Introduction to Algebraic Topology and Category Theory

Á. del Pino Gómez

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Contents

The goals of this course

Welcome to Topologie en Meetkunde! In this course we will be interested in *topological spaces* and in the *continuous functions* between them. The main question we want to look into is the following:

Question 0.1. *Given topological spaces A and B, can we distinguish them? That is, can we tell whether there exists a homeomorphism $f : A \to B$ between the two?*

There are many tools meant to provide some answers to Question 0.1. The following questions summarise some of the ideas that you have encountered already (perhaps in Inleiding Topologie, the course leading up to this one):

- Are A and B both Hausdorff?
- Are A and B both connected? Even further: do A and B have the same number of connected components[1]?
- Are A and B both compact?

Being Hausdorff/connected/compact is *invariant under homeomorphisms*. It follows that if the answer to any of these questions is "no", then A and B cannot possibly be homeomorphic.

0.1 The themes of this course

There are three main themes/goals that we will pursue in this course.

0.1.1 Algebraic invariants of spaces

Our first learning goal reads:

To become familiar with new invariants of topological spaces, particularly the so-called **fundamental group**.

The nature of the fundamental group is quite different from the nature of the invariants we have mentioned above. Observe that:

[1] Recall that the connected component of a point $a \in A$ is the largest connected subset of A containing x. Connected components partition A and we can therefore ask what the cardinality is of the set of all connected components.

- Being compact is a *binary* invariant. I.e. you are either compact or not.
- The same applies to Hausdorffness.
- Connectedness is slightly different. By counting the connected components we obtain an invariant that is a number (or, in general, a cardinality).

By contrast, the fundamental group is a group! This will allow us to encode more subtle information about our topological space.

Defining the fundamental group rigorously will involve some work, but an intuitive idea is given in Section 0.3 below. As the course progresses we will encounter different techniques that will allow us to compute it. Namely:

- The *theorem of van Kampen*. The rough idea is as follows: Under certain assumptions, whenever a topological space A is presented as a union $A = B \cup C$, we will be able to compute the fundamental group of A from the fundamental groups of B, C, and $B \cap C$.
- The *theory of covering spaces*. The intuition is that, again under suitable assumptions, a space A can be "unwrapped" to produce another space $\widetilde{A}$ (called the universal cover) together with a map $\pi : \widetilde{A} \to A$ that "wraps" $\widetilde{A}$ around A. The fundamental group is precisely a measure of how much A can be unwrapped.

0.1.2 More examples

The second learning goal of the course is:

> To become familiar with more examples of topological spaces.

Two families of examples will be particularly important to us:

- *Surfaces*. Using the tools introduced in this course (mostly the fundamental group), we will be able to classify them.
- *CW-complexes*. These are spaces that can be built by iteratively gluing discs of various dimensions. The theorem of van Kampen can be applied algorithmically, as we go along gluing, to compute the fundamental group.

0.1.3 Category theory as the organising principle in Mathematics

The last learning goal of the course is:

> To understand how the ideas of this course fit within the framework of **Category Theory**.

Category Theory permeates many (or all?) areas of Mathematics, providing a unifying perspective. Being aware of this will make many of the topological concepts appearing in this course more transparent. We introduce some of the basic definitions in Section 1.1 below, but we will continue introducing new concepts as they become relevant to our topological goals.

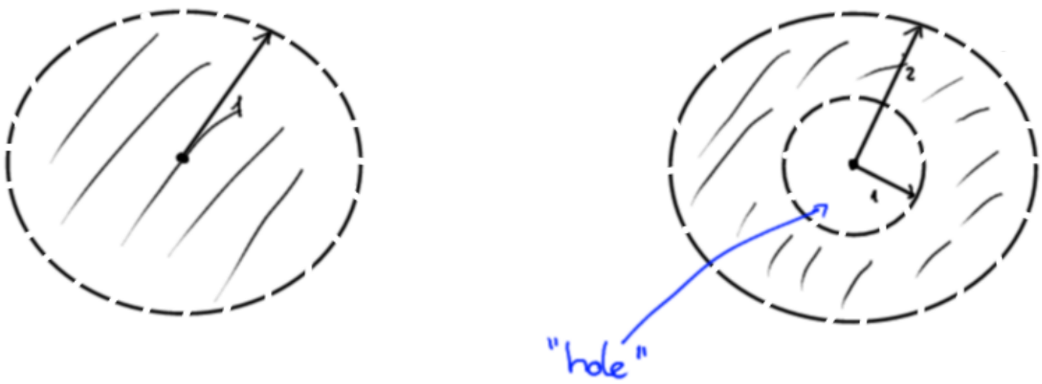


Figure 1: The open ball $\mathbb{B}^2$ on the left and the open annulus $\mathbb{A}$ on the right. Both are Hausdorff, connected, and non-compact. How do we tell them apart?

0.2 A motivating example

The premise of this course is that we need new techniques to tell topological spaces apart (Question 0.1). However, before we introduce new fancy techniques, it is probably best if we convince ourselves that our existing tools don't quite cut it.

Thus: consider the spaces shown in Figure 1. On the left we see the 2-dimensional open ball:

$$\mathbb{B}^2 = \{(x, y) \in \mathbb{R}^2 \mid |(x, y)| = \sqrt{x^2 + y^2} < 1\}.$$

On the right we have the 2-dimensional open annulus (of inner radius 1 and outer radius 2):

$$\mathbb{A} = \{(x, y) \in \mathbb{R}^2 \mid 1 < |(x, y)| < 2\}.$$

The claim is that:

Problem 0.2: The ball $\mathbb{B}^2$ and the annulus $\mathbb{A}$ are *not homeomorphic*.

We can try to address Problem 0.2 with the tools we already have:

Lemma 0.3. *$\mathbb{B}^2$ and $\mathbb{A}$ are Hausdorff and non-compact.*

Proof. According to the Heine-Borel theorem, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. $\mathbb{B}^2$ and $\mathbb{A}$ are bounded but open, so they are not compact. Furthermore, $\mathbb{R}^n$ is metrisable (i.e. we can endow it with its usual metric space structure) and thus Hausdorff. It follows that all its subspaces are metrisable and Hausdorff. □

Lemma 0.4. *$\mathbb{B}^2$ and $\mathbb{A}$ are path-connected and therefore connected.*

Proof. We discuss path-connectedness in detail below, in Subsection 2.1.2. For now, just recall the definition (a space is path-connected if any two points can be connected via a continuous path), as well as the fact that being path-connected implies being connected (Lemma 2.9 below).

$\mathbb{B}^2$ is convex and therefore path-connected (Corollary 2.8 below). The annulus $\mathbb{A}$ is neither convex nor star-shaped, but we can show that it is path-connected by hand. Namely, any point in $\mathbb{A}$ can be connected via a radial path to a point in $S \subset \mathbb{A}$, the circle of radius $3/2$. It remains to show that S is path-connected, which follows from the fact that any two points $\frac{3}{2}e^{\theta_0 i}, \frac{3}{2}e^{\theta_1 i} \in S$ can be connected by a path that interpolates linearly between their angles. □

That is to say, we cannot use connectedness, Hausdorffness, or compactness to tell $\mathbb{B}^2$ and $\mathbb{A}$ apart. What distinguishes them is the fact that $\mathbb{A}$ has a big hole in the middle and $\mathbb{B}^2$ does not. Which leads us to the following general idea:

We should formalise the intuition that the "number of holes" that a space has, as well as their "type", are invariants of topological spaces.

The fundamental group will be such an invariant (but it is not the only one!).

0.3 Introducing the fundamental group (loosely)

How do we formalise the fact that $\mathbb{B}^2$ and $\mathbb{A}$ are not homeomorphic because $\mathbb{A}$ has a hole in the middle but $\mathbb{B}^2$ does not? The idea is that we can "tie a rope" around the hole of $\mathbb{A}$ in a manner that cannot be undone. To be precise:

Lemma 0.5. *Let $\gamma : \mathbb{S}^1 \to \mathbb{A}$ be the inclusion of the circle of radius $3/2$. This loop cannot be contracted to a constant map in a continuous way.*

The intuition is that any attempt to contract γ (by "pulling it") will result in the loop getting "stuck" in the hole. The hole serves thus as an obstacle. See Figure 2.

In contrast, any "rope" in $\mathbb{B}^2$ can be contracted to a point. The intuition is that $\mathbb{B}^2$ has no holes and therefore no loop in $\mathbb{B}^2$ can get stuck in them:

Lemma 0.6. *Let $X \subset \mathbb{R}^n$ be convex. Any loop $\nu : \mathbb{S}^1 \to X$ is homotopic (i.e. can be deformed continuously) to a constant map.*

Proof. Fix a point $x \in X$ and consider the map $F(t, s) := (1 - s)\nu(t) + sx$. This is a continuous map $F : \mathbb{S}^1 \times [0, 1] \to X$, which we imagine as a "movie" of maps $\mathbb{S}^1 \to X$. The first map in this movie is $F(t, 0) = \nu(t)$. The last map $F(t, 1)$ is the constant loop with value x. We have therefore exhibited a continuous way of connecting the two. □

It will take us several lectures to formalise all these ideas. For now, you can keep in mind the following intuitions:

- Some topological spaces have holes, which we are interested in detecting.
- Some of these holes can be detected using loops. Namely, if a loop cannot be deformed into some other one, it must be because it is getting "stuck somewhere".
- This motivates us to define the fundamental group to be the set of loops, up to deformation. Concatenation of loops will provide the group multiplication.

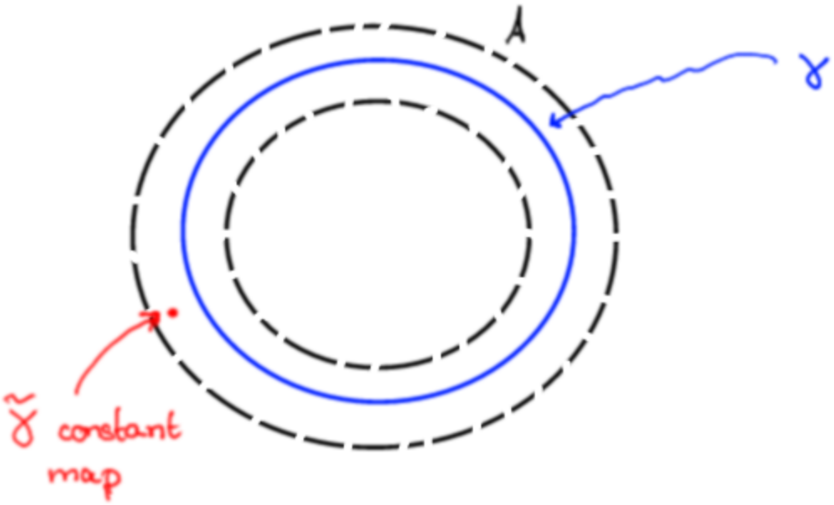


Figure 2: Two loops in the annulus $\mathbb{A}$. One of them is constant, the other is the usual inclusion of the circle of radius 3/2. The two cannot be continuously deformed to one another.

- In the case of the annulus $\mathbb{A}$, loops can wrap in $\mathbb{Z}$ different ways. Namely, we can count how often the loop goes around the hole. Two loops can be deformed to one another if and only if they wrap the same number of times. We will prove this in Theorem 3.2.
- Some spaces have holes that cannot be detected via the fundamental group. You can think of these as "higher dimensional holes".

And remember:

The main tool to be used in this course is the fundamental group. This means that most exercises will require that you compute/use it!

Homotopies and categories

Lecture 1

Our goals in this first chapter are as follows:

- To introduce the language of category theory (Subsection 1.1).
- To define what it means to "continuously deform" a map (Definition 1.17).

Note: We have been doing it all along, but just to be clear:

- Whenever we write "map", we mean continuous function.
- Whenever we say "space", we mean topological space.

1.1 What are categories?

Developing some familiarity with Category Theory is also one of the learning goals of this course. The basic idea behind it is that certain patterns and structural reasoning can be found everywhere in Mathematics, not being particular to any concrete field. Categorical reasoning is not about concrete mathematical objects, but rather about how mathematical reasoning itself is structured.

Consider for instance the following thinking pattern: We define a certain mathematical object of interest (in this course, topological spaces). These *objects* are often sets endowed with some extra structure. This leads us to focus, among all possible functions, on those that interact nicely with the extra structure (for us, continuous maps); we call these *morphisms*. Some morphisms are bijections whose inverse is also a morphism (for us, homeomorphisms); we call these *isomorphisms*.

You have seen this pattern in many other courses. The first case you encountered was the world of sets; here morphisms are simply functions and isomorphisms are bijections. You are probably also familiar with the world of groups; a morphism is then called a *group homomorphism* and an isomorphism is called, for emphasis, a *group isomorphism*. You may have encountered the same idea for rings, vector spaces, modules, manifolds...

It may now be apparent why it is handy to learn some Category Theory: it pinpoints the conceptual parallels shared by different mathematical fields.

1.1.1 The definition

Let us now abstract away the previous discussion:

Definition 1.1. *A* ***category*** $\mathcal{C}$ *is a tuple consisting of:*

- *A class*[1] $\mathrm{Ob}(\mathcal{C})$ *whose elements are called* ***objects****.*
- *For each pair of objects* $x, y \in \mathrm{Ob}(\mathcal{C})$*, there is a class* $\mathrm{Hom}_{\mathcal{C}}(x, y)$ *whose elements are called the* ***morphisms*** *from* x *to* y*.*
- *For each object* $x \in \mathrm{Ob}(\mathcal{C})$*, there is a unique element* $\mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x, x)$ *called the* ***identity*** *at* x*.*
- *For each triple of objects* $x, y, z \in \mathrm{Ob}(\mathcal{C})$*, there is a* ***composition*** *map*

$$\circ : \mathrm{Hom}_{\mathcal{C}}(x, y) \times \mathrm{Hom}_{\mathcal{C}}(y, z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(x, z).$$

Given a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ *we often write it as* $f : x \to y$*. We also say that* $x = \mathfrak{s}(f)$ *is the* ***source*** *of* f *and* $y = \mathfrak{t}(f)$ *is its* ***target****. Given another morphism* g*, with* $\mathfrak{s}(g) = \mathfrak{t}(f)$*, we say that* g *is* ***composable*** *with* f *and we write* $g \circ f := \circ(f, g)$*. We require that:*

- *The composition* $\circ$ *is associative. That is:* $h \circ (g \circ f) = (h \circ g) \circ f$*. This allows us to drop the parentheses.*
- *The identities behave like identities for the composition. That is:* $f \circ \mathrm{id}_x = f$ *and* $id_x \circ h = h$ *for any* f *and* h *morphisms with* $\mathfrak{s}(f) = \mathfrak{t}(h) = x$*.*

The morphisms are often also called *arrows*; see Figure 1.1. For notational simplicity we often write $x \in \mathcal{C}$ to mean an object in $\mathcal{C}$, instead of writing $x \in \mathrm{Ob}(\mathcal{C})$.

Example 1.2: The first example of a category is Set. Its class of objects Ob(Set) consists of all sets. The class $\mathrm{Hom}_{\mathrm{Set}}(x, y)$ consists of all functions from the set x to the set y. Do note that, in this case, $\mathrm{Hom}_{\mathrm{Set}}(x, y)$ is in fact a set (namely, y^x). The morphism $\mathrm{id}_x : x \to x$ is the usual identity from x to itself. The composition is the usual composition of functions. △

Example 1.3: Another important category is Grp. Its class of objects Ob(Grp) contains all groups. The set $\mathrm{Hom}_{\mathrm{Grp}}(x, y)$ consists of all group homomorphisms from x to y. The composition and the identity are inherited from Set, so they satisfy the required axioms. △

[1] You may recall that writing something like $\{x \mid A(x)\}$ ("the set of all sets satisfying property A") is not allowed when we formalise the theory of sets using Zermelo-Fraenkel. The reason is that we would then be able to write $\{x \mid x \notin x\}$ ("the set of all sets that do not contain themselves"), i.e. Russell's paradox.
However, we want to be able to discuss the "category of sets", whose objects should be all possible sets. The way to deal with this is to say: The collection of all sets that ZF can produce is what we call a "class" (which you can think of as being like a set, but larger!). This class is not a set so it is not an element of itself, avoiding paradoxes. If you find this interesting, you can read more here and here, but for the purposes of this course you can safely ignore this issue.

Example 1.4: A small variation of the previous example is the category of abelian groups Ab. Its objects Ob(Ab) are the abelian groups and the morphisms $\mathrm{Hom}_{\mathrm{Ab}}(x, y)$ are the group homomorphisms from x to y. △

Example 1.5: Given a field F we can consider the category Vect_F of F-vector spaces. Its objects $\mathrm{Ob}(\mathrm{Vect}_F)$ are F-vector spaces and the morphisms $\mathrm{Hom}_{\mathrm{Vect}_F}(x, y)$ are the F-linear maps from x to y. △

Example 1.6: Lastly, we have the category Top. Its objects Ob(Top) are topological spaces and the morphisms $\mathrm{Hom}_{\mathrm{Top}}(x, y)$ are the continuous maps from x to y. △

Important warning! In all these examples we think of an element $x \in \mathrm{Ob}(\mathcal{C})$ as a set with additional structure. However, do note that this is not part of the definition of a category! In particular, for a general category, it makes no sense[a] to talk about $a \in x$, for a given element $x \in \mathrm{Ob}(\mathcal{C})$. Since talking about the "elements of an object" may not make sense, it may not make sense either to "evaluate" a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ on an element of x.

