and real constants $\lambda_1, \ldots, \lambda_n$ such that

$$Aa_j = \lambda_j a_j, \quad (1 \leq j \leq n).$$

Exercise 6.22. Consider the parabola $P \subset \mathbb{R}^2$ given by the equation $x_1^2 - 4x_2 = 0$.

(a) Show that for every $t \in \mathbb{R}$ the function $x \mapsto d_t(x) = \|x - (0,t)\|$ attains a minimum on P.

(b) Predict how the function $t \mapsto d((0,t), P)$ behaves on $\mathbb{R}$ and sketch the expected graph.

(c) Compute $d((0,t), P)$ for every $t \in \mathbb{R}$ using the Lagrange method. Compare the obtained result with your prediction in (b).

Exercise 6.23. Given $a_1, \ldots, a_k \in \mathbb{R}$, show that the function $f : x \mapsto \sum_{j=1}^k a_j x_j$ attains a maximum and a minimum on the unit sphere $S = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$. Determine these maximum and minimum values in two ways:

(a) using the Lagrange method;

(b) using the Cauchy-Schwarz inequality.

Exercise 6.24. Consider the plane V in $\mathbb{R}^3$ given by the equation $\langle b, x \rangle = c$, with $b \in \mathbb{R}^3 \setminus \{0\}$ and $c \in \mathbb{R}$. Determine the point on V with minimal distance to a given $a \in \mathbb{R}^3$. Verify your answer for $c = 0$.

Exercise 6.25. Given two planes V and W in $\mathbb{R}^3$ given by the equations $V : \langle a, x \rangle = p$ and $W : \langle b, x \rangle = q$, with $a, b \in \mathbb{R}^3$ linearly independent and $p, q \in \mathbb{R}$, determine the point $x \in V \cap W$ with the smallest distance to the origin.

Exercise 6.26. (a) Determine the maximum of $(x_1 x_2 \cdots x_n)^2$ under the condition $\|x\|^2 = 1$.

(b) Use (a) to show that for all $a_1, \ldots, a_n > 0$:

$$(a_1 \cdots a_n)^{1/n} \leq \frac{1}{n}(a_1 + \cdots + a_n).$$

6.4 Extra

Exercise 6.27. Consider $U = V = \mathbb{R}^2$; we introduce these names to clearly distinguish the source and target of the change of coordinates $\phi : U \to V$ given by $\phi(x, y) := (x, y + x)$. Denote the coordinates in V by (a, b), to distinguish them from $(x, y) \in U$.

- Compute the differential $D\phi : U \to \mathrm{Lin}(U, V)$.
- Verify that ϕ is a C^∞-diffeomorphism.

Suppose we are given a smooth vector field $X : U \to U$. We want to produce a vector field $Y : V \to V$ corresponding to X under ϕ, via:

$$Y(a, b) := D\phi(\phi^{-1}(a, b))(X(\phi^{-1}(a, b))) : V \to V. \tag{6.1}$$

- Write the entries Y_1 and Y_2 of Y explicitly in terms of the entries of X. Show that Y is C^∞.

Consider the function $f : U \longrightarrow \mathbb{R}$ given by $f(x, y) := x$ and its gradient $\mathrm{grad}(f) : U \to U$. Compute $\mathrm{grad}(f)$ and show it is smooth.

- Compute $f \circ \phi^{-1} : V \to \mathbb{R}$.
- Apply Equation 6.1 to $X = \mathrm{grad}(f)$ and ϕ to produce a vector field $Y : V \to V$. Write it explicitly.
- Verify that $Y \neq \mathrm{grad}(f \circ \phi^{-1})$.

Exercise 6.28. Let $H := \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = a\}$ with $a > 0$. Consider the subsets $H_+ := \{x \in H \mid x_j \geq 0 \text{ for all } j\}$ and $H_{++} := \{x \in H \mid x_j > 0 \text{ for all } j\}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) := \sum_{j=1}^n x_j^k$, $k \in \{2, 3, \ldots\}$. Show that f attains its minimum $\mathbf{m}_n := n^{1-k}a^k$ on H_+.

(a) Show that f attains the value $\mathbf{m}_n$ on H_+.

(b) Show that f attains a minimal value μ_n on H_+.

(c) Show that $\mu_1 = \mathbf{m}_1$.

(d) Show that the minimum μ_n is attained in H_{++}.

(e) Use Lagrange multipliers to show that $\mu_n = \mathbf{m}_n$.