In fact, if you are arguing about a category and you have to take elements of some $x \in \mathrm{Ob}(\mathcal{C})$ (or, more generally, perform an operation that uses the precise nature of x and not just the fact that it is an object in a category), then you are not arguing categorically. Category theory is about reasoning with categories abstractly, and not about "looking what is inside the objects".

[a] Of course, everything in Zermelo-Fraenkel is formalised through the language of sets, but if I gave you a number you would not start taking elements from it.

Example 1.7: To illuminate the previous warning further, here is a familiar example. Suppose G is a group. We can then form a category $\mathcal{C}$ by setting $\mathrm{Ob}(\mathcal{C}) := \{p\}$ and $\mathrm{Hom}_{\mathcal{C}}(p, p) := G$. That is, $\mathcal{C}$ has a single element whose morphisms are the group G. Group elements are thus morphisms and their product is interpreted to be the composition of morphisms. Since G has an identity and the product is associative, the axioms of a category are verified.

Observe that, as in the warning, we are not thinking of $p \in \mathrm{Ob}(\mathcal{C})$ as a set. Similarly, we are calling the elements of $G = \mathrm{Hom}_{\mathcal{C}}(p, p)$ "morphisms", but it does not make sense to "evaluate them" on anything. △

1.1.2 Isomorphisms

We will introduce new categorical concepts as the course goes along, hopefully allowing you to relate the topological ideas of the course to other mathematical concepts you may have seen already.

As a first example of "reasoning categorically", consider the following definition:

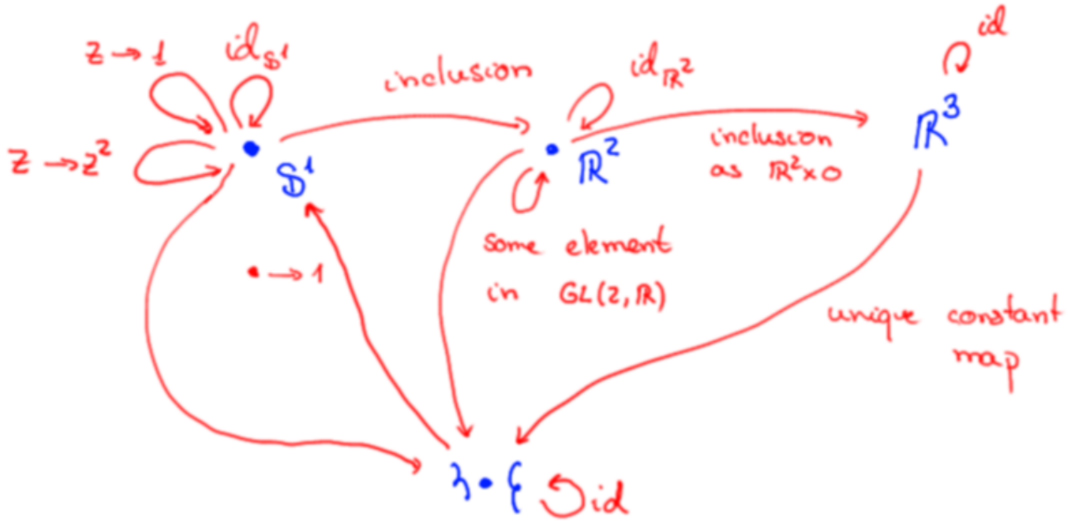


Figure 1.1: A crude depiction of the category Top. As is customary, we draw the objects as points. The Figure shows four objects, representing the spaces $\mathbb{S}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$ and the one-point set $\{.\}$. Each morphism is drawn as an arrow starting at the source and finishing at the target. For instance, every topological space maps to $\{.\}$ via the unique constant map. The category contains many more objects and morphisms (but we do not draw them due to lack of space!)

Definition 1.8. *Let $\mathcal{C}$ be a category. Let A, B be objects in $\mathcal{C}$ and $f : A \to B$ and $g : B \to A$ be morphisms. We say that:*

- *g is a **left-inverse** of f if $g \circ f = \mathrm{id}_A$.*
- *g is a **right-inverse** of f if $f \circ g = \mathrm{id}_B$.*
- *g is the **inverse** of f if it is both a right and a left inverse.*
- *f is an **isomorphism** if it has an inverse.*

This definition does not use the precise nature of the objects A and B, but nonetheless generalises more familiar notions:

Lemma 1.9. *Isomorphisms are:*

- *Bijections, in the category* Set*.*
- *Group isomorphisms, in* Grp *and* Ab*.*
- *Homeomorphisms, in* Top*.*

Furthermore, in a group, seen as a category, all elements are isomorphisms.

The proof is a reality-check and left to the reader. Furthermore, we can prove that the inverse is unique:

Lemma 1.10. *Let $f : A \to B$ and $g, h : B \to A$ be morphisms. Suppose that g is a left-inverse of f. Suppose that h is a right-inverse of f. Then $g = h$.*

Proof. We compute $g = g \circ \mathrm{id}_B = g \circ (f \circ h) = (g \circ f) \circ h = \mathrm{id}_A \circ h = h$. □

You have probably seen already the uniqueness of inverses in Set theory (where we talk about functions and their inverses) and Group theory (where we talk about the uniqueness of the inverse of a group element). Those two statements (and this one!) have the exact same proof.

Example 1.11: In Set, a morphism $f : A \to B$ is injective if and only if it has a left-inverse (which is itself surjective). Dually, f is surjective if and only if it has a right-inverse. △

1.1.3 Functors

Now that we have defined what categories are, we can ask ourselves: *"What is a map between categories?"*. A category $\mathcal{C}$ consists of the class of objects $\mathrm{Ob}(\mathcal{C})$ and the classes of morphisms $\mathrm{Hom}_{\mathcal{C}}(x, y)$, so a map between categories should take objects to objects and morphisms to morphisms, in a manner that is coherent (i.e. it should respect the composition and identity rules).

Definition 1.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A (covariant)* ***functor*** *$F : \mathcal{C} \to \mathcal{D}$ consists of:*

- *A function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, which we still denote by F.*
- *For each pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$, a function $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$.*

Given $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ we denote its image by $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$. The following properties must hold:

- *Given $x \in \mathrm{Ob}(\mathcal{C})$, it holds that $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.*
- *Given objects $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f : x \to y$ and $g : y \to z$, it holds that $F(g \circ f) = F(g) \circ F(f)$.*

Let us look at some examples:

Example 1.13: Suppose that G and H are groups. As in Example 1.7, we can regard G and H as categories with a single object. Then, there is a bijection between group homomorphisms $G \to H$ and functors from G to H, seen as categories. △

Example 1.14: Consider the category Top of topological spaces. We can define a functor $F : \mathrm{Top} \to \mathrm{Set}$ into the category of sets as follows. Every topological space A is in particular a set, so we can let $F(A) \in \mathrm{Ob}(\mathrm{Set})$ be the set underlying A. Similarly, we can forget that a map between topological spaces $f : A \to B$ is continuous and simply see it as a function $F(A) \to F(B)$. We denote this function by $F(f)$. It is clear that the identity and composition properties are preserved, so F is indeed a functor. We often say that it is a **forgetful functor**, because we are forgetting the topological structure. △

Example 1.15: The previous reasoning works in any category whose objects are sets with extra structure and whose morphisms are functions respecting that structure. I.e. there are also forgetful functors from Grp and Ab into Set.

Similarly, there is a forgetful functor $\mathrm{Ab} \to \mathrm{Grp}$; this amounts to forgetting the fact that an abelian group is abelian. You can think of this as "the inclusion of abelian groups into all groups". △

A crucial property of functors, that we will invoke very often, is the following:

Lemma 1.16. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.*

- *Let $f : a \to b$ and $g : b \to a$ be morphisms in $\mathcal{C}$. Suppose that $g \circ f = \mathrm{id}_a$. Then*

$$F(g) \circ F(f) = \mathrm{id}_{F(a)}\,.$$

- *Suppose f is an isomorphism. Then $F(f)$ is an isomorphism.*

Proof. The first property is immediate from the fact that F preserves the composition and the identities:

$$F(g) \circ F(f) = F(g \circ f) = F(\mathrm{id}_a) = \mathrm{id}_{F(a)}\,.$$

This implies that left and right inverses are preserved by F. This implies the second item. □

1.2 Homotopies of maps

In the introduction we said that our goal is to distinguish topological spaces by detecting that some have "holes". The intuition was that some holes (e.g. the one in the annulus) can be detected by looking at maps $\gamma : \mathbb{S}^1 \to X$ and proving that some of them cannot be continuously deformed to be constant. We interpret this as saying that γ is somehow wrapped around a hole in a manner that cannot be untied.

Let us formalise first the meaning of "continuously deforming" a map:

Definition 1.17. *Two maps $f, g : A \to B$ are* ***homotopic*** *to each other if there is a* ***homotopy*** *between them. That is, a map $F : A \times [0,1] \to B$ such that $f(a) = F(a, 0)$ and $g(a) = F(a, 1)$.*

It is common to think of F as a family/movie of maps $f_t : A \to B$ parametrised by the interval $[0, 1]$. These functions are given by the formula $f_t(a) := F(a, t)$.

Do note that a homotopy F is more that just a family of (continuous) functions $(f_t = F(-, t))_{t\in[0,1]}$. It is crucial that F is also continuous in the t-variable. This continuity in t is precisely what we mean by "continuously" deforming a map. If you imagine f_t as a "movie" of maps, continuity in t means that there are no sudden "jumps" as we watch the movie.

1.2.1 Being homotopic is an equivalence relation

Our first result about homotopies reads:

Proposition 1.18. *Being homotopic is an equivalence relation.*

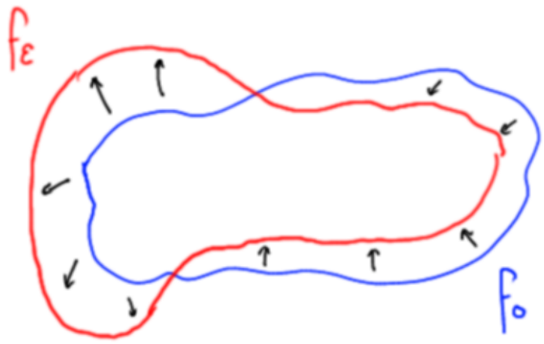


Figure 1.2: A homotopy of loops $F : \mathbb{S}^1 \times [0,1] \to \mathbb{R}^2$. As we move in the second variable (i.e. the interval), the loops change in a continuous way. This is indicated in the figure using arrows.

The rest of this Subsection is dedicated to proving this statement. We do so one axiom at a time, introducing some important definitions along the way.

Definition 1.19. *Let $f : A \to B$ be a map. The* ***constant*** *homotopy $F : A \times [0,1] \to B$ associated to f is $F(a,t) := f(a)$.*

We will now prove that the constant homotopy is indeed a homotopy and therefore it will follow that:

Lemma 1.20. *Being homotopic is a reflexive relation.*

Proof. Let $f : A \to B$ be a map and let $F : A \times [0,1] \to B$ be $F(a,t) = f(a)$. We claim that F is continuous because it is a composition of continuous functions. Indeed, the projection to the first factor $\pi_A : A \times [0,1] \to A$ is continuous and so is f by assumption. We conclude by writing $F = f \circ \pi_A$.

Since $F(a,0) = F(a,1) = f(a)$ we see that F is a homotopy from f to itself, proving reflexivity. □

Similarly, we introduce:

Definition 1.21. *Let $F : A \times [0,1] \to B$ be a homotopy. The* ***reverse*** *homotopy $\bar{F} : A \times [0,1] \to B$ is $\bar{F}(a,t) := F(a, 1-t)$.*

Which is a generalisation of the case of paths. If f is homotopic to g by F, then g is homotopic to f using the reverse, proving:

Lemma 1.22. *Being homotopic is a symmetric relation.*

Proof. Let $F : A \times [0,1] \to B$ be a homotopy from f to g. We claim that its reverse $\bar{F}$ is a homotopy from g to f. We readily verify $\bar{F}(a,0) = F(a,1) = g(a)$ and $\bar{F}(a,1) = F(a,0) = f(a)$. It remains to prove that $\bar{F}$ is continuous. We write $\bar{F} = F \circ (\mathrm{id}_A, \rho)$ with $\rho : [0,1] \to [0,1]$ the map $t \mapsto 1-t$ that reverses the interval, which is continuous. The product (id_A, ρ) of continuous maps is continuous, and thus $\bar{F}$ is a composition of continuous maps. □

Lastly, we introduce, generalising the concatenation of paths:

Definition 1.23. *A homotopy $F : A \times [0,1] \to B$ is **concatenable** with a homotopy $G : A \times [0,1] \to B$ if $F(a,1) = G(a,0)$.*

*Suppose F is concatenable with G. Then, the **concatenation** of F with G is defined as:*

$$G \bullet F(a,t) \quad = \quad \begin{cases} F(a,2t) & \text{for } t \in [0,1/2] \\ G(a,2t-1) & \text{for } t \in [1/2,1] \end{cases}$$

That is, if we think of F and G as movies of maps $A \to B$, the concatenation $G \bullet F$ amounts to watching F first and then G. Being concatenable means that the last frame $F(a,1)$ of F is the same as the first frame $G(a,0)$ of G, so the two glue seamlessly.

Do note that concatenation *is not commutative*. $G \bullet F$ means F first and then G. Often, the symmetric expression $F \bullet G$ will not even make sense, because F being concatenable with G does not imply that G is concatenable with F.

We are now ready to prove transitivity and thus conclude the proof of Proposition 1.18.

Lemma 1.24. *Being homotopic is a transitive relation.*

Proof. Suppose $f, g, h : A \to B$ are maps such that F is a homotopy from f to g and G is a homotopy from g to h. Then, the concatenation $G \bullet F$ is a homotopy from f to h, proving transitivity.

By construction we have $G \bullet F(a,0) = F(a,0) = f$ and $G \bullet F(a,1) = G(a,1) = h$, so it only remains to prove continuity. This follows from the pasting Lemma 1.25. We check its hypotheses: Over $A \times [0,1/2]$, the concatenation $G \bullet F$ is given by $F(a,2t)$, which is continuous (it is the composition of F and $(\mathrm{id}_A, t \mapsto 2t)$). Over $A \times [1/2,1]$, it is given by $G(a,2t-1)$, which is continuous for the same reasons. These two definitions agree in the overlap $A \times \{1/2\}$, since either definition yields $G \bullet F(a,1/2) = g(a)$. The pasting lemma applies, proving the claim. □

The **Pasting Lemma** is an incredibly useful tool to show that maps are continuous. We will use it a lot in this course and you should be familiar with it:

Lemma 1.25. *Let A be a topological space presented as a finite union $A = \cup_i B_i$ of (not necessarily disjoint) subspaces, all of them closed or all of them open. Then, a function $f : A \to C$ is continuous if and only if $f|_{B_i}$ is continuous for all i.*

1.2.2 Homotopy classes

Using Proposition 1.18 we can then define:

Definition 1.26. *Let A and B be spaces. Given a map $f : A \to B$, we denote by $[f]$ its equivalence class according to the homotopy equivalence relation. We say that $[f]$ is a* ***homotopy class*** *and that f is a representative of the class.*

The set of all homotopy classes of maps is denoted by

$$[A, B] := \{[f] \mid f : A \to B \text{ map } \}.$$

Recall that $\mathrm{Hom}_{\mathrm{Top}}(A, B)$ denotes the set of all maps from A to B, so $[A, B]$ is its quotient by the homotopy equivalence relation.

As a first example, we now generalise the introductory Lemma 0.6 from the convex setting to the setting of star-shaped subspaces. First we recall the definition:

Definition 1.27. *A subspace $B \subset \mathbb{R}^n$ is* ***star-shaped*** *if there is a point $c \in B$ such that, for every $b \in B$, the straight segment $[b, c]$ is contained in B.*

See Figure 1.3. If $c \in B$ is as in the definition, we will say that c is a **central point** for B. Do note that such a c is not unique in general.