(f) Interpret this geometrically for $k = 2$.

Exercise 6.29 (Hadamard's Estimate)**.** Let M_n denote the set of real $n \times n$ matrices $x = (x_{ij})$. Identify M_n with $\mathbb{R}^{n^2}$. For a differentiable function $f : M_n \to \mathbb{R}$, the gradient $\mathrm{grad} f : M_n \to M_n$ is defined componentwise.

(a) Show that $f(x) := \det(x)$ is differentiable with gradient

$$(\mathrm{grad} f(x))_{ij} = (-1)^{i+j} D_{ij}(x),$$

where $D_{ij}(x)$ is the determinant of the matrix obtained by deleting row i and column j.

(b) Show that for $g_i(x) := \|R_i(x)\|^2$ (the squared norm of the i-th row), its gradient is

$$(\mathrm{grad} g_i(x))_{jk} = 2\delta_{ij} x_{jk}.$$

Let $d_1, \ldots, d_n > 0$ and define

$$S_i := \{x \in M_n \mid \|R_i(x)\|^2 = d_i^2\}.$$

(c) Show that f attains a maximum $M > 0$ on $S := S_1 \cap \cdots \cap S_n$, and that if $f(x) = M$, there exist $\lambda_1, \ldots, \lambda_n \neq 0$ such that

$$x^c = \text{diag}(\lambda_1, \ldots, \lambda_n) x^t.$$

(d) Show that for such x, $x^t x = \text{diag}(d_1^2, \ldots, d_n^2)$ and conclude $M = d_1 \cdots d_n$.

(e) Prove Hadamard's inequality for every $x \in M_n$:

$$|\det x| \leq \|R_1(x)\| \cdots \|R_n(x)\|.$$

7 Line integrals of covector fields

7.1 Computing line integrals

Exercise 7.1. Compute the line integrals $\int_\gamma \alpha$ for the following covector fields and curves in $\mathbb{R}^2$ or $\mathbb{R}^3$.

(a) $\alpha(x, y, z) = (x, y, xy - z)$ and $\gamma(t) = t(1, 2, 4)$, with $0 \le t \le 1$.

(b) $\alpha(x, y) = (x^2 - y^2, 2xy)$ and γ a piecewise C^1 parametrization of the boundary of the rectangle $R := \{0 \le x, y \le a\}$, traversed counterclockwise, $a > 0$.

(c) $\alpha(x, y, z) = (y, z, x)$ and γ a C^1 parametrization of the intersection of the sphere $\{x^2 + y^2 + z^2 = a^2\}$ with the cylinder $\{x^2 + y^2 = \frac{1}{2}a^2, z > 0\}$, where $a > 0$.

Exercise 7.2. Consider the vector field $\alpha : \mathbb{R}^3 \to \mathrm{Lin}(\mathbb{R}^3, \mathbb{R})$ defined by $\alpha(x, y, z) = (yz, xz, xy)$, and the curve $\gamma : [0, \pi] \to \mathbb{R}^3$ defined by $\gamma(t) = (\cos t, \sin t, \sin^2 t)$.

(a) Compute the line integral $\int_\gamma \alpha$ by direct calculation.

(b) Show that α is closed, and find a potential on $\mathbb{R}^3$.

(c) Compute the line integral $\int_\gamma \alpha$ again using the potential from (b).

Exercise 7.3. Define the covector field $\alpha : \mathbb{R}^3 \longrightarrow \mathrm{Lin}(\mathbb{R}^3, \mathbb{R})$ by $\alpha(x, y, z) := (y, x, z^2)$. Let $\gamma : [0, 1] \longrightarrow \mathbb{R}^3$ be the curve $\gamma(t) := (t^2, t^3, t)$.

- Compute $\int_\gamma \alpha$.
- Does α have a primitive?

Exercise 7.4. Let $v : \mathbb{R}^3 \longrightarrow \mathrm{Lin}(\mathbb{R}^3, \mathbb{R})$ be the covector field $\alpha(x, y, z) := (\cos(z), \sin(z), 0)$. Let $\gamma : [0, 1] \longrightarrow \mathbb{R}^3$ be the curve $\gamma(t) := (t, 0, 0)$.

- Does α have a potential? If yes, describe one explicitly.
- Is there a C^1 curve $\nu : [0, 1] \longrightarrow \mathbb{R}^3$ such that $\nu(0) = \gamma(0)$, $\nu(1) = \gamma(1)$, and $\int_\gamma v(p)dp \neq \int_\nu v(p)dp$?