Lemma 1.28. *Suppose that $B \subset \mathbb{R}^n$ is star-shaped. Let A be a topological space. Then:*

- *Any map $f : A \to B$ is homotopic to a constant map.*
- *Any two maps $f, g : A \to B$ are homotopic to each other.*
- *$[A, B]$ consists of a single element.*

Proof. We will prove that any map $f : A \to B$ is homotopic to the constant map with value $p \in B$, where p is a central point in B. This implies the three statements, the second one following from the transitivity of the homotopy relation. To prove the claim, consider the homotopy $F(a, t) := f(a)(1-t)+tp$. The continuity of F follows by writing it as $F = G \circ H$, with $H : A \times [0, 1] \to B \times \mathbb{R}$ given by $(a, t) \mapsto (f(a), t)$, whose entries are continuous, and $G : B \times \mathbb{R} \to B$ given by $(x, t) \mapsto x(1 - t) + tp$, which is a polynomial map. We also verify $F(a, 0) = f(a)$ and $F(a, 1) = p$. □

Remark 1.29: Understanding $[A, B]$ (for whatever given topological spaces A and B) is one of the main goals of Algebraic Topology (and of this course in particular). An important caveat is that this turns out to be extremely difficult. For instance, you can consider the case in which A and B are (higher-dimensional) spheres. Arguably, these are some of the "easier" topological spaces one may encounter. However, computing $[\mathbb{S}^k, \mathbb{S}^l]$ is, in general, an open problem! (But there has been a lot of research about it and we can compute it in many cases). △

1.3 Exercises

1.3.1 Recap of Inleiding Topologie

Exercise 1.1: Prove that the following spaces are homeomorphic:

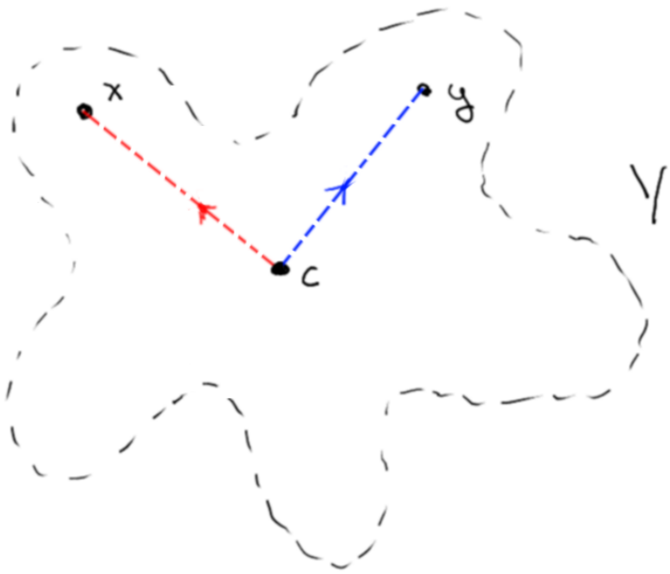


Figure 1.3: A star-shaped subset Y of $\mathbb{R}^2$, with c serving as a central point. If x and y are points in Y, the straight segment between them does not lie in Y necessarily. However, the segments $[x, c]$ and $[y, c]$ do.

- $\mathbb{S}^n$.
- $\mathbb{D}^n/\mathbb{S}^{n-1}$.
- $[-1, 1]^n/(\partial[-1, 1]^n)$. Here ∂ denotes the boundary, i.e. those points that are not in the interior.
- $(\mathbb{R}^{n+1} \setminus \{0\})/(x \cong \lambda x \text{ for } \lambda > 0)$

Exercise 1.2: Consider the following topological spaces:

1. $\mathbb{R}^n$ with its usual Euclidean topology.
2. The sphere $\mathbb{S}^n$ with the subset topology.
3. Projective space $\mathbb{RP}^n := \{x \in \mathbb{S}^n\}/\{x \cong -x\}$ with the quotient topology.
4. The open hypercube $(0, 1)^n \subset \mathbb{R}^n$ with the subset topology.
5. The hypercube $[0, 1]^n \subset \mathbb{R}^n$ with the subset topology.
6. The closed unit ball $\mathbb{D}^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ with the subset topology.
7. The closed upper half-space $\mathbb{H}^n := \{x \in \mathbb{R}^n \mid x_1 \geq 0\}$ with the subset topology.
8. A set A with the discrete topology (i.e. every subset is open).
9. The torus $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$ with the product topology.
10. The union of two circles at a point $\{x \in \mathbb{R}^2 \mid |x-(1, 0)| = 1\} \cup \{x \in \mathbb{R}^2 \mid |x+(1, 0)| = 1\}$.
11. The line with two origins: Set $A, B = \mathbb{R}$. Then consider $(A \coprod B)/\{A \ni x \cong x \neq 0 \in B\}$ with the quotient topology induced from the disjoint union of A with B.
12. The hawaiian earring $\bigcup_{n\in\mathbb{Z}^+} \mathbb{S}^1_{1/n}((1/n, 0))$ with the topology induced from $\mathbb{R}^2$.

Check whether they are connected, locally connected, Haussdorff, compact, and locally compact.

1.3.2 Homotopy classes

Exercise 1.3: Let X be a space. Let $f : X \to \mathbb{S}^n$ be a map such that $f(X) \neq \mathbb{S}^n$, i.e. the image of f misses at least one point in the sphere. Prove that f is homotopic to a constant map. **Hint:** Use the stereographic projection.

1.3.3 Categories

Exercise 1.4: Let $\mathcal{C}$ be a category. Prove that being isomorphic is a equivalence relation in $\mathrm{Ob}(\mathcal{C})$.

Exercise 1.5: Let X be a topological space. We define a category $\mathcal{C}$ (often denoted by $\mathcal{O}(X)$) as follows. The objects are the open subsets of X. If U, V are opens in X with $U \subset V$, we define $\mathrm{Hom}_{\mathcal{C}}(U, V) := \{\iota\}$, where $\iota : U \to V$ is the inclusion. Otherwise we set $\mathrm{Hom}_{\mathcal{C}}(U, V) := \emptyset$.

- Verify that $\mathcal{C}$ satisfies the axioms of a category (along the way, you should explicitly say what the identities and the composition are).
- Prove that $U, V \in \mathrm{Ob}(\mathcal{C})$ are isomorphic if and only if $U = V$.

Exercise 1.6: Let G be a group. We define a category $\mathcal{C}$ as follows. The objects are the subgroups of G. If H and I are subgroups with $I \subset H$, we define $\mathrm{Hom}_{\mathcal{C}}(I, H) := \{\iota\}$, where $\iota : I \to H$ is the inclusion. Otherwise we set $\mathrm{Hom}_{\mathcal{C}}(I, H) := \emptyset$.

- Verify that $\mathcal{C}$ satisfies the axioms of a category.
- Prove that $I, H \in \mathrm{Ob}(\mathcal{C})$ are isomorphic if and only if $I = H$.

Exercise 1.7: A **poset** is a pair $(A, \leq)$ consisting of a set A and a partial order. We can define a category $\mathcal{C}$ as follows: $\mathrm{Ob}(\mathcal{C}) := A$ and $\mathrm{Hom}_{\mathcal{C}}(a, b) = \{.\}$ if $a \leq b$, otherwise we set $\mathrm{Hom}_{\mathcal{C}}(a, b) := \emptyset$. Then:

- Verify that $\mathcal{C}$ is indeed a category.
- Prove that the previous two examples are posets.

The homotopy category

Lecture 2

Our goals in this Chapter are as follows:

- To compute homotopy classes of maps in some simple examples (Sections 2.1 and 2.3.3).
- To define what it means for two spaces to be equivalent "up to homotopy" (Definition 2.15).
- To define a category in which the morphisms are homotopy classes of maps (Definition 2.11).
- To discuss the idea of comparing two objects in a category by looking at how they interact with other objects (Section 2.3).

2.1 Examples of homotopies

At the end of the previous lecture we noted that computing $[A, B]$, the set of homotopy classes of maps $A \to B$, is generally very difficult. Nonetheless, here are some examples.

2.1.1 Nullhomotopic maps

Since constant maps will keep appearing in our discussion, it is convenient to introduce the notation:

Definition 2.1. *Let A, B be topological spaces. Fix an element $b \in B$. We write $c_b : A \to B$ for the constant map $c_b(a) := b$.*

A map $f : A \to B$ is ***nullhomotopic*** *if it is homotopic to a constant map.*

Do observe that two constant maps need not be homotopic to each other:

Example 2.2: Let $A = \{p\}$ and $B = \{a, b\}$, both endowed with the discrete topology. There are two maps $A \to B$, namely c_a and c_b. Both of them are constant and thus nullhomotopic.

However, they are not homotopic to one another. It follows that $[A, B] = \{[c_a], [c_b]\}$ has also two elements. △

As a follow-up of Lemma 1.28 we can show:

Lemma 2.3. *Suppose $B \subset \mathbb{R}^n$ is star-shaped. Let A be a topological space. Then any map $f : B \to A$ is nullhomotopic.*

Proof. Let $p \in B$ be a central point. Given a map $f : B \to A$, we consider the homotopy $F(a, t) := f(a(1-t) + tp)$. We write F as $F = f \circ H$, with $H : B \times [0, 1] \to B$ given by $(a, t) \mapsto a(1-t) + tp$ polynomial, so F is continuous. We also verify $F(a, 0) = f(a)$ and $F(a, 1) = f(p)$. □

2.1.2 Path-components

It makes sense to focus on the "simplest non-trivial space", the point. We can now look at morphisms from and out of it. First, note that maps $A \to \{p\}$ are not very interesting. Indeed, $\mathrm{Hom}_{\mathrm{Top}}(A, \{p\})$ contains a single map; namely, the one that sends all of A to the unique point p. The same is true for $[A, \{p\}]$.

The case $\{p\} \to A$ is more interesting but can nonetheless be fully understood. First note:

Lemma 2.4. *Let A be a topological space. Then, there is a bijection:*

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Hom}_{\mathrm{Top}}(\{p\}, A) \\ a & \mapsto & c_a. \end{array}$$

Do note that this does not use the structure of A as a topological space, just as a set.

What about homotopies then? Given points $a, b \in A$, a homotopy between c_a and c_b is a map $F : \{p\} \times [0, 1] \to A$ that satisfies $F(p, 0) = a$ and $F(p, 1) = b$. We can use the homeomorphism $[0, 1] \to \{p\} \times [0, 1]$ given by $t \mapsto (p, t)$ to deduce that:

Lemma 2.5. *A homotopy between c_a and c_b is equivalent to a path $\gamma : [0, 1] \to A$ starting at $a = \gamma(0)$ and finishing at $b = \gamma(1)$.*

Moreover, we can particularise the concatenation/reversal of homotopies in order to recover the analogous notions for paths (that you may have seen in your courses on multivariate analysis).

Definition 2.6. *Let A be a space. We introduce the following equivalence relation: two points $a, b \in A$ are related $a \sim b$ if there is a path from a to b.*

- *Each equivalence class is said to be a **path-component**.*
- *The set of path-components of A is denoted by $\pi_0(A)$.*
- *If A has a single path-component, we say that A is **path-connected**.*

Which we can relate to homotopy classes of maps from $\{p\}$:

Lemma 2.7. *Let A be a topological space. Then, there is a bijection:*

$$\begin{array}{rcl} \pi_0(A) & \longrightarrow & [\{p\}, A] \\ [a] & \mapsto & [c_a]. \end{array}$$

Proof. According to 2.4 there is a bijection $A \simeq \mathrm{Hom}_{\mathrm{Top}}(\{p\}, A)$. On the right we are relating maps if they are homotopic. On the left we are relating points if they can be connected by a path. These two relations correspond to one another according to Lemma 2.5, proving the claim.

In particular, note that being homotopic is an equivalence relation (Proposition 1.18). From this, it follows that the relation given in Definition 2.6 is indeed a equivalence relation (which is something we had to verify!). □

In particular:

Corollary 2.8. *Suppose $A \subset \mathbb{R}^n$ is star-shaped. Then $\pi_0(A) \cong \{.\}$.*

Finally, we provide a couple of useful lemmas:

Lemma 2.9. *A path-connected space X is connected.*

Proof. Consider a partition $X = Y \coprod Z$, with Y and Z both open. If both are non-empty, we can find points $y \in Y$ and $z \in Z$. We can then use path-connectedness to find a path $\gamma : [0,1] \to X$ starting at y and finishing at z. The subsets $\gamma^{-1}(Y)$ and $\gamma^{-1}(Z)$ partition $[0,1]$ and are open (by continuity of γ). This is a contradiction with the fact that $[0,1]$ is connected. It follows that Y or Z had to be empty, implying that X is itself connected. □

The following statement is quite useful for spaces presented piece by piece (which is something we do often in this course). See Figure 2.1 for an illustration.

Lemma 2.10. *Let $X = \cup U_i$ be a topological space presented as a finite union of subspaces. Suppose that:*

- *All U_i are path-connected.*
- *For every pair of indices i and j, there is a sequence of indices*

$$i = i_0, i_1, i_2, \cdots, i_n = j$$

  verifying $U_{i_k} \cap U_{i_{k+1}} \neq \emptyset$ for all $k = 0, \cdots, n-1$.

Then, X is path-connected.

Proof. Given any two points $x, y \in X$, we want to exhibit a path between them. We have that x is contained in some U_i and y in some U_j. Let $i_0, i_1, i_2, \cdots, i_n$ be a sequence from i to j given by the second hypothesis. We can fix points $x_{i_k} \in U_{i_k} \cap U_{i_{k+1}}$. The first hypothesis then says that there is a path from x to x_{i_0} (because U_{i_0} is path-connected), a path between x_{i_k} and $x_{i_{k+1}}$ (because $U_{i_{k+1}}$ is path-connected) for each k, and a path from x_{i_n} to y. All these paths can be concatenated to produce the claimed path from x to y. □

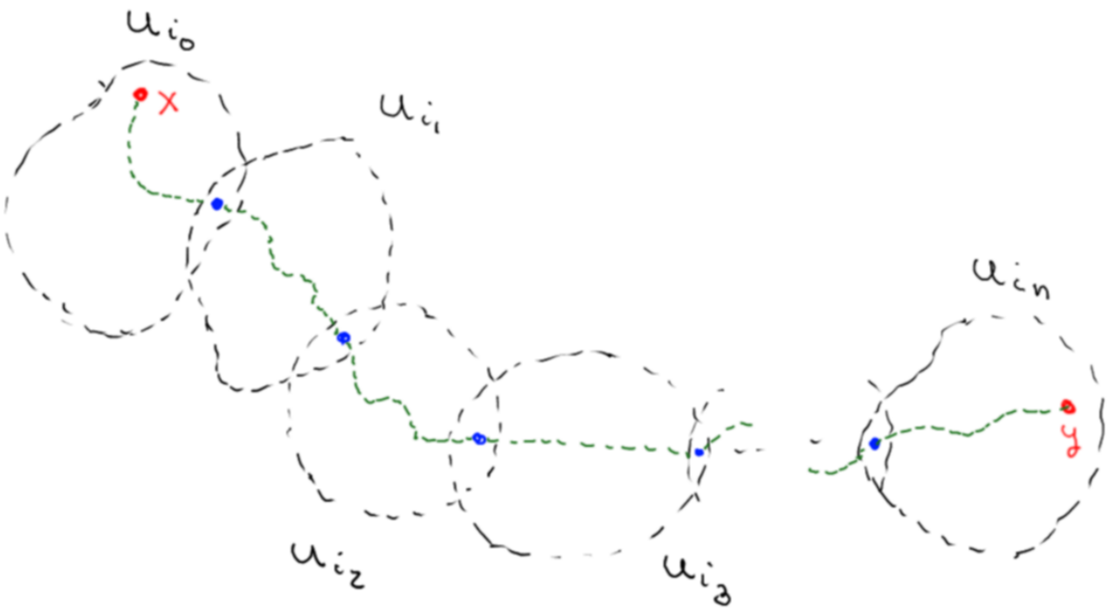


Figure 2.1: A space X satisfying the hypothesis of Lemma 2.10. In order to go from one point x to some other point y, we concatenate paths passing through intermediate points lying in the overlaps of the pieces.

2.2 The (naive) homotopy category

One of the motto's of this course reads:

We only care about maps up to homotopy.

That is: we are interested in detecting the "holes" that a space B may have and, to do so, we look at maps $A \to B$. However, the concrete map is not important, only the fact that it gets stuck somewhere (for instance, whether it is nullhomotopic or not) matters. I.e. all the information that we need is encoded in the sets $[A, B]$.

This philosophy we can implement as follows:

Definition 2.11. *The (naive)* ***homotopy category*** *of topological spaces, which we denote* hTop*:*

- *Has the same objects as* Top*. I.e.* $\mathrm{Ob}(\mathrm{hTop}) := \mathrm{Ob}(\mathrm{Top})$.
- *Its morphisms are homotopy classes of maps. I.e. for any two spaces A and B, we set* $\mathrm{Hom}_{\mathrm{hTop}}(A, B) := [A, B]$.
- *The identity at $A \in \mathrm{Ob}(\mathrm{hTop})$ is the homotopy class* $[\mathrm{id}_A]$ *of the identity map.*
- *The composition of morphisms is defined by composing representatives. I.e.*

$$\begin{aligned} \circ : \mathrm{Hom}_{\mathrm{hTop}}(A, B) \times \mathrm{Hom}_{\mathrm{hTop}}(B, C) &\longrightarrow \mathrm{Hom}_{\mathrm{Top}}(A, C) \\ ([f], [g]) &\mapsto [g] \circ [f] := [g \circ f]. \end{aligned}$$

We need to verify that hTop is indeed a category. First:

Lemma 2.12. *The composition is well-defined.*

Proof. We must check that the composition does not depend on the choice of representatives. That is, suppose that there are maps $f, f' : A \to B$ and $g, g' : B \to C$ such that $[f'] = [f]$ and $[g'] = [g]$. This means that there is a homotopy F between f and f' and a homotopy G from g to g'. We claim that $H(a,t) := G(F(a,t),t)$ is a homotopy between $g \circ f$ and $g' \circ f'$, proving that $[g \circ f] = [g' \circ f']$, as desired.

First note that $H(a,0) = G(F(a,0),0) = g \circ f(a)$. Similarly $H(a,1) = G(F(a,1),1) = g' \circ f'(a)$. Lastly, H is a composition of continuous functions and thus continuous. □

Second:

Lemma 2.13. *The composition is associative.*

Proof. Associativity for the composition of homotopy classes follows from associativity at the level of representatives:

$$([g] \circ [f]) \circ [h] = [g \circ f] \circ [h] = [g \circ f \circ h] = [g] \circ [f \circ h] = [g] \circ ([f] \circ [h]).$$

□

Lastly:

Lemma 2.14. *Let A be a space. Then $[\mathrm{id}_A]$ is the identity, in* hTop*, at the object A.*

Proof. Given maps $f : A \to B$ and $g : B \to A$ we have:

$$[f] \circ [\mathrm{id}_A] = [f \circ \mathrm{id}_A] = [f], \qquad [\mathrm{id}_A] \circ [g] = [\mathrm{id}_A \circ g] = [g].$$

□

We conclude that indeed hTop is a well-defined category.

Warning! Previously we remarked that, in a category, it may not make sense to "evaluate" morphisms. This is precisely the case in hTop. A morphism is an equivalence class of functions so writing $[f](a)$, with $[f] \in \mathrm{Hom}_{\mathrm{hTop}}(A,B)$ and $a \in A$, makes no sense.

2.2.1 Homotopy equivalences

In Definition 1.8 we defined what an isomorphism is in an arbitrary category. How does this particularise to hTop?

If $[f] \in [A,B] = \mathrm{Hom}_{\mathrm{hTop}}(A,B)$ is an isomorphism, this means that there exists some other class $[g] \in [B,A]$ such that $[g \circ f] = [\mathrm{id}_A]$ and $[f \circ g] = [\mathrm{id}_B]$. We give this a name:

Definition 2.15. *A map $f : A \to B$ is a* ***homotopy equivalence*** *if there is a map $g : B \to A$ such that*

- *$g \circ f$ is homotopic to* id_A.
- *$f \circ g$ is homotopic to* id_B.

*Then, we say that f and g are **homotopy inverses** and that A and B are **homotopy equivalent**.*

That is, f is a homotopy equivalence if and only if $[f]$ is an isomorphism in hTop. Loosely speaking, you may want to think of homotopy equivalent spaces as spaces that have "the same holes". We will formalise this fully in Corollary 2.35.

We now prove a couple of properties. Observe first that homotopy inverses need not be unique, but they are unique up to homotopy:

Lemma 2.16. *Let $f : A \to B$ be a map. Suppose that $g, h : B \to A$ are homotopy inverses of f. Then $[g] = [h]$.*

Proof. f is a homotopy equivalence iff $[f]$ is an isomorphism in hTop. Inverses are unique in any category (Lemma 1.10), so the inverse of $[f]$ (which in this case exists) is unique. It follows that $[g] = [f]^{-1} = [h]$. □

Categorical reasoning also shows:

Lemma 2.17. *Homotopy equivalence is an equivalence relation.*

Proof. Being homotopy equivalent simply means being isomorphic in hTop. The result then follows from the fact that being isomorphic (in any category) is a equivalence relation. □

Moreover:

Lemma 2.18. *A homeomorphism is a homotopy equivalence.*

Proof. This follows from the fact that a continuous inverse is in particular a homotopy inverse. More categorically, you could instead say that the quotient functor Top $\to$ hTop takes isomorphisms to isomorphisms. □

However, the converse is not true. For instance, the point and $\mathbb{R}$ are homotopy equivalent, but not homeomorphic (see Lemma 2.21 below).

Remark 2.19: Consider the functor $F : \mathrm{Top} \to \mathrm{hTop}$ that is the identity on objects (recall that both categories have the class of topological spaces as the class of objects) and that takes each map $f \in \mathrm{Hom}_{\mathrm{Top}}(A, B)$ to $[f] \in [A, B] = \mathrm{Hom}_{\mathrm{hTop}}(A, B)$. Checking that this is a functor amounts to checking that identities and compositions are taken to identities and compositions, which is precisely what we showed in the lemmas following Definition 2.11. You may want to think of this functor as a quotient map. We call it the **localisation functor**. △

2.2.2 Contractible spaces

Since the point is the "simplest non-trivial space", it seems natural to look at those spaces that are homotopy equivalent to it.

Definition 2.20. *A space A is* ***contractible*** *if it is homotopy equivalent to $\{.\}$ (the set with one point).*

From the point of view of hTop, a point and a contractible space are indistinguishable.

Concrete examples are given by the following lemma:

Lemma 2.21. *Any $X \subset \mathbb{R}^n$ star-shaped is contractible.*

Proof. Consider the unique constant map $f : X \to \{.\}$ and an inclusion $g : \{.\} \to X$. It trivially follows that $f \circ g = \mathrm{id}_{\{.\}}$ so, upon taking homotopy classes, $[f]$ is a left-inverse of $[g]$. Conversely, since $[X, X] = \{[\mathrm{id}_X]\}$ (Lemma 1.28) we deduce that $[g \circ f] = [\mathrm{id}_X]$ and thus $[f]$ is a right-inverse of $[g]$. We conclude that f and g are homotopy inverses. □

With this example in mind:

Warning! g and f being homotopy inverses does not mean that they are set-theoretical inverses. In particular, neither of them need to be injective or surjective. A concrete example is given by A a point and B a convex set with non-empty interior in $\mathbb{R}^n$, with $n > 0$. Using Lemma 2.21 we see that A and B are homotopy equivalent. However, the two are not homeomorphic (left for the reader). Furthermore, no map $A \to B$ is surjective and no map $B \to A$ is injective. More generally, you may want to recall Lemma 2.16 and note that being injective/surjective are not properties preserved under homotopies.

2.3 Pushforwards and pullbacks

Let us go back to the motivating example of the disc $\mathbb{B}^2$ and the annulus $\mathbb{A}$. Our goal is to use Lemma 1.28 (which says that $[A, \mathbb{B}^2]$ is a singleton for all A) and find some space A such that $[A, \mathbb{A}]$ has more than one element. The claim is that $A = \mathbb{S}^1$ does the job (Corollary 3.5). Moreover, note that it is natural to look for some A that is as simple as possible, since computing $[A, B]$ is very difficult for general spaces.

More generally, whenever we encounter spaces B and C that we want to compare, we can fix some other auxiliary space A and check whether $[A, B]$ and $[A, C]$ have the same size. I.e. we compare the ways one can "wrap A within B" to the ways one can "wrap A within C". This depends heavily on A (meaning that a "hole" in B may be detectable using A but not detectable if we use some other space A'). Additionally, observe that one could proceed in a dual manner! We could instead compare the sizes of $[B, A]$ and $[C, A]$.

These ideas in fact work in any category, as we now explain.

2.3.1 The pushforward

Consider the following definition:

Definition 2.22. *Let $\mathcal{C}$ be a category, w, x, y objects in $\mathcal{C}$ and $g : x \to y$ a morphism. The* ***pushforward*** *of g is the function:*

$$\begin{aligned} g_* : \mathrm{Hom}_{\mathcal{C}}(w, x) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(w, y) \\ h &\mapsto g_*(h) := g \circ h. \end{aligned}$$

Do note that g defines a pushforward for each object $w \in \mathcal{C}$, but we omit it from the notation g_*. For this reason, it is important to specify what the domain of g_* is.

We now verify a number of properties of the pushforward:

Lemma 2.23. *Let $\mathcal{C}$ be a category, x, y, z objects in $\mathcal{C}$ and $g : x \to y$ and $f : y \to z$ morphisms. Then, for any other object $w \in \mathcal{C}$, the pushforwards f_* and g_* satisfy:*

$$(f \circ g)_* = f_* \circ g_*.$$

Proof. This follows from the fact that both maps, when applied to $h : w \to x$, yield $f \circ g \circ h$. □

Since the pushforward respects compositions, we deduce:

Corollary 2.24. *Let f, g, and h be morphisms in $\mathcal{C}$. Then:*

- *If f is a left-inverse of g, then f_* is a left-inverse of g_*.*
- *If f is a right-inverse of h, then f_* is a right-inverse of h_**

Which in turn means that two isomorphic objects cannot be distinguished by other objects:

Corollary 2.25. *Suppose $f : y \to z$ is an isomorphism. Then $f_* : \mathrm{Hom}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(x, z)$ is a bijection, for all $x \in \mathcal{C}$.*

Lastly:

Lemma 2.26. *The following are equivalent:*

a. *$f : y \to z$ has a right-inverse.*
b. *$f_* : \mathrm{Hom}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(x, z)$ is surjective, for all $x \in \mathcal{C}$.*
c. *$f_* : \mathrm{Hom}_{\mathcal{C}}(z, y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(z, z)$ is surjective.*

Proof. Item (a) implies item (b) by Corollary 2.24. Item (c) is a particular case of (b). To show that (c) implies (a), observe that $\mathrm{Hom}_{\mathcal{C}}(z, z)$ contains the map id_z, and f_* being surjective means that there is a map $g \in \mathrm{Hom}_{\mathcal{C}}(z, y)$ such that $f \circ g = f_*(g) = \mathrm{id}_z$. I.e. g is a right-inverse of f. □

All these statements should be interpreted as saying that y is more complicated than z. In the setting of spaces, we will call the map f a retraction (Definition 3.6).

2.3.2 The pullback

We can now reason dually and try to understand how a category $\mathcal{C}$ sees a given object $z \in \mathrm{Ob}(\mathcal{C})$. That is, we want to consider morphisms *into* z:

Definition 2.27. *Let $\mathcal{C}$ be a category, x, y, z objects in $\mathcal{C}$ and $g : x \to y$ a morphism. The* ***pullback*** *of g is the function:*

$$\begin{aligned} g^* : \mathrm{Hom}_{\mathcal{C}}(y,z) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(x,z) \\ f &\mapsto g^*(f) := f \circ g. \end{aligned}$$

Do note, once again, that g defines a pullback for each z we consider. As such, one has to specify which concrete z we are referring to when we talk about g^*.

Now we prove the same lemmas as before, in this dual setting:

Lemma 2.28. *Let $\mathcal{C}$ be a category, w, x, y objects in $\mathcal{C}$ and $h : w \to x$ and $g : x \to y$ morphisms. Then, for any other object $z \in \mathcal{C}$, the pullbacks g^* and h^* satisfy:*

$$(g \circ h)^* = h^* \circ g^*.$$

Proof. Both sides, when applied to $f : y \to z$, yield $f \circ g \circ h$. □

The crucial observation here is that the pullback reverses the way in which the maps compose. This will show up again in all upcoming statements.

Corollary 2.29. *Let f, g, and h be morphisms in $\mathcal{C}$. Then:*

- *If f is a left-inverse of g, then f^* is a right-inverse of g^*.*
- *If f is a right-inverse of h, then f^* is a left-inverse of h^*.*

Corollary 2.30. *Suppose $f : y \to z$ is an isomorphism. Then $f^* : \mathrm{Hom}_{\mathcal{C}}(z,x) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(y,x)$ is a bijection, for all $x \in \mathcal{C}$.*

The following, which says that z is more complicated than y, is left as an exercise for you:

Lemma 2.31. *The following are equivalent:*

a. *$f : y \to z$ has a left-inverse.*
b. *$f^* : \mathrm{Hom}_{\mathcal{C}}(z,x) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(y,x)$ is surjective, for all $x \in \mathcal{C}$.*
c. *$f^* : \mathrm{Hom}_{\mathcal{C}}(z,y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(y,y)$ is surjective.*

Lastly, we can put together our analysis about pullback and pushforward to show:

Proposition 2.32. *Let $f : y \to z$ be a morphism such that f_* and f^* are bijections for all $x \in \mathcal{C}$. Then f is an isomorphism.*

Proof. According to Lemma 2.26, f_* being surjective, means that f has a right-inverse. Similarly, Lemma 2.31 says that f has a left-inverse because f^* is surjective. It follows that f is invertible by Lemma 1.10. □

Which we interpret as saying that two objects are isomorphic if and only if they are indistinguishable when seen from any other object.

2.3.3 Pushforward and pullback in hTop

We now particularise the previous discussion to the study of spaces.

First observe that, given a map $f : B \to C$ in Top, we can consider its homotopy class $[f] : B \to C$ in hTop. In turn, we can then take the pushforward $[f]_* : [A, B] \to [A, C]$. This amounts to "scanning" B and C using A and comparing the result via f. We can compose these two operations and thus define:

Definition 2.33. *Given a map $f : B \to C$ in* Top*, we write $f_* := [f]_* : [A, B] \to [A, C]$ for its pushforward at the level of homotopy classes.*

Do note that this is different from the pushforward $f_* : \mathrm{Hom}_{\mathrm{Top}}(A, B) \to \mathrm{Hom}_{\mathrm{Top}}(A, C)$ in Top, which takes a map g to $f \circ g$ (i.e. no homotopy classes here!).

Similarly:

Definition 2.34. *Given a map $f : B \to C$ in* Top*, we write $f^* := [f]^* : [C, A] \to [B, A]$ for its pullback at the level of homotopy classes.*

An extremely useful consequence of the theory we have developed (concretely of Proposition 2.32 as a particular case) is that:

Corollary 2.35. *A map $f : B \to C$ in* Top *is a homotopy equivalence if and only if:*

- *$f_* : [A, B] \to [A, C]$ is a bijection and,*
- *$f^* : [C, A] \to [B, A]$ is a bijection,*

for all $A \in$ Top*.*

Which particularises to:

Corollary 2.36. *If $B \in$* Top *is contractible then it holds, for any A:*

- *$[A, B] \cong \{.\}$ for all A.*
- *$[B, A] \cong \pi_0(A)$.*

Proof. The statement is true if B is the point. For any other contractible B, we can apply the previous corollary using the fact that B and the point are isomorphic. ☐

And with the same reasoning:

Lemma 2.37. *Let B and C be homotopy equivalent spaces. Then, $\pi_0(B) \cong \pi_0(C)$.*

2.4 Exercises

2.4.1 Nullhomotopies

Exercise 2.1: Prove that $\pi : \mathbb{S}^n \times [0, 1] \to \mathbb{D}^{n+1}$ given by $(x, t) \mapsto xt$ is a quotient map. **Hint:** You may use Lemma 10.30.

Exercise 2.2: Let X be a space and let $f : \mathbb{S}^n \to X$ be a map. Prove that the following are equivalent:

- f is nullhomotopic.
- There is a map $g : \mathbb{D}^{n+1} \to X$ such that $g|_{\mathbb{S}^n} = f$.

2.4.2 Path-connectedness

Exercise 2.3: Let X and Y be path-connected. Prove that $X \times Y$ is path-connected.

Exercise 2.4: For each of the spaces in Exercise 1.2, check whether it is path-connected.

Exercise 2.5: Let A be a path-connected topological space; fix a point $p \in A$. Show that any map $\gamma_0 : \mathbb{S}^1 \to A$ is homotopic to a map γ_1 such that $\gamma_1(1) = p$.

Exercise 2.6: Let A and B be non-empty topological spaces.

- Prove that the set of homotopy classes of constant maps $A \to B$ is isomorphic to $\pi_0(B)$ (as sets).
- Use this to define an injective map $\iota : \pi_0(B) \to [A, B]$.
- Recall that a map $f : A \to B$ can be evaluated on a point $a \in A$. Use this to define a map $\phi_a : [A, B] \to \pi_0(B)$. Prove that ϕ_a is well-defined and surjective.
- Prove that $\phi_a = \phi_{a'}$ if $[a] = [a'] \in \pi_0(A)$.
- Prove that ϕ_a is a left-inverse of ι.

2.4.3 Pullbacks

Exercise 2.7: Prove Corollaries 2.29 and 2.29, and Lemma 2.31.

2.4.4 Homotopy equivalences

Exercise 2.8: Find a topological space X that is not contractible.

Exercise 2.9: Let $L_1 \cong L_2 \cong \mathbb{R}$ be two copies of the real line. Let L be the quotient of $L_1 \coprod L_2$ modulo the equivalence relation $L_1 \ni x \cong x \in L_2$ if $x > 0$. That is, we glue L_1 to L_2 along the positive numbers.

- Verify that L is not Hausdorff.
- Verify that L is contractible.
- Observe that Hausdorffness is not preserved by homotopy equivalences.

Exercise 2.10: Let $N, S \in \mathbb{S}^2$ be the north and south poles. Show that the following spaces are homotopy equivalent to one another:

$$A := (\mathbb{S}^2 \coprod [0,1])/\{\mathbb{S}^2 \ni N \sim 0 \in [0,1]; \mathbb{S}^2 \ni S \sim 1 \in [0,1]\}$$

$$B := \mathbb{S}^2/\{N \sim S\}.$$

You should write the desired homotopy inverses $f : A \to B$ and $g : B \to A$ as explicitly as possible, as well as the two relevant homotopies.

Hint: A crucial part of this exercise is that you choose nice coordinates to work on. Some options are:

- Note that $\mathbb{S}^2$ consists of two copies of the disc $\mathbb{D}$ (the hemispheres) with their boundaries identified (the equator). Use the polar coordinates in these discs.
- Identify the complement of the north pole with $\mathbb{R}^2$, using the stereographic projection. Do the same for the complement of the south pole.
- Use spherical coordinates.

Whatever you do, be explicit in defining f, g, and the necessary homotopies.

Retracts, Hom functors, (co)products

Lecture 3

Our goals in this Chapter are as follows:

- To define the functors $[A, -]$ and $[-, A]$, that take as input a topological space and output a set (Section 3.3). The concrete case of $[\mathbb{S}^1, -]$ computes the homotopy classes of loops.
- To introduce retracts and deformation retracts (Section 3.2), which are subspaces that sit particularly nicely in their ambient space.
- To explain how taking homotopy classes relates to taking products (Section 3.4) and unions (Section 3.5) of spaces. This fits into the more general theme of checking whether a functor preserves (co)products.