Exercise 7.5. Consider a C^1 curve $\gamma : [0, 1] \longrightarrow \mathbb{R}^n$ and a continuous vector field $\alpha : \mathbb{R}^n \longrightarrow \mathrm{Lin}(\mathbb{R}^n, \mathbb{R})$. Define $\gamma_s := \gamma|_{[0,s]} : [0, s] \to \mathbb{R}^n$. Let $h : [0, 1] \to \mathbb{R}$ be the function defined by $h(s) := \int_{\gamma_s} \alpha$.

- Show that h is a C^1 function. Compute its derivative.
- Suppose that α is the total derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$. Show that h is identically zero if and only if there exists a $c \in \mathbb{R}$ such that $\gamma(t) \in f^{-1}(c)$ for every $t \in [0, 1]$.

Exercise 7.6. Let $U \subset \mathbb{R}^n$ be an open set and $\alpha : U \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R})$ a closed covector field. Let $\gamma : [a, b] \to U$ be a continuous curve. Show that

$$f(\tau) := \int_{\gamma|_{[a,\tau]}} \alpha$$

defines a potential $f : [a, b] \to \mathbb{R}$ of α along γ.

7.2 (Non-)simply sonnected spaces

Exercise 7.7. The goal of this opgave is to show that $U := \mathbb{R}^2 \setminus \{0\}$ is not simply connected. This is similar to the proof given in the lecture notes. The difference is that we will now use paths relative endpoints, instead of loops.

On U consider the covector field $v : U \to \mathbb{R}^2$ defined by

$$\alpha(x) = \|x\|^{-2}(-x_2 \quad x_1).$$

(a) Show that α is closed.

(b) Let $c_\pm : [0, \pi] \to U$ be defined by $c_\pm(t) = (\cos t, \pm \sin t)$. Compute the integrals

$$\int_{c_+} \alpha \qquad \int_{c_-} \alpha.$$

(c) Prove that c_+ and c_- are not homotopic relative endpoints.

(d) Show that α has no potential.

(e) Conclude that U is not simply connected.

Exercise 7.8. Write $U := \mathbb{R}^2 \setminus \{0\}$ for the complement of the origin in $\mathbb{R}^2$. Let $\gamma \in C^\infty([0, 1], U)$ be a loop.

a. Given any positive real number r, define $\gamma_r(t) := r\gamma(t)$. Show that it is a smooth loop.

b. Show that γ and γ_r are homotopic as loops for all $r > 0$.

Suppose now that $\alpha \in C^\infty(U, \mathrm{Lin}(\mathbb{R}^2, \mathbb{R}))$ is a closed covector field. Assume that $\int_\gamma \alpha = C$, with $C \neq 0$. Then:

c. Show that α has no potential.

d. Show that $\int_{\gamma_r} \alpha = C$ for every $r > 0$.

Suppose that $\beta \in C^\infty(\mathbb{R}^2, \mathrm{Lin}(\mathbb{R}^2, \mathbb{R}))$ is a covector field such that $\beta|_U = \alpha$.

e. Show that β is also closed. **Hint:** Consider the function $\mathbb{R}^2 \to \mathrm{Lin}(\mathbb{R}^2, \mathrm{Lin}(\mathbb{R}^2, \mathbb{R}))$ given by $x \mapsto D\beta(x) - (D\beta(x))^t$, where the second term is the transpose.

f. Deduce that such a β cannot exist.

Fix a point $q \in \mathbb{R}^2$, different from the origin. Write $U_q := \mathbb{R}^2 \setminus \{0, q\}$ for the complement of q in U.

g. Construct a covector field $\alpha_q \in C^\infty(U_q, \mathrm{Lin}(\mathbb{R}^2, \mathbb{R}))$ and a loop $\eta_q \in C^\infty([0, 1], U_q)$ such that $\int_{\eta_q} \alpha_q = C$ but $\int_{\eta_q} \alpha = 0$.

h. Conclude that η_q is not contractible but η_q and γ are not homotopic as loops in U_q.

Exercise 7.9. The goal of this opgave is to show that $U := \mathbb{R}^n \setminus \{0\}$ is simply connected for $n \geq 3$. Let $\gamma : [0, 1] \to \mathbb{R}^n \setminus \{0\}$ be a closed continuous curve.

(a) Show that there exists $r > 0$ such that $\|\gamma(t)\| \geq r$ for all $t \in [0, 1]$.