3.1 Some more computations

To recap the previous lecture, let us go through some applications of the ideas we have seen.

Lemma 3.1. *The following conditions are equivalent for a space X:*

i. *X is contractible.*
ii. *$[A, X]$ consists of a single element, for all spaces A.*
iii. *$[X, X]$ consists of a single element.*
iv. *id_X is nullhomotopic.*

Proof. Item (i) implies (ii) according to Corollary 2.36. Item (iii) is a concrete instance of (ii). According to Item (iii), any two maps from X to X are homotopic; in particular, this is true for id_X and any constant map. To see that (iv) implies (i), we consider the unique map $c_p : X \to \{p\}$ and the constant map $c_x : \{p\} \to X$ with value $x \in X$. We have that $c_p \circ c_x = \mathrm{id}_{\{p\}}$. Conversely, $c_x \circ c_p : X \to X$ is the constant map with value x, which is homotopic to the identity. It follows that c_x and c_p are homotopy inverses. □

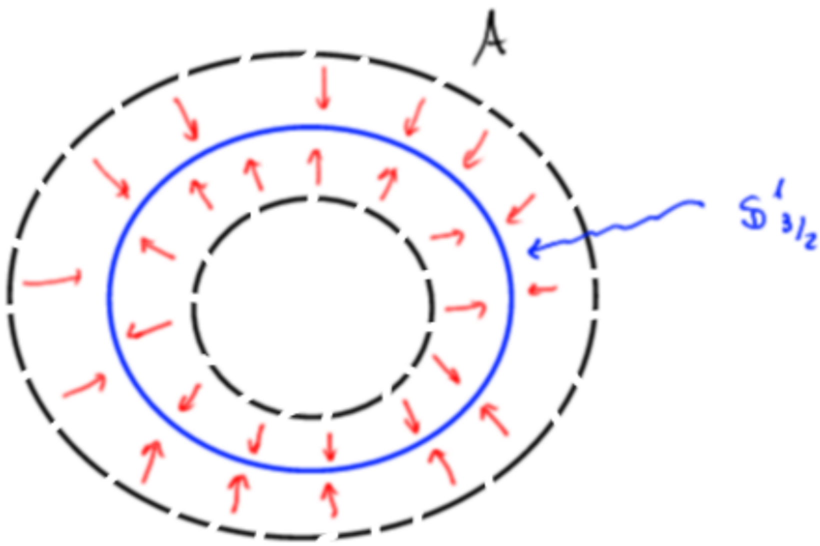


Figure 3.1: The circle of radius 3/2 is a deformation retract of the annulus $\mathbb{A}$.

Let us now move on to more complicated spaces.

The following is the first important computation to be done in this course:

Theorem 3.2. $[\mathbb{S}^1, \mathbb{S}^1] \simeq \mathbb{Z}$.

We do not have the tools to prove it yet. We will tackle it in Section 10.3. For now, it is sufficient that you have in mind the intuition that each homotopy class $[\gamma] \in [\mathbb{S}^1, \mathbb{S}^1]$ is identified, under this bijection, with the number of turns made by the curve $\gamma : \mathbb{S}^1 \to \mathbb{S}^1$.

Lemma 3.3. *The annulus $\mathbb{A}$ and the circle are homotopy equivalent.*

Proof. We can use polar coordinates (r, θ) in $\mathbb{R}^2$. Then $\mathbb{S}^1 = \{r = 3/2\}$ (note that any two circles are homeomorphic) and $\mathbb{A} = \{1 < r < 2\}$. We let $f : \mathbb{S}^1 \to \mathbb{A}$ be the inclusion and $g : \mathbb{A} \to \mathbb{S}^1$ be given by $g(r, \theta) = (3/2, \theta)$. Then $g \circ f = \mathrm{id}_{\mathbb{S}^1}$. Furthermore, $f \circ g(r, \theta) = (3/2, \theta)$ is homotopic to the identity thanks to $(r, \theta; t) \mapsto (tr + (1-t)3/2, \theta)$, which amounts to interpolating linearly in the radius. See Figure 3.1. □

Corollary 3.4.

$$[\mathbb{A}, \mathbb{A}] \cong [\mathbb{S}^1, \mathbb{S}^1] \cong \mathbb{Z}$$

Proof. According to Corollary 2.35, when evaluating $[-, -]$, we can replace each entry by an homotopy equivalent space without changing the result (up to isomorphism). We thus apply Lemma 3.3 to replace $\mathbb{A}$ by $\mathbb{S}^1$, proving the first equality. The second equality is Theorem 3.2. □

Corollary 3.5. *The annulus and $\mathbb{S}^1$ are not contractible. In particular, the annulus is not homotopy equivalent to the ball $\mathbb{B}^2$.*

Proof. Since $[\mathbb{S}^1, \mathbb{S}^1] \cong [\mathbb{S}^1, \mathbb{A}] \simeq \mathbb{Z}$ (Theorem 3.2), we deduce from Corollary 2.36 that they cannot be homotopy equivalent to a point. In contrast, the ball is convex and thus contractible. □

3.2 (Deformation) retracts

We have seen in Lemma 3.3 that the circle sits nicely inside the annulus. It is not just a subspace, there is additionally a projection map $r : \mathbb{A} \to \mathbb{S}^1$ that serves as an inverse to the inclusion ι. Moreover, $\iota \circ r$ is homotopic to $\mathrm{id}_{\mathbb{A}}$ in a manner that fixes $\mathbb{S}^1$. These notions are somewhat general, as we now explain.

3.2.1 Retractions

The following notion works in any category:

Definition 3.6. *Let $\mathcal{C}$ be a category. A morphism $r : x \to y$ is a* ***retraction*** *if it has a right-inverse $\iota : y \to x$. Then we say that y is a* ***retract*** *of x.*

This means that x is more complicated than y. Compare this to Lemmas 2.26 and 2.31, which say that r_* and ι^* are surjective and r^* and ι_* are injective (for any test object).

Example 3.7: Let $\iota : H \to G$ be an injective homomorphism of groups. If ι is a right inverse of some $r : G \to H$, we can consider the kernel $K := \ker(r) \subset G$ and, by the first isomorphism theorem, we have that $H \cong G/K$. This means that G is the semidirect product of H and K. △

Now we specialise to spaces.

Lemma 3.8. *Suppose $r : X \to Y$ is a retraction and $\iota : Y \to X$ is a right-inverse. Then ι is an inclusion (i.e. a homeomorphism with its image).*

Proof. Seeing ι as a map $Y \to \iota(Y)$, we see that $r|_{\iota(Y)}$ is a continuous inverse. □

A particularly simple case is:

Lemma 3.9. *Let A be a topological space and $a \in A$ a point. Then, $\{a\}$ is a retract of A.*

Proof. It is immediate that the unique map $r : A \to \{a\}$ is a left-inverse to the inclusion. □

For completeness we repeat the content of Lemmas 2.26 and 2.31:

Proposition 3.10. *Let A, B and C be topological spaces. Suppose that $\iota : B \to C$ is an inclusion and $r : C \to B$ is a retraction serving as a left-inverse. Then:*

- $r_* : [A, C] \to [A, B]$ *is a left-inverse of* $\iota_* : [A, B] \to [A, C]$.

- *In particular, r_* is surjective and ι_* is injective.*
- *$r^* : [B, A] \to [C, A]$ is a right-inverse of $\iota^* : [C, A] \to [B, A]$.*
- *In particular, r^* is injective and ι^* is surjective.*

This proposition is the tool that we will use repeatedly to prove that a subspace is not a retract:

Lemma 3.11. *Not every inclusion ι is a right-inverse of a retraction.*

Proof. Consider the inclusion $\iota : \{0, 1\} \to \mathbb{R}$. Suppose that there is a retraction $r : \mathbb{R} \to \{0, 1\}$. Then we would have a surjection

$$r_* : [\{.\}, \mathbb{R}] \to [\{.\}, \{0, 1\}].$$

However, this is not possible, since the domain is the singleton set and the target has two elements.

More generally, let $\iota : A \to B$ be an inclusion such that $\iota_* : \pi_0(A) \to \pi_0(B)$ is not injective. I.e. one of the path-components of B contains more than one path-component of A. Then, A cannot be a retract of B. □

The examples seen in the previous proof rely on computing the number of path-components. Instead, we can look at loops:

Corollary 3.12. *$\mathbb{S}^1$ is not a retract of $\mathbb{R}^2$.*

Proof. Suppose for contradiction that we can find an inclusion $\iota : \mathbb{S}^1 \to \mathbb{R}^2$ (which need not be the standard one) and a retraction $r : \mathbb{R}^2 \to \mathbb{S}^1$ left-inverting ι. Then, the pushforward $r_* : [\mathbb{S}^1, \mathbb{R}^2] \to [\mathbb{S}^1, \mathbb{S}^1]$ is surjective and ι_* is injective. This contradicts the fact that $[\mathbb{S}^1, \mathbb{R}^2] \simeq \{.\}$ and $[\mathbb{S}^1, \mathbb{S}^1] \simeq \mathbb{Z}$ (Theorem 3.2). □

3.2.2 Deformation retractions

The circle $\mathbb{S}^1$ is a retract of the annulus $\mathbb{A}$, but there is some extra structure. Namely, there is a homotopy equivalence between the two that amounts to deforming $\mathbb{A}$ within itself, keeping $\mathbb{S}^1$ fixed, until $\mathbb{A}$ collapses onto $\mathbb{S}^1$. We give this a name:

Definition 3.13. *Let A be a space and $B \subset A$ a retract. We write ι and r for the inclusion and the retraction, respectively. We say that B is a* ***deformation retract*** *if there is a homotopy $(f_t)_{t\in[0,1]} : A \to A$ such that $f_0 = \mathrm{id}_A$, $f_1 = \iota \circ r$, and $f_t|_B = \mathrm{id}_B$ for all t. We then say that r is a* ***deformation retraction****.*

A deformation retraction is both a retraction and a homotopy equivalence, but it satisfies the additional property that the relevant homotopy is the identity over the subspace B.

You can go through the proof of Lemma 3.3 and verify that:

Corollary 3.14. *The circle of radius $3/2$ is a deformation retract of the annulus $\mathbb{A}$.*

Similarly:

Lemma 3.15. *The sphere $\mathbb{S}^{n-1}$, included in the standard manner in $\mathbb{R}^n$, is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.*

Proof. The map $r(x) = x/|x|$ is a retraction $r : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$, since it keeps unit vectors fixed. The linear homotopy $(r_t)_{t\in[0,1]} : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$ given by $r_t(x) = (1-t)x + tr(x)$ provides the desired deformation retraction. □

3.3 Hom functors

We now revisit the pullback and pushforward (Section 2.3), packaging them as part of the so-called Hom functors.

3.3.1 The covariant Hom

Definition 3.16. *Fix a category*[1] $\mathcal{C}$ *and an object* x. *Using this data, we define a functor* $F = \mathrm{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Set}$ *as follows:*

- *Given* $y \in \mathrm{Ob}(\mathcal{C})$, *we map it to* $F(y) := \mathrm{Hom}_{\mathcal{C}}(x, y)$.
- *Given objects* $y, z \in \mathrm{Ob}(\mathcal{C})$, *we take a morphism* $f : y \to z$ *to* $F(f) : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$ *by setting* $F(f)(g) := f_* g = f \circ g$.

We say that $\mathrm{Hom}_{\mathcal{C}}(x, -)$ *is the* ***covariant hom-functor*** *associated to* x.

$\mathrm{Hom}_{\mathcal{C}}(x, -)$ packages, at the level of objects, all the morphisms from x into other elements of $\mathcal{C}$. At the level of morphisms, it tells us how such maps from x relate to one another. As such, you can think of $\mathrm{Hom}_{\mathcal{C}}(x, -)$ as a device that allows us to see $\mathcal{C}$ from the perspective of x.

Before we move on, we should verify that:

Lemma 3.17. *The functor* $F = \mathrm{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Set}$ *is well-defined.*

Proof. The composition axiom holds: Indeed, given morphisms $f : y \to z$ and $g : z \to u$, we have that $g \circ f : y \to u$ and therefore its image by the functor is the morphism

$$F(g \circ f) : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, u)$$

which verifies:

$$F(g \circ f)(d) = g \circ f \circ d = g \circ (f \circ d) = F(g)(F(f)(d)).$$

The identity axiom follows similarly. □

[1] For this definition to make sense we need to assume that $\mathcal{C}$ is **locally small**: This means that $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is really a set and not just a class, for every two objects x and y in $\mathcal{C}$.

3.3.2 The contravariant Hom

We can now reason dually and try to understand how a category $\mathcal{C}$ sees a given object $x \in \mathrm{Ob}(\mathcal{C})$. That is, we want to consider morphisms *into* x. Before we proceed, let us define:

Definition 3.18. *Let $\mathcal{C}$ be a category. Its* ***opposite category*** *$\mathcal{C}^{\mathrm{op}}$:*

- *Has the same objects as $\mathcal{C}$.*
- *For each pair of objects x and y, it satisfies $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) := \mathrm{Hom}_{\mathcal{C}}(y, x)$. That is, we regard each morphism $f : y \to x$ in $\mathcal{C}$ as a "flipped morphism" $f^{\mathrm{op}} : x \to y$ in $\mathcal{C}^{\mathrm{op}}$.*
- *$f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$.*

It is left to reader to verify that $\mathcal{C}^{\mathrm{op}}$ does satisfy the axioms of a category.

Definition 3.19. *Fix a category $\mathcal{C}$ and an object x. Using this data, we define a functor $F = \mathrm{Hom}_{\mathcal{C}}(-, x) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ as follows:*

- *Given $y \in \mathrm{Ob}(\mathcal{C})$, we map it to $F(y) := \mathrm{Hom}_{\mathcal{C}}(y, x)$.*
- *Given objects $y, z \in \mathrm{Ob}(\mathcal{C})$, we take a morphism $f : y \to z$ to $F(f) : \mathrm{Hom}_{\mathcal{C}}(z, x) \to \mathrm{Hom}_{\mathcal{C}}(y, x)$ by setting $F(f)(g) := f^* g = g \circ f$.*

We say that $\mathrm{Hom}_{\mathcal{C}}(-, x)$ is the ***contravariant hom-functor*** *associated to x.*

This functor (we leave to the reader to verify that this is indeed a functor) packages how all the objects in $\mathcal{C}$ see x. Do note that we put $\mathcal{C}^{\mathrm{op}}$ as the source category because $\mathrm{Hom}_{\mathcal{C}}(-, x)$ inverts the direction of the morphisms compared to $\mathcal{C}$.

3.3.3 The functors $[A, -]$ and $[-, A]$

Having introduced the relevant formalism, we can now specialise to (the homotopy category of) topological spaces:

$$\begin{array}{rcl} [A, -] : \mathrm{hTop} & \to & \mathrm{Set} \\ B & \mapsto & [A, B] \\ [f] : B \to C & \mapsto & f_* : [A, B] \to [A, C], \end{array}$$

where A is some topological space we have fixed to "test" all other topological spaces.

Remark 3.20: Given the localisation functor $F : \mathrm{Top} \to \mathrm{hTop}$ (Remark 2.19) we can consider the functor $[A, -] \circ F : \mathrm{Top} \to \mathrm{Set}$. It takes a morphism f to f_*. We will often abuse notation and still denote it by $[A, -]$. △

There is also the dual notion, which is contravariant:

$$\begin{array}{rcl} [-, A] : \mathrm{hTop} & \to & \mathrm{Set} \\ B & \mapsto & [B, A] \\ [f] : B \to C & \mapsto & f^* : [C, A] \to [B, A], \end{array}$$

As a reality check, we prove that the pushforward and the pullback are invariant under homotopies:

Lemma 3.21. *Suppose that $f, g : B \to C$ are homotopic. Then:*

- $f_*, g_* : [A, B] \to [A, C]$ *are equal.*
- $f^*, g^* : [C, A] \to [B, A]$ *are equal.*

Proof. The classes $[f]$ and $[g]$ are the same by assumption. It follows that

$$f_* = [f]_* = \mathrm{Hom}_{\mathrm{hTop}}(A, -)([f]) = \mathrm{Hom}_{\mathrm{hTop}}(A, -)([g]) = [g]_* = g_* : [A, B] \to [A, C],$$

where the first and last equalities are simply the definition. The second statement is proven similarly. □

Important: $[A, -]$ and $[-, A]$ are Set-valued invariants of spaces. I.e. we can evaluate them on spaces and obtain sets that we can then compare. We are particularly interested in the invariant $[\mathbb{S}^1, -]$, that computes the homotopy classes of loops in each space. Very soon we will instead look at Grp-valued invariants, which have a richer structure (since groups have more structure than sets).

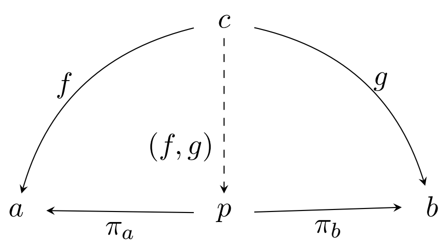
3.4 Products

Much like you can take cartesian products in Set, you can also consider them in Top or Grp. In this section we will show that the product interacts nicely with the functor $[A, -]$. Namely, we will be able to compute $[A, \prod_i M_i]$ immediately once we know each $[A, M_i]$.