(b) Show that for $x, y \in \mathbb{R}^n \setminus B(0; r)$

$$\|x - y\| < r \implies 0 \notin [x, y].$$

(c) Show that there exists $\delta > 0$ such that for all $t_1, t_2 \in [0, 1]$

$$|t_1 - t_2| < \delta \implies 0 \notin [\gamma(t_1), \gamma(t_2)].$$

Hint: use a theorem on uniform continuity from *Introduction to Analysis*.

(d) Show that γ is homotopic in U to a closed piecewise linear curve $c : [0, 1] \to U$, meaning there exists a partition $0 = t_0 < t_1 < \ldots < t_k = 1$ such that for each $1 \leq j \leq k$

$$c(t_{j-1} + \tau(t_j - t_{j-1})) = c(t_{j-1}) + \tau(c(t_j) - c(t_{j-1})), \quad (\tau \in [0, 1]).$$

(e) Show that for each $1 \leq j \leq k$ there exists a unit vector n_j such that $n_j \perp c(t_{j-1})$ and $n_j \perp c(t_j)$.

(f) Show that there exists $p \in \mathbb{R}^n$ with $\langle p, n_j \rangle \neq 0$ for all $1 \leq j \leq k$.

(g) Show that for all $\xi \in c([0, 1])$, $0 \notin [p, \xi]$.

(h) Show that c is homotopic in U to the constant curve $t \mapsto p$.

(k) Conclude that U is simply connected.

7.3 Extra

Exercise 7.10. The length $L(\gamma)$ of a C^1 curve $\gamma : [a, b] \to \mathbb{R}^n$ is defined by

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \, dt.$$

(a) Let $\phi : [c, d] \to [a, b]$ be a monotone C^1 map. Show that the reparametrization $\gamma \circ \phi : [c, d] \to \mathbb{R}^n$ is a C^1 curve and satisfies

$$L(\gamma \circ \phi) = L(\gamma).$$

(b) Show that for every continuous vector field v defined on $\gamma([a, b])$, there exists a constant $M > 0$ such that $\|v(x)\| \leq M$ for all $x \in \gamma([a, b])$.

(c) Show that for v and M as in (b):

$$\left| \int_\gamma v(x) \cdot dx \right| \leq L(\gamma) M.$$

(d) Explain how the above extends to piecewise C^1 curves in $\mathbb{R}^n$, and prove the corresponding statements.

Exercise 7.11. Let $\mathcal{O} =]0, \infty[\times \mathbb{R}$ and consider the polar coordinates map $\Phi : \mathcal{O} \to \mathbb{R}^2$ given by $\Phi(r, \phi) = (r \cos \phi, r \sin \phi)$. Previously, the inverse function theorem was used to show that for each $(r_0, \phi_0) \in \mathcal{O}$ there exists an open neighborhood $U \subset \mathcal{O}$ such that $\Phi(U)$ is open in $\mathbb{R}^2$ and Φ is a C^1 diffeomorphism from U onto $\Phi(U)$.

(a) Show that for each $x^0 \in \mathbb{R}^2 \setminus \{0\}$ there exists an open set $U \subset \mathcal{O}$ such that $\Phi(U)$ is an open neighborhood of x^0 and $\Phi : U \to \Phi(U)$ is a diffeomorphism.

Denote the inverse of $\Phi|_U$ by Ψ and define the function $\psi = \psi_U : \Phi(U) \to \mathbb{R}$ by $\psi(x_1, x_2) = \Psi(x_1, x_2)_2$.

(b) Show that ψ is a C^1 function on $\Phi(U)$ and that $\Phi(\|x\|, \psi(x)) = x$ for all $x \in \Phi(U)$.

(c) Show that the vector field $\nabla\psi$ on $\Phi(U)$ equals v, where the vector field $v : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ is given by

$$v(x) = \|x\|^{-2}(-x_2, x_1).$$

Hint: use a formula for $D\Psi(x)$.

Let $R : [0,1] \to]0, \infty[$ and $\phi : [0,1] \to \mathbb{R}$ be two continuous functions. Define the continuous curve $\gamma : [0,1] \to \mathbb{R}^2 \setminus \{0\}$ by

$$\gamma(t) = R(t)(\cos\phi(t), \sin\phi(t)).$$

(d) Show that $f(t) = \phi(t)$ defines a potential of v along γ.