3.4.1 The categorical definition

For completeness, we first define what a product should be in any category:

Definition 3.22. *Let $\mathcal{C}$ be a category, and let $a, b \in \mathcal{C}$ be objects. A tuple $(p \in \mathcal{C}, \pi_a : p \to a$ and $\pi_b : p \to b)$ is said to satisfy the* ***universal property of the product*** *if for any other tuple $(c \in \mathcal{C}, f : c \to a, g : c \to b)$, there exists a unique morphism $(f, g) : c \to p$ such that the following diagram commutes:*



I.e. $f = \pi_a \circ (f, g)$ and $g = \pi_b \circ (f, g)$.

You should think of π_a and π_b as the usual projections to each of the factors (although in many categories this does not really make sense). Then, the universal property says that any space c that maps into a and b separately in particular maps to the product by putting the two maps together.

Remark 3.23: Refer to the exercises for the product in Set, Top, Grp, and Ab. △

Remark 3.24: In general, the product of two objects is not unique, but almost. Even though there may be many objects that play the role of the product, in Exercise 3.10 you are asked to prove that such objects are all isomorphic (and the isomorphism is, additionally, unique). △

Remark 3.25: Once we have defined the product of two objects, we can iterate the construction and consider the product of finitely many objects. A corresponding universal property can be phrased, which we will not do. At this point we encounter some further non-uniqueness (as in Exercise 3.10). If A, B and C are sets, then $(A \times B) \times C$ is bijective to $A \times (B \times C)$ (in a canonical way) but they are not literally the same. Both can play the role of the product of the three objects. △

3.4.2 Interaction with $[A, -]$

We now prove:

Proposition 3.26. *Let $\{M_i\}_{i=1}^n$ be a finite collection of spaces. Fix an auxiliary space A. Then:*

$$[A, \prod_i M_i] \simeq \prod_i [A, M_i].$$

Proof. Using the definition of the product topology we see that

$$f = (f_1, \cdots, f_n) : A \to \prod_i M_i$$

is continuous if and only if each f_i is continuous. It follows that $f \mapsto (f_1, \cdots, f_n)$ yields a bijection:

$$\Phi : \mathrm{Hom}_{\mathrm{Top}}(A, \prod_i M_i) \simeq \prod_i \mathrm{Hom}_{\mathrm{Top}}(A, M_i).$$

We can then define a function

$$\Psi : [A, \prod_i M_i] \to \prod_i [A, M_i]$$

using the expression $[f] \mapsto ([f_1], \cdots, [f_n])$. We claim that it is well-defined and bijective. To show this, we reason as above and note that

$$\mathrm{Hom}_{\mathrm{Top}}(A \times [0,1], \prod_i M_i) \simeq \prod_i \mathrm{Hom}_{\mathrm{Top}}(A \times [0,1], M_i),$$

meaning that the continuity of homotopies can be checked entry by entry. This implies that a homotopy of $[f]$ defines a homotopy of each entry, proving that Ψ is well-defined. Surjectivity is clear, since Φ was surjective. For injectivity we observe that a collection of homotopies for the entries $[f_i]$ yields a homotopy of $[f]$. □

We can then deduce that:

Corollary 3.27. *The n-dimensional torus $\mathbb{T}^n = (\mathbb{S}^1)^n$ is not contractible.*

Proof. We use Proposition 3.26 to show $[\mathbb{S}^1, \mathbb{T}^n] \simeq [\mathbb{S}^1, \mathbb{S}^1]^n \simeq \mathbb{Z}^n$, which is non-trivial. □

Remark 3.28: At this point we want to ask ourselves whether $\mathbb{T}^n$ and $\mathbb{T}^m$ are homotopy equivalent if $n \neq m$. Computing as in Corollary 3.27 we see that

$$[\mathbb{S}^1, \mathbb{T}^n] \simeq \mathbb{Z}^n, \qquad [\mathbb{S}^1, \mathbb{T}^n] \simeq \mathbb{Z}^m.$$

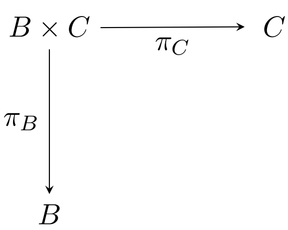
However, this tells us nothing at all! The reason is that $\mathbb{Z}^n$ and $\mathbb{Z}^m$ have *the same cardinality*, meaning that they are isomorphic as sets.

The issue is that sets do not have as much structure as we would wish for. In contrast, $(\mathbb{Z}^n, +)$ and $(\mathbb{Z}^m, +)$ are *not* isomorphic as groups (with the usual additive structure). This motivates us to look for invariants valued in Grp instead of in Set. This extra structure will allow us to distinguish $\mathbb{T}^n$ and $\mathbb{T}^m$ (i.e. show they are not homotopy equivalent and thus not homeomorphic). We will see this in the next chapter. △

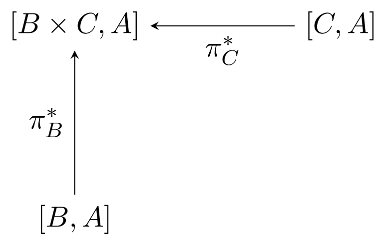
Lemma 3.29. *Let $\{M_i\}_{i=1}^n$ be a finite collection of non-empty spaces, with M_1 not contractible. Then $\prod_i M_i$ is not contractible.*

Proof. According to Proposition 3.26 it holds that $[M_1, \prod_i M_i] \simeq \prod_i [M_1, M_i]$. The cardinality of this product is at least two, since $[M_1, M_1]$ contains more than one element (since M_1 is not contractible), and each $[M_1, M_i]$ contains at least one element (since M_i is non-empty). □

Remark 3.30: We may wonder how products interact with the functor $[-, A]$. Given spaces B and C, we have a diagram of the type:



which, upon applying $[-, A]$ transforms to:



In particular, observe that arrows get reversed, since $[-, A]$ is contravariant. If you compare this to Definition 3.22, you can see that $[B \times C, A]$ does not look like a good candidate to be the product of $[C, A]$ and $[B, A]$, since the natural maps we have in this setting go in the opposite direction with respect to the projections that a product should come with. This corresponds to the intuition that we cannot naturally associate to a pair of maps $(f : B \to A, g : C \to A)$ a map $B \times C \to A$ in general. This discussion continues in Exercise 3.35 below. △

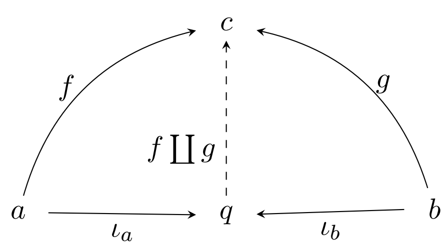
3.5 Coproducts

We can now reason dually and consider the disjoint union of topological spaces, instead of the product. In a general category, the analogue of the union is called a *coproduct*.

3.5.1 The categorical definition

In an arbitrary category, the coproduct is defined as:

Definition 3.31. *Let $\mathcal{C}$ be a category, and let $a, b \in \mathcal{C}$ be objects. A tuple $(q \in \mathcal{C}, \iota_a : a \to q$ and $\iota_b : b \to q)$ is said to satisfy the* ***universal property of the coproduct*** *if for any other tuple $(c \in \mathcal{C}, f : a \to c, g : b \to c)$, there exists a unique morphism $f \coprod g : q \to c$ such that the following diagram commutes:*



Meaning that $f = (f \coprod g) \circ \iota_a$ and $g = (f \coprod g) \circ \iota_b$.

You should think of ι_a and ι_b as the inclusions of a and b into their disjoint union. The universal property says that once you know how to map a and b into c, you also know how to map their union into c.

Remark 3.32: Refer to the exercises for the coproduct in Set, Top, Ab, and $\text{Vect}_{\mathbb{R}}$. The coproduct in Grp is more involved and will be discussed in Section 9.3. △

3.5.2 Results about homotopy classes

We now state and prove the dual to Proposition 3.26:

Proposition 3.33. *Let $\{M_i\}_{i=1}^n$ be a finite collection of spaces. Fix a connected space A. Then:*

$$[A, \coprod_i M_i] \simeq \coprod_i [A, M_i].$$

Proof. An element $[f]$ in $\coprod_i [A, M_i]$ is an element in one of the $[A, M_i]$, for some i. We can see it as an element in $[A, \coprod_i M_i]$ by composing f with the inclusion $M_i \to \coprod_i M_i$. This defines a map

$$\Psi : \coprod_i [A, M_i] \to [A, \coprod_i M_i]$$

that is injective. To prove surjectivity we consider $[g] \in [A, \coprod_i M_i]$ and we observe that $\{A_i = g^{-1}(M_i)\}$ is a partition of A by open sets. By connectedness of A it follows that all of them but one, say A_i, must be empty. It follows that g takes values in M_i and thus $[g]$ is in the image of Ψ. □

You can check that the result need not be true anymore if A is not connected; consider for instance $A = \{p, q\}$ and $M_1 \coprod M_2$ with both M_i non-empty.

If we consider the functor $[-, A]$ instead, we see that it takes coproducts to products:

Proposition 3.34. *Let $\{M_i\}_{i=1}^n$ be a finite collection of spaces. Fix an auxiliary space A. Then:*

$$[\coprod_i M_i, A] \simeq \prod_i [M_i, A].$$

Proof. Given a map $f : \coprod_i M_i \to A$ we can consider $f_i := f|_{M_i}$. Continuity of f is equivalent to the continuity of each f_i (for instance, by the Pasting Lemma 1.25). The function $f \mapsto (f_1, \cdots, f_n)$ is thus a bijection:

$$\Phi : \mathrm{Hom}_{\mathrm{Top}}(\coprod_i M_i, A) \simeq \prod_i \mathrm{Hom}_{\mathrm{Top}}(M_i, A).$$

Similarly we define

$$\Psi : [\coprod_i M_i] \to \prod_i [M_i, A]$$

using the expression $[f] \mapsto ([f_1], \cdots, [f_n])$. Homotopies of f correspond to homotopies of each of the $[f_i]$, proving that Ψ is well-defined and bijective. □

Remark 3.35: Let us go back to Remark 3.30. There we argued that $[-, A] : \mathrm{Top} \to \mathrm{Set}$ does not take products to products, since it is contravariant. We now claim that it does not take products to coproducts either. I.e. $[B \times C, A]$ is, in general, not the union of $[C, A]$ and $[B, A]$.

Assuming that $B, C \neq \emptyset$, you can prove that $[B, A]$ and $[C, A]$ map injectively to $[B \times C, A]$ via π_B^* and π_C^*. This is because we can exhibit both as retracts of $B \times C$ and then apply Proposition 3.10. However, not every element in $[B \times C, A]$ is in the image of one of these maps. You can take $A = B = C = \mathbb{S}^1$ as an example. $A = B = C = \{p, q\}$ also works. △

3.6 Exercises

3.6.1 Retracts

Exercise 3.1: Let X be contractible space. Let $A \subset X$ be a retract. Prove that A is also contractible.

Exercise 3.2: Let $A \subset \mathbb{R}^n$ be not closed. Show that A is not a retract of $\mathbb{R}^n$.

3.6.2 Deformation retracts

Exercise 3.3: Show that $C = [-1, 0]^2 \cup [0, 1]^2 \subset \mathbb{R}^2$ is contractible. Show that any point $p \in C$ is a deformation retract.

Exercise 3.4: Find a topological space X that does not deformation retract to a point.

Exercise 3.5: Show that there is a deformation retract of $\mathbb{T}^2 \setminus \{p\}$ (the torus minus a point) which is homeomorphic to a wedge of two circles (two circles joined at a point).

Exercise 3.6: Find an example of a subset $A \subset \mathbb{R}$ which is not a deformation retract of any neighbourhood $B \supset A$.

3.6.3 The fundamental group of the circle

Exercise 3.7: Let A be a non-empty topological space. Show that there is a map $\gamma : \mathbb{S}^1 \to A \times \mathbb{S}^1$ which is not homotopic to a constant map.

Exercise 3.8: Let $T^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (n times). Show that for all $n, m \in \mathbb{Z}^+$, the set $[T^n, T^m]$ has more than one element.

3.6.4 Products and coproducts

Exercise 3.9: Show that the usual product satisfies the universal property of the product in Set, Top, Grp, and Ab.

Exercise 3.10: Let $\mathcal{C}$ be a category, and let $a, b \in \mathcal{C}$ be objects. Suppose that the triple $(q, f : q \to a, g : q \to b)$ satisfies the universal property of the product. Suppose, similarly, that there is another triple $(q', f' : q' \to a, g' : q' \to b)$ that also satisfies it. Then, there is a unique isomorphism $\phi : q \to q'$ such that $f = f' \circ \phi$ and $g = g' \circ \phi$.

Exercise 3.11: The coproduct in Set and Top is the disjoint union.

Exercise 3.12: The coproduct in Ab and in Vect_F is the same as the product[2].

Exercise 3.13: Prove that the coproduct is unique up to isomorphism. That is: Let $\mathcal{C}$ be a category, and let $a, b \in \mathcal{C}$ be objects. Suppose that the tuples $(q \in \mathcal{C}, f : a \to q, g : b \to q)$ and (q', f', g') satisfy the universal property of the coproduct. Then, there is a unique isomorphism $\phi : q \to q'$ such that $f' = \phi \circ f$ and $g' = \phi \circ g$.

Exercise 3.14: Let X be a space and let $\mathrm{SO}(X)$ be the corresponding category of opens (Exercise 1.5). Let U and V be opens in X.

- Observe that $U \cap V$ is the (unique!) product of U and V, as elements of $\mathrm{SO}(X)$.
- Observe that $U \cup V$ is the (unique!) coproduct of U and V, as elements of $\mathrm{SO}(X)$.

Exercise 3.15: We continue with the previous exercise. Let $F : \mathrm{SO}(X) \to \mathrm{Top}$ be the inclusion functor. I.e. it includes the opens of X into the category of all topological spaces.

- Prove that it is indeed a functor.
- Verify that it does not map products to products.
- Verify that it does not map coproducts to coproducts.

Exercise 3.16: Let $\mathcal{C}$ be a poset, seen as a category. Describe the product and coproduct of two arbitrary elements $x, y \in \mathrm{Ob}(\mathcal{C})$. **Note:** Do observe that the (co)product may not always exist. Find an example where this is the case.

3.6.5 Extra: maps into the circle

The following exercises explore the functor $[-, \mathbb{S}^1]$.

Exercise 3.17: The circle $\mathbb{S}^1$ is a group under multiplication as complex numbers. Prove that:

- The product $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ is a continuous map.
- The inverse $\mathbb{S}^1 \to \mathbb{S}^1$ is a continuous map.

We say that $\mathbb{S}^1$ is a **topological group** (because the structures as group and topological space are compatible).

Exercise 3.18: Let A be a topological group. Then:

- The set of all maps $\mathrm{Hom}_{\mathrm{Top}}(X, A)$ can be endowed with a group structure, for all X.
- $[X, A]$ has a group structure as well, for all X.

[2] We can also consider (co)products of infinite cardinality. In this case, the coproduct and the product differ from one another both in Ab and Vect_F.

- $[-, A]$ can be regarded as a functor Top $\to$ Grp.
- If A is abelian, $[-, A]$ takes values in Ab.

The fundamental group(oid)

Lecture 4

In Remark 3.28 we observed that $[\mathbb{S}^1, -] : \text{Top} \to \text{Set}$ is not powerful enough to distinguish the n-dimensional torus from the m-dimensional one, $n \neq m$. The reason was that $[\mathbb{S}^1, \mathbb{T}^n] \simeq \mathbb{Z}^n$ which, as a set, is isomorphic to $\mathbb{Z}^m$.

This motivates us to construct functors $\text{Top} \to \text{Grp}$ instead, i.e. invariants of spaces that take values in groups. We cannot put a group structure on $[\mathbb{S}^1, X]$ directly, but with a few tweaks we will be able to see the concatenation of paths as the operation in a certain group (the fundamental group) consisting of classes of loops in X.

To summarise, our goals in this Chapter are as follows:

- To define the *fundamental groupoid* (Definitions 4.23), an invariant of spaces with values in groupoids (Definition 4.26).
- To introduce the *fundamental group* (Definitions 4.24), an invariant of pointed spaces (Definition 4.4) with values in groups.

These two concepts will be the main characters of this course moving forward.

4.1 Homotopies relative to subsets

The crucial observation is that two loops in a space X, starting and finishing at the same point $x \in X$, can be concatenated. Upon taking homotopy classes, this will produce for us a group operation in the set of homotopy classes of loops based at x. This idea forces us to consider homotopies that keep the endpoints fixed at x. We now explain how this goes, in a bit more generality.

4.1.1 Maps and homotopies of pairs

Definition 4.1. *Let A and B be spaces with subspaces $A' \subset A$ and $B' \subset B$, respectively. A map of pairs $f : (A, A') \to (B, B')$ is a continuous map $f : A \to B$ satisfying $f(A') \subset B'$.*

*Two maps $f, g : (A, A') \to (B, B')$ are **homotopic** if there is a map of pairs $F : (A \times [0,1], A' \times [0,1]) \to (B, B')$ with $F(-,0) = f$ and $F(-,1) = g$.*

In particular, $F|_{A'}$ is a homotopy between $f|_{A'}$ and $g|_{A'}$.