(e) Compute $\int_\gamma v(x) \cdot dx$.

8 Extra: Reeksen

8.1 Reeksen

Exercise 8.1. Ga na of de volgende reeksen convergent of divergent zijn;

(a) $\sum_{n\geq 1} \frac{2^n}{n!}$; (b) $\sum_{n\geq 1} \cos \frac{1}{n^4}$; (c) $\sum_{n\geq 2} \frac{1}{(\log n)^n}$;

(d) $\sum_{n\geq 1} \frac{(2n)!}{(n!)^2}$; (e) $\sum_{n\geq 1} \sin \frac{1}{n^3}$; (f) $\sum_{n\geq 1} \sin \frac{1}{n}$.

Exercise 8.2. Beschouw de reeks $\sum_{k\geq 1} a_k$ in $\mathbb{R}$ met $a_k = (-1)^{k+1}\frac{1}{k}$.

(a) Bewijs dat de reeks convergent is.

(b) Bewijs dat de reeks niet absoluut convergent is.

(c) Definieer een bijectie $k \mapsto n(k)$ van $\mathbb{N}_+ = \{1, 2, \ldots\}$ op zichzelf, zo dat de reeks

$$\sum_{k\geq 1} a_{n(k)}$$

convergent is met som 0.

Exercise 8.3. Toon aan dat de reeks

$$\sum_{n\geq 2} \log\left(1 - \frac{1}{n^2}\right)$$

convergent is. *Hint*: majoreer door Taylor met rest te gebruiken.

8.2 Reeksen met een parameter

Exercise 8.4. (a) Bepaal alle $a \in \mathbb{R}$ waarvoor de reeks $\sum_{k\geq 1} \frac{\log k}{k^a}$ convergent is. *Hint*: gebruik het bekende feit dat $\lim_{k\to\infty} k^{-\geq} \log k = 0$ voor $\geq > 0$.

(b) Beantwoord dezelfde vraag voor de reeks $\sum_{k\geq 2} \frac{1}{k^a \log k}$.

Exercise 8.5. Beschouw de reeks $\sum_{n=1}^{\infty} \dfrac{\sin(n)}{n} z^n$, $z \in \mathbb{C}$.

- Toon aan dat de reeks convergeert als $|z| < 1$.
- Toon aan dat er een z is zodat de reeks divergeert.

Exercise 8.6. Voor welke complexe waarden $z \in \mathbb{C}$ zijn de volgende reeksen convergent?

- $\sum_{n=0}^{\infty} \dfrac{n}{(3n+1)!} z^n$.
- $\sum_{n=0}^{\infty} \dfrac{n^2+n}{n^2+1} z^n$.

Exercise 8.7. Voor welke reële waarden van x zijn de volgende reeksen convergent?

(a) $\sum_{n\geq 1} \frac{\cos(x^n)}{n^2}$; (b) $\sum_{n\geq 1} \frac{\sqrt{n}+\cos nx}{n^2+1}$; (c) $\sum_{n\geq 1} \frac{x^n}{x^2+n^2}$.

8.3 Extra

Exercise 8.8. Laat met een resultaat uit het dictaat zien dat er een constante $C > 0$ en een rij (r_n) reële getallen bestaan zo dat

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + C + r_n, \qquad \lim_{n\to\infty} r_n = 0.$$

(a) Toon aan dat

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \log 2 + r_{2n} - r_n.$$

(b) Toon dat $\sum_{k\geq 1} \frac{(-1)^{k+1}}{k}$ convergent is en dat

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \log 2.$$

Exercise 8.9. Gegeven zijn complexe rijen $(a_k)_{k\geq 0}$ en $(b_k)_{k\geq 0}$. We definiëren

$$B_n := \sum_{k=0}^{n} b_k, \qquad (n \geq 0).$$

(a) Toon aan dat voor alle $n \geq 0$ geldt:

$$\sum_{k=0}^{n} a_k b_k = \sum_{k=0}^{n-1} (a_k - a_{k+1}) B_k + a_n B_n.$$

Hint: schrijf $b_k = B_k - B_{k-1}$, voor $k \geq 1$.

In het vervolg veronderstellen we dat (a_k) een monotoon dalende rij positieve reële getallen is met $\lim_{k\to\infty} a_k = 0$. Verder veronderstellen we dat een $M > 0$ bestaat zo dat $|B_n| \leq M$ voor alle $n \geq 0$.