The same proof that we used for usual homotopies (Proposition 1.18) also shows:

Proposition 4.2. *Being homotopic is an equivalence relation for maps of pairs.*

Remark 4.3: We invite the reader to check the following: We can define the category Top^2 of pairs of spaces, as well as its corresponding homotopy category by taking homotopy classes of maps of pairs. There is then a functor $\mathrm{Top} \to \mathrm{Top}^2$ sending X to $(X, \emptyset)$. Conversely, there is a functor the other way $\mathrm{Top}^2 \to \mathrm{Top}$ sending (X, X') to the quotient X/X'. Alternatively, one could consider the functor $(X, X') \mapsto X$ or the functor $(X, X') \mapsto X'$.

Let us spell out two concrete instances that will be of interest for us.

4.1.2 Pointed spaces

When the subspace under consideration is a point, we obtain:

Definition 4.4. *A **pointed space** is a pair (X, x) consisting of a space and a point $x \in X$. The point x is often called the **basepoint**. A **pointed map** $f : (X, x) \to (Y, y)$ is a continuous function $X \to Y$ that preserves the basepoint, i.e. $f(x) = y$.*

Example 4.5: The only map from the point $(\{p\}, \{p\})$ to a pointed space (X, x) is the one sending p to x. △

Then:

Definition 4.6. *The category of pointed spaces Top_* is defined by:*

- *Its objects are pointed spaces.*
- $\mathrm{Hom}_{\mathrm{Top}_*}((X, x), (Y, y))$ *is the set of all pointed maps $(X, x) \to (Y, y)$.*
- *The composition is the usual composition of functions.*
- *The identity of (X, x) is id_X, which is indeed pointed.*

Observe that there is a forgetful functor $\mathrm{Top}_* \to \mathrm{Top}$ that sends (X, x) to X. This implies that:

Lemma 4.7. *A pointed map $f : (X, x) \to (Y, y)$ is an isomorphism if and only if it is a homeomorphism.*

Proof. If f is an isomorphism, it is sent to a homeomorphism in Top by the forgetful functor (because functors preserve isomorphisms). Conversely, if f is a homeomorphism, it has a continuous inverse that is thus a bijection and thus pointed. This means that f is invertible, so it is an isomorphism. □

Example 4.8: The empty space $\emptyset$ is not an element of Top_*, because it contains no points and thus no possible basepoint. Instead, the simplest pointed space is the point $(\{p\}, p)$. △

Example 4.9: Let $p, q \in \mathbb{R}^n$. We can consider the pointed map $f : (\mathbb{R}^n, p) \to (\mathbb{R}^n, q)$ given by translating $f(x) = x + (q - p)$. This has $x \mapsto x + (p - q)$ as pointed inverse. We deduce that $(\mathbb{R}^n, p)$ and $(\mathbb{R}^n, q)$ are isomorphic. △

Example 4.10: Let $p, q \in \mathbb{S}^n$. We can then take $A \in \mathrm{SO}(n+1)$ an orthogonal matrix satisfying $Ap = q$. Since A defines a homeomorphism of $\mathbb{S}^n$ with inverse A^{-1}, we deduce that $(\mathbb{S}^n, p)$ and $(\mathbb{S}^n, q)$ are isomorphic. △

4.1.3 The homotopy category of pointed spaces

We can particularise homotopies of maps of pairs (Definition 4.1) to yield:

Definition 4.11. *Two pointed maps $f, g : (X, x) \to (Y, y)$ are* ***homotopic*** *to each other if there is a* ***pointed homotopy*** *between them. That is, a map $F : X \times [0,1] \to Y$ such that $f(a) = F(a, 0)$, $g(a) = F(a, 1)$, and $F(x, s) = y$ for all $s \in [0,1]$.*

Pointed homotopy is an equivalence relation, so we can talk about the pointed homotopy class $[f]$ of a pointed map $f : (X, x) \to (Y, y)$. The set of all pointed homotopy classes of maps is denoted by $[(X, x), (Y, y)]$.

Definition 4.12. *The* ***pointed homotopy category*** hTop_* *has pointed spaces as objects and pointed homotopy classes of maps as morphisms.*

In analogy with the non-pointed case, we can consider those pointed maps f such that $[f]$ is an isomorphism in hTop_*:

Lemma 4.13. *A* ***pointed homotopy equivalence*** *$f : (X, x) \to (Y, y)$ is a map that has a pointed homotopy inverse $g : (Y, y) \to (X, x)$, meaning that $g \circ f$ is pointed homotopic to id_X and $f \circ g$ is pointed homotopic to id_Y. We then say that f and g are* ***pointed homotopy inverses****.*

Example 4.14: Suppose a pointed space (X, x) is pointed homotopy equivalent to the point $(\{p\}, p)$. This means that the constant maps c_x and c_p are pointed homotopy inverses. This is the case if and only if $c_x \circ c_p$ is pointed homotopic to id_X. I.e. if and only if X deformation retracts to x. △

In the pointed setting we can also consider the Hom functors (Section 3.3). Namely:

Definition 4.15. *Let (A, a) be a pointed space. We write $[(A, a), -]$ for the covariant Hom functor*

$$\mathrm{Hom}_{\mathrm{hTop}_*}((A, a), -) : \mathrm{hTop}_* \to \mathrm{Set}.$$

One can similarly consider the contravariant version $[-, (A, a)]$.

The most important case for us is $[(\mathbb{S}^1, 1), (X, x)]$, the (pointed) homotopy classes of loops based at x. However, do note that this is still a set, not a group. We will address this next.

Remark 4.16: More generally, for each positive integer n there is a group $\pi_n(X, x)$, called the **nth homotopy group**, whose underlying set is $[(\mathbb{S}^n, 1), (X, x)]$. These are used to study higher dimensional holes in X. △

4.1.4 Homotopies of paths relative endpoints

The following is a special case of maps of pairs $([0, 1], \{0, 1\}) \to (X, \{x, y\})$ and homotopies thereof:

Definition 4.17. *Given a space X and points $x, y \in X$ we can consider the set of paths*

$$\mathcal{P} := \{\gamma : [0, 1] \to X \mid \gamma(0) = x, \quad \gamma(1) = y\}.$$

Two paths $\gamma, \nu \in \mathcal{P}$ are said to be ***homotopic relative to the endpoints*** *if there is a homotopy $\Gamma : [0, 1] \times [0, 1] \to X$ between $\Gamma(t, 0) = \gamma(t)$ and $\Gamma(t, 1) = \nu(t)$ that additionally satisfies*

$$\Gamma(0, s) = x \quad \text{and} \quad \Gamma(1, s) = y \quad \text{for all } s.$$

We write $\pi_1(X, x, y)$ for the quotient of $\mathcal{P}$ by the equivalence relation of homotopy relative to the endpoints. We write

$$\Pi_1(X) := \coprod_{x,y \in X} \pi_1(X, x, y)$$

for the disjoint union of all these sets.

That is, we are considering paths with fixed endpoints and we only allow ourselves to homotope paths keeping said endpoints fixed. See Figure 4.1. As a first useful example:

Lemma 4.18. *Let $X \subset \mathbb{R}^n$ be convex. Then $\pi_1(X, x, y) \simeq \{.\}$ for all pairs of points $x, y \in X$.*

Proof. Given $\gamma, \nu : [0, 1] \to X$, both starting at x and finishing at y, we have that the usual linear homotopy $F(t, s) := (1 - s)\gamma(t) + s\nu(t)$ preserves the endpoints $F(0, s) = (1 - s)\gamma(0) + s\nu(0) = (1 - s)x + sx = x$ and similarly $F(1, s) = y$, proving the statement. □

4.2 The fundamental groupoid

In Chapter I we encountered the operations of reversal (Definition 1.21) and concatenation (Definition 1.23) for homotopies. These particularise to the case of paths (Subsection 2.1.2). These operations suggest that there is a group-like structure lying around for paths. Namely, concatenation resembles the group multiplication and reversal resembles taking the inverse. Our goal is to formalise this idea.

Spoiler alert: Using paths, we are going to construct something called a groupoid and, from it, we will obtain a group structure on the set of loops with endpoints in p, for each point p in our space.

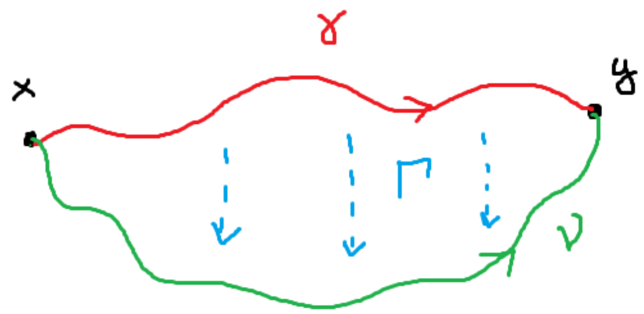


Figure 4.1: Two paths γ and ν, with the same endpoints x and y, connected by a homotopy Γ relative endpoints.

4.2.1 Concatenation up to homotopy

Recall that two paths can be concatenated if the first ends where the second begins. The paths were said to be *concatenable*. It follows that, at the level of homotopy classes, elements in $\pi_1(X, x, y)$ are concatenable with elements in $\pi_1(X, y, z)$:

Lemma 4.19. *Fix points $x, y, z \in X$ and paths $\gamma, \gamma', \nu, \nu' : [0, 1] \to X$ such that $[\gamma] = [\gamma'] \in \pi_1(X, x, y)$ and $[\nu] = [\nu'] \in \pi_1(X, y, z)$. Then:*

$$[\nu \bullet \gamma] = [\nu' \bullet \gamma'] \in \pi_1(X, x, z).$$

Therefore, $([\gamma], [\nu]) \mapsto [\nu \bullet \gamma]$ is a well-defined function

$$\Psi : \pi_1(X, x, y) \times \pi_1(X, y, z) \longrightarrow \pi_1(X, x, z).$$

Proof. Suppose that G is a homotopy relative endpoints between γ and γ' and N is a homotopy relative endpoints between ν and ν'. Then:

$$N \hat{\bullet} G(t, s) := \begin{cases} G(2t, s) & \text{for } t \in [0, 1/2] \\ N(2t - 1, s) & \text{for } t \in [1/2, 1] \end{cases}$$

is a well-defined function, since $G(1, s) = y$ and $N(0, s) = y$. Furthermore, it satisfies

$$(N \hat{\bullet} G)(t, 0) = N(-, 0) \bullet G(-, 0) = \nu \bullet \gamma$$

and

$$(N \hat{\bullet} G)(t, 1) = N(-, 1) \bullet G(-, 1) = \nu' \bullet \gamma'.$$

To prove that it is continuous we apply the Pasting Lemma, noting that it is continuous in each of the pieces $[0, 1/2] \times [0, 1]$ and $[1/2, 1] \times [0, 1]$, thanks to the continuity of G and N. $\square$

You can observe that $N \hat{\bullet} G$ is not quite the concatenation of G with N. The usual concatenation would take place in the s-variable, which is the homotopy variable. However, $N \hat{\bullet} G$ takes place in the t-variable.

4.2.2 Associativity up to homotopy

We now prove that the product defined in Lemma 4.19 for concatenable classes is associative:

Lemma 4.20. *Fix points $x, y, z, u \in X$ and paths $\gamma, \nu, \beta : [0,1] \to X$ with $[\gamma] \in \pi_1(X, x, y)$, $[\nu] \in \pi_1(X, y, z)$, and $[\beta] \in \pi_1(X, z, u)$. Then:*

$$[\beta \bullet (\nu \bullet \gamma)] = [(\beta \bullet \nu) \bullet \gamma].$$

Proof. Let us observe first that $\beta \bullet (\nu \bullet \gamma)$ and $(\beta \bullet \nu) \bullet \gamma$ are not the same path a priori. Indeed:

$$((\beta \bullet \nu) \bullet \gamma)(t) := \begin{cases} \gamma(2t) & \text{for } t \in [0, 1/2] \\ \nu(4t-2) & \text{for } t \in [1/2, 3/4] \\ \beta(4t-3) & \text{for } t \in [3/4, 1] \end{cases}$$

meaning that ν and β are run at double the speed compared to γ. In comparison:

$$(\beta \bullet (\nu \bullet \gamma))(t) := \begin{cases} \gamma(4t) & \text{for } t \in [0, 1/4] \\ \nu(4t-1) & \text{for } t \in [1/4, 1/2] \\ \beta(2t-1) & \text{for } t \in [1/2, 1] \end{cases}$$

which shows that we are basically performing the same path, just at a different pace.

It follows that we have to homotope between the two "speeds". We claim that there exists an increasing homeomorphism $\chi : [0,1] \to [0,1]$ such that:

$$((\beta \bullet \nu) \bullet \gamma)(t) = (\beta \bullet (\nu \bullet \gamma))(\chi(t)).$$

Indeed:

$$\chi(t) := \begin{cases} t/2 & \text{for } t \in [0, 1/2] \\ t - 1/4 & \text{for } t \in [1/2, 3/4] \\ 2t - 1 & \text{for } t \in [3/4, 1] \end{cases}$$

does the job. See Figure 4.2. To conclude the proof it is enough to show that the identity $\mathrm{id}_{[0,1]}$ is homotopic to χ, relative endpoints. Indeed, given such a homotopy $\chi_s : [0,1] \to [0,1]$, it holds that $\beta \bullet (\nu \bullet \gamma)(\chi_s(t))$ is a homotopy starting at $\beta \bullet (\nu \bullet \gamma)$ and finishing at $(\beta \bullet \nu) \bullet \gamma$. The existence of such a homotopy follows from Lemma 4.18, since the interval is convex. □

When used this way, we will often say that a map $\chi : [0,1] \to [0,1]$ is a *reparametrisation* (in this case, orientation-preserving) of the interval.

4.2.3 Identities up to homotopy

We furthermore observe that the constant paths behave like identities up to homotopy:

Lemma 4.21. *Fix points $x, y, z \in X$. Then, for all paths $[\gamma] \in \pi_1(X, x, y)$ and $[\nu] \in \pi_1(X, z, x)$:*

$$[\gamma \bullet c_x] = [\gamma], \qquad [c_x \bullet \nu] = [\nu] \in \pi_1(X, z, x).$$

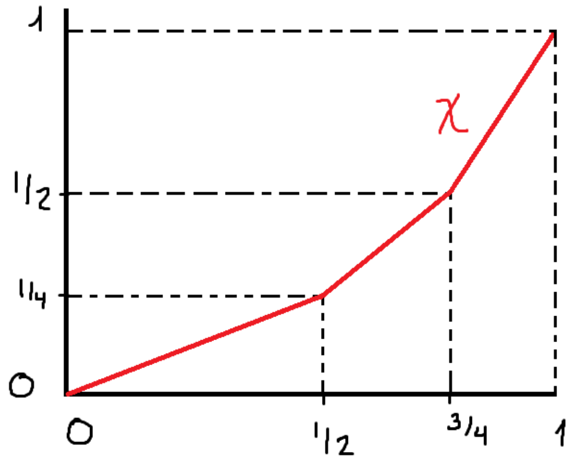


Figure 4.2: The graph of the function χ appearing in the proof of Lemma 4.20. A homotopy χ_s between χ and $\mathrm{id}_{[0,1]}$ amounts to interpolating linearly between the graphs.

Proof. We can write $\gamma \bullet c_x(t) = \gamma(\alpha(t))$, where

$$\alpha(t) := \begin{cases} 0 & \text{for } t \in [0, 1/2] \\ 2t - 1 & \text{for } t \in [1/2, 1] \end{cases},$$

as shown in Figure 4.3. The function $\alpha : [0,1] \to [0,1]$ begins at 0 and ends at 1, so it is homotopic relative endpoints to $\mathrm{id}_{[0,1]}$, according to Lemma 4.18. If we write $(\alpha_s)_{s\in[0,1]}$ for such a homotopy, we have that $\gamma \circ \alpha_s$ homotopes between $\gamma \bullet c_x$ and γ. □

4.2.4 Inverses up to homotopy

Similarly, we have that the reversal is the inverse for concatenation, up to homotopy:

Lemma 4.22. *Fix points $x, y \in X$ and a path $\gamma : [0,1] \to X$ such that $[\gamma] \in \pi_1(X, x, y)$. Then:*

$$[\bar{\gamma} \bullet \gamma] = [c_x] \in \pi_1(X, x, x)$$

and

$$[\gamma \bullet \bar{\gamma}] = [c_y] \in \pi_1(X, y, y)$$

Proof. Observe that $\bar{\gamma} \bullet \gamma(t) = \gamma(\rho(t))$, where

$$\rho(t) := \begin{cases} 2t & \text{for } t \in [0, 1/2] \\ 2 - 2t & \text{for } t \in [1/2, 1] \end{cases}$$

is shown in Figure 4.4. Observe that $\rho : [0,1] \to [0,1]$ begins and ends at 0, so it is homotopic relative endpoints to the constant path c_0, according to Lemma 4.18. If we let $(\rho_s)_{s\in[0,1]}$ be such a homotopy, we have that $(\gamma \circ \rho_s)_{s\in[0,1]}$ homotopes between $\bar{\gamma} \bullet \gamma$ and the constant path c_x, as claimed. The second item follows from the first by swapping γ and $\bar{\gamma}$. □

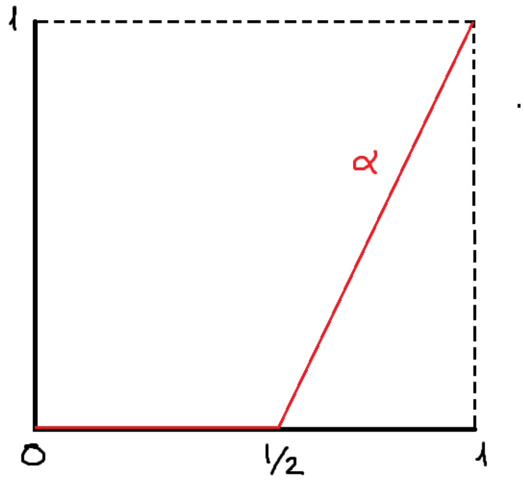


Figure 4.3: The graph of the function α appearing in the proof of Lemma 4.21.