(b) Toon aan dat de reeks $\sum_{k\geq 0} (a_k - a_{k+1}) B_k$ convergent is en dat

$$\left| \sum_{k=0}^{\infty} (a_k - a_{k+1}) B_k \right| \leq a_0 M.$$

(c) Toon aan dat de reeks $\sum_{k\geq 0} a_k b_k$ convergent is en dat

$$\left| \sum_{k=0}^{\infty} a_k b_k \right| \leq 2 a_0 M.$$

(d) Toon aan dat de reeks

$$\sum_{k\geq 0} a_k z^k$$

convergent is voor alle $z \in \mathbb{C}$ met $|z| \leq 1$, $z \neq 1$. Geef een voorbeeld waaruit blijkt dat de reeks divergent kan zijn voor $z = 1$.

9 Extra: Oneigenlijke integralen

Exercise 9.1. Gegeven is een continue functie $f : [0,1] \to \mathbb{R}$. Toon aan dat de integraal

$$\int_0^1 f(t)\, t^x (1-t)^y \, dt$$

convergent is voor $x, y > -1$, en op dat gebied een continue functie van (x, y) definieert.

Exercise 9.2. Gegeven is een continue functie $f : [0,1] \to \mathbb{R}$ met $f(0) = 1$. Toon aan dat de integraal

$$\int_0^1 \frac{f(t)}{t} \, dt$$

divergeert.

Exercise 9.3. Toon aan de oneigenlijke integraal

$$\int_0^\infty \frac{\sin t}{t\sqrt{t}} \, dt$$

convergeert.

Exercise 9.4. (a) Toon aan dat de oneigenlijke integraal

$$\int_1^\infty \frac{\cos x}{x^2} \, dx$$

convergeert.

(b) Toon aan dat de oneigenlijke integraal

$$\int_0^\infty \frac{\sin t}{t} \, dt$$

convergeert. Hint: dit lukt niet met het majorantie-criterium. Beschouw de integraal $\int_1^\beta \frac{\sin t}{t} \, dt$ en gebruik partiële integratie om de integraal te vergelijken met de integraal in (a).

Exercise 9.5. We bekijken nogmaals de volgende oneigenlijke integraal uit Opgave 2.6:

$$F(t) := \int_0^\infty \frac{1}{x^2 + t} \, dx, \qquad (t > 0).$$

Gebruik in de volgende onderdelen direct de behandelde stellingen over oneigenlijke integratie.

(a) Laat zien dat de integraal convergeert voor iedere $t > 0$.

(b) Bewijs dat de functie F continu differentieerbaar is, met afgeleide

$$F'(t) = -\int_0^\infty \frac{1}{(x^2 + t)^2} \, dx.$$

(c) Toon aan dat voor $k \in \mathbb{N}$ geldt dat

$$\int_0^\infty \frac{1}{(1 + x^2)^{k+1}} \, dx = \frac{(2k)!\pi}{2^{2k+1}(k!)^2}.$$

Exercise 9.6. (a) Laat zien dat door

$$f(x) = \int_{-\infty}^{\infty} e^{-t^2} \cos(xt)\, dt$$

een continu differentieerbare functie gedefinieerd wordt.

(b) Toon aan dat $xf(x) = -2f'(x)$ voor alle $x \in \mathbb{R}$.

(c) Toon aan dat

$$f(x) = \sqrt{\pi} e^{-x^2/4},$$

voor alle $x \in \mathbb{R}$. Hint: differentieer de functie $g(x) = f(x)e^{x^2/4}$.

Exercise 9.7. In deze opgave zullen we laten zien dat de integraal

$$\int_0^{\infty} \frac{\sin x}{x}\, dx$$

niet absoluut convergent is. We doen dit door middel van een tegenspraak. Veronderstel dus dat de integraal wel absoluut convergent is.

(a) Toon aan dat uit de aanname volgt dat de integraal

$$\int_1^{\infty} \frac{(\sin x)^2}{x}\, dx$$

convergent is.

(b) Toon aan dat voor alle $R > 1$ geldt dat

$$\int_1^{R} \frac{(\sin x)^2}{x}\, dx \geq \int_{1+\pi/2}^{R+\pi/2} \frac{(\cos x)^2}{x}\, dx.$$

(c) Toon aan dat uit de aanname ook volgt dat de integraal

$$\int_1^{\infty} \frac{(\cos x)^2}{x}\, dx$$

convergeert.

(d) Laat zien dat (a) en (c) tot een tegenspraak leiden.