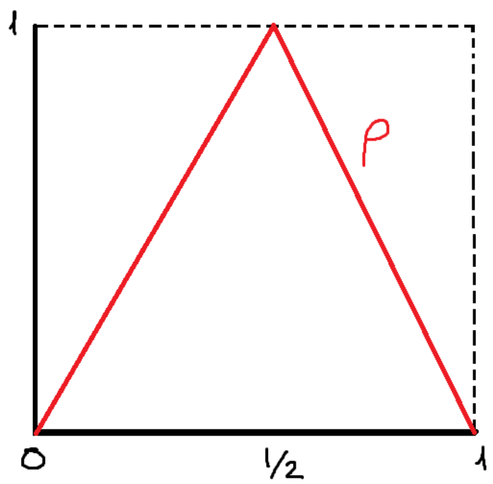


Figure 4.4: The graph of the function ρ appearing in the proof of Lemma 4.22.

4.2.5 Defining the fundamental groupoid

We have now all the pieces in place to define:

Definition 4.23. *The* ***fundamental groupoid*** *of a space X is the category $\Pi_1(X)$ such that:*

- *Its class of objects is X, seen as a set.*
- *Given points $x, y \in X$, the morphisms between them are the set $\pi_1(X, x, y)$ (Definition 4.17).*
- *The composition is given by concatenation of homotopy classes (Lemma 4.19).*
- *The identity at x is the class of the constant path $[c_x]$.*
- *Each $[\gamma] \in \pi_1(X, x, y)$ has an inverse $[\bar{\gamma}] \in \pi_1(X, y, x)$ given by reversing.*

The composition is well-defined by Lemma 4.19, it is associative by Lemma 4.20, it has the claimed identities by Lemma 4.21, and has inverses given by the class of the reversal by Lemma 4.22.

Definition 4.24. *The* ***fundamental group*** *of a space X at a point $x \in X$ is the group $\pi_1(X, x) := \pi_1(X, x, x)$, endowed with the composition coming from $\Pi_1(X)$.*

The fundamental group is indeed a group by Lemma 4.33 below.

Remark 4.25: Elements in $\pi_1(X, x)$ are classes of paths that begin and finish at x. I.e. each class can be represented by a map

$$\gamma : [0, 1] \to X.$$

However, a path that begins and finishes at the same point is equivalent to a **loop**. To be precise, under the assumption $\gamma(0) = \gamma(1)$, we can consider the quotient map $[0, 1] \to \mathbb{S}^1$, realising $\mathbb{S}^1$ as the quotient $[0, 1]/\{0 \simeq 1\}$. Under this quotient map, γ uniquely defines a loop $\widetilde{\gamma} : \mathbb{S}^1 \to X$, which we say is **based** at x. △

The group structure in $\pi_1(X)$ arises then from the fact that concatenation of two loops yields yet again another loop based at the same point. Similarly, the reversal of a loop, is a loop.

4.3 Groupoids

It turns out that $\Pi_1(X)$ is in fact a concrete example of:

Definition 4.26. *A* ***groupoid*** *$\mathcal{G}$ is a category*[1] *in which all morphisms are isomorphisms.*

According to the definition, each morphism in a groupoid has an inverse. This implies that:

Lemma 4.27. *Let $\mathcal{G}$ be a groupoid. Let $x, y \in \mathrm{Ob}(\mathcal{G})$ be objects. Then, either $\mathrm{Hom}_{\mathcal{G}}(x, y)$ is empty or x and y are isomorphic.*

[1] We will usually assume that $\mathcal{G}$ is a *small* category. Smallness means that $\mathrm{Ob}(\mathcal{G})$ is a set and each $\mathrm{Hom}_{\mathcal{G}}(x, y)$ is also a set. $\Pi_1(X)$ is indeed small.

Proof. If a morphism $f : x \to y$ between the two exists, it is an isomorphism. □

However, do note that $\mathrm{Hom}_{\mathcal{G}}(x, y)$ may consist of multiple elements, which are thus different ways in which x and y may be isomorphic.

Notation 4.28: When dealing with groupoids it is customary to use the notation $\mathcal{G} \rightrightarrows B$. Here B is the set of objects, which is sometimes called the **base** and $\mathcal{G}$ is the union of all the morphism sets. The two arrows represent the source function $\mathfrak{s} : \mathcal{G} \to B$ (which takes a morphism to its source object) and the target function $\mathfrak{t} : \mathcal{G} \to B$ (which takes a morphism to its target). The composition is left implicit in this notation. △

Notation 4.29: Given a groupoid $\mathcal{G} \rightrightarrows B$ we write $\mathcal{G}_{x,y} = \mathfrak{s}^{-1}(x) \cap \mathfrak{t}^{-1}(y)$ to mean the morphisms from x to y. This is shorter than writing $\mathrm{Hom}_{\mathcal{G}}(x, y)$. The set $\mathcal{G}_{x,x}$ is often denoted by $\mathcal{G}_x$ and is a group (see Lemma 4.33 below), which we call the **isotropy group** of the object x. △

Example 4.30: If G is a group and we see it as a category, we readily see that it is a groupoid, since all morphisms are invertible. We can emphasise that we see it as a groupoid by writing $G \rightrightarrows \{p\}$. I.e. there is a single object, denoted by p, and thus $\mathfrak{s}(g) = \mathfrak{t}(g) = p$ for all elements $g \in G$. The group G is thus the isotropy of p. △

Example 4.31: If X is a set, we can consider its **pair groupoid**, which we denote by $X \times X \rightrightarrows X$. This means that the objects are given by X and the morphisms by $X \times X$. Each element $(x, y) \in X \times X$ is the unique morphism from x to y. It follows that $(y, z) \circ (x, y) = (x, z)$. The identities are the elements of the form (x, x). Similarly, the inverse of (x, y) is (y, x). △

We already proved above that:

Lemma 4.32. *Let X be a space. Then, $\Pi_1(X) \rightrightarrows X$ is a groupoid.*

You should think of $\Pi_1(X)$ as a "discrete" version of X. The set S underlying X encodes all the points in X, but somehow forgets how they are assembled topologically. $\Pi_1(X)$ has S as its space of objects and additionally encodes how points are connected thanks to the morphisms. This is richer, but we are still forgetting plenty of information (the "higher dimensional holes of X", so to speak). You can then imagine that there are more complicated objects ("higher groupoids") that encode more and more information about X.

As promised:

Lemma 4.33. *Let $\mathcal{G} \rightrightarrows B$ be a groupoid and let $x \in B$ be an object. Then, the isotropy $\mathcal{G}_x$ is a group.*

Proof. (1) All the morphisms in $\mathcal{G}_x$ have x as source and target, so all of them can be composed with one another. It follows that there is a well-defined operation. (2) The identity id_x belongs to $\mathcal{G}_x$. (3) Since $\mathcal{G}$ is a groupoid, every morphism is invertible, so $\mathcal{G}_x$ contains inverses for all the elements. (4) The composition in $\mathcal{G}_x$ is associative, since it is induced from the composition in $\mathcal{G}$, which is itself associative. □

In particular, the fundamental group $\pi_1(X, x)$ of X at a point x is the isotropy group of $\Pi_1(X)$ at the object x.

4.4 Exercises

4.4.1 Pointed topological spaces

Exercise 4.1: Let $\mathbb{T}^n := (\mathbb{S}^1)^n$ be the n-**torus**. Fix points $p, q \in \mathbb{T}^n$. Show that $(\mathbb{T}^n, p)$ is isomorphic to $(\mathbb{T}^n, q)$ as pointed spaces.

Exercise 4.2: Find an example of a pair of pointed topological spaces (X_1, p_1) and (X_2, p_2) that are non-isomorphic as pointed spaces but X_1 is homeomorphic to X_2. Can you find an example where X_1 and X_2 are path-connected?

4.4.2 Groupoids

Exercise 4.3: Let $\mathcal{C}$ be the subcategory of hTop defined by $\mathrm{Ob}(\mathcal{C}) = \{X \text{ contractible space }\}$ and $\mathrm{Hom}_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathrm{hTop}}(A, B)$ for each A and B. Prove that $\mathcal{C}$ is a groupoid.

Exercise 4.4: Let $\mathcal{G} \rightrightarrows B$ be a connected groupoid. Since it is a category, we can ask ourselves whether (co)products exist in $\mathcal{G}$. Prove that $\mathcal{G}$ has all (co)products if and only if it is isomorphic to the pair groupoid.

The fundamental group(oid) functor

Lecture 5

Our goals in this lecture are as follows:

- To partition a groupoid into its connected components (Section 5.1), in analogy to the case of spaces.
- Given any morphism in a groupoid, to define its so-called conjugation (Section 5.2), which is analogous to the conjugation in a group.
- We will prove that the fundamental groupoid defines a functor from spaces to groupoids (Section 5.3).
- We will prove that the fundamental group defines a functor from the homotopy category of pointed spaces to groups (Section 5.4).

5.1 Connectivity of groupoids

$\Pi_1(X)$ is meant to be a more discrete/algebraic version of X. We now want to define a notion of connectedness for groupoids so that path-connectedness of X corresponds to connectedness of $\Pi_1(X)$:

Definition 5.1. *Let $\mathcal{G} \rightrightarrows B$ be a groupoid.*

- *Two objects $x, y \in B$ are said to be* ***connected*** *if $\mathcal{G}_{x,y} \neq \emptyset$.*
- *We write $\pi_0(\mathcal{G})$ for the set of equivalence classes of B with respect to the connectedness equivalence relation.*
- *$\mathcal{G}$ is* ***connected*** *if all its elements are connected.*

Proving that connectedness is an equivalence relation is left to the reader.

Since a groupoid always has inverses, we deduce:

Lemma 5.2. *All the objects in a connected groupoid are isomorphic to one another.*

We can then partition a groupoid into its components:

Definition 5.3. *Let $\mathcal{G} \rightrightarrows B$ be a groupoid. Given an equivalence class $\alpha \in \pi_0(\mathcal{G})$, we can consider the groupoid $\mathcal{G}_\alpha \rightrightarrows B_\alpha$ defined as:*

- $B_\alpha = \{x \in B \mid [x] = \alpha\}$.
- $\mathrm{Hom}_{\mathcal{G}_\alpha}(x, y) = \mathrm{Hom}_{\mathcal{G}}(x, y)$ *for every* $x, y \in B_\alpha$.

$\mathcal{G}_\alpha \rightrightarrows B_\alpha$ is said to be a ***component*** *of $\mathcal{G} \rightrightarrows B$.*

That is, we are effectively partitioning

$$\mathcal{G} \rightrightarrows B = \coprod_{\alpha \in \pi_0(\mathcal{G})} \mathcal{G}_\alpha \rightrightarrows B_\alpha$$

into (disjoint) groupoids, all of which are connected.

5.1.1 Consequences for the fundamental groupoid

We now spell out what the previous results mean for the case of $\Pi_1(X)$.

Lemma 5.4. *Let X be a space. Then:*

- $\pi_0(\Pi_1(X)) \simeq \pi_0(X)$.
- *In particular, X is path-connected if and only if $\Pi_1(X) \rightrightarrows X$ is connected.*
- *The partition $X = \coprod_i X_i$ into path-components corresponds to the partition $\Pi_1(X) = \coprod_i \Pi_1(X_i)$ into components.*

Proof. The bijection claimed in the first item is given by the function $f : \pi_0(X) \to \pi_0(\Pi_1(X))$ that takes $[x] \in \pi_0(X)$ to $[x] \in \pi_0(\Pi_1(X))$. We see that this is well-defined, since the equivalence class on the left means that $[x] = [y] \in \pi_0(X)$ if a path exists between the two, which is equivalent to $\pi_1(X, x, y) \neq \emptyset$, which defines the equivalence class on the right. Surjectivity is clear, since X is the base of the groupoid. Injectivity follows from the argument we gave for well-defined. The other two items follow immediately. □

That is, $\Pi_1(X)$ knows which points in X can be connected to each other by a path (and also in how many distinct ways up to homotopy).

5.2 Conjugation, pushforward, pullback

Let $\mathcal{G} \rightrightarrows B$ be a groupoid. Let x, y, z be objects. Let $f \in \mathcal{G}_{y,z}$ be a morphism from y to z. Recall

- the pushforward $f_* : \mathcal{G}_{x,y} \to \mathcal{G}_{x,z}$,
- and the pullback $f^* : \mathcal{G}_{z,x} \to \mathcal{G}_{y,x}$.

The following notion generalises the idea of conjugating in a group:

Definition 5.5. *The **conjugation** of $f \in \mathcal{G}_{y,z}$ at z is the function:*

$$\beta_f : \mathcal{G}_{z,z} \to \mathcal{G}_{y,y}$$

given by $\beta_f(g) = (f^{-1})_* \circ f^*(g) = f^{-1} \circ g \circ f$.

Lemma 5.6. *Let $\mathcal{G} \rightrightarrows B$ be a groupoid. Let $f \in \mathcal{G}_{y,z}$. Then:*

- *f_* and f^* are bijections.*
- *β_f is a group isomorphism.*

Proof. First observe that $\beta_f : \mathcal{G}_{z,z} \to \mathcal{G}_{y,y}$ is indeed a function between isotropy groups, so it makes sense for us to study whether it is a group isomorphism. Note furthermore, that $\mathcal{G}_{x,y}$ and $\mathcal{G}_{x,z}$ are simply sets if $x \neq y, z$, so f_* and f^* can at most be bijections.

To prove the first claim observe that f has an inverse f^{-1}, since $\mathcal{G}$ is a groupoid, and f_* and f_*^{-1} are inverses, meaning that both are bijections. The same reasoning applies to the pullback.

For the second claim we readily see from the formula $\beta_f = (f^{-1})_* \circ f^*$ that β_f is a bijection. It remains to prove that it is a group homomorphism. We see that it preserves the identity

$$\beta_f(\mathrm{id}_z) = f^{-1} \circ \mathrm{id}_z \circ f = f^{-1} \circ f = \mathrm{id}_y,$$

and furthermore it preserves the composition:

$$\beta_f(h \circ g) = f^{-1} \circ h \circ g \circ f = (f^{-1} \circ h \circ f) \circ (f^{-1} \circ g \circ f) = \beta_f(h) \circ \beta_f(g).$$

□

Remark 5.7: The Lemma tells us a key idea: In a groupoid $\mathcal{G}$, there is a lot of redundant information. All isotropy groups in a component are isomorphic to each other. All morphism sets in a component are similarly bijective to one another. We will revisit this in Sections 6.1 and 7.1. △

Remark 5.8: Suppose $f : a \to b$ is an invertible morphism in some category $\mathcal{C}$. We can then define $\beta_f : \mathrm{Hom}_{\mathcal{C}}(b, b) \to \mathrm{Hom}_{\mathcal{C}}(a, a)$ exactly as above. Since $\mathcal{C}$ is an arbitrary category and not a groupoid, $\mathrm{Hom}_{\mathcal{C}}(b, b)$ need not be a group. It is however a monoid (a set/class with a multiplication that satisfies associativity and has an identity). You can then verify that β_f is an isomorphism of monoids. △

5.2.1 Non-triviality of conjugations

Let $f : a \to a$ be a morphism in the groupoid $\mathcal{G}$. Then $\beta_f : \mathcal{G}_a \to \mathcal{G}_a$ is a group isomorphism. However, it need not be the identity.

Lemma 5.9. *Let $\mathcal{G}$ be the groupoid associated to the group G. Then the conjugation $\beta_f : G \to G$ given by $f \in G$ is trivial if and only if f commutes with every $g \in G$.*

Proof. We see that $\beta_f(g) = f^{-1}gf$ is equal to g if and only if $gf = fg$, which was the claim. □

We deduce:

Corollary 5.10. *A group G is abelian if and only if all conjugations β_f are trivial.*

Example 5.11: Suppose $\mathcal{G}$ is the groupoid defined by the group general linear group $\mathrm{GL}(n)$ (i.e. the group of invertible $n \times n$ matrices). If $n > 1$, $\mathrm{GL}(n)$ is not abelian, since not all matrices commute. This implies that the there are non-trivial conjugations in $\mathcal{G}$. △

5.3 The fundamental groupoid functor

Our next goal is to describe Π_1 as an invariant of spaces with values in groupoids, i.e. a functor $\Pi_1 : \mathrm{Top} \to \mathrm{Grpoid}$.

5.3.1 Morphisms of groupoids

Our motto in this course is that, whenever we introduce a new mathematical construct, we should ask ourselves what the corresponding morphisms are. This then allows us to define the category of such things. We will now go through these steps for groupoids. We need this to talk about Π_1 as a functor.

Since a groupoid is a particular case of a category:

Definition 5.12. *The category* Cat *of small*[1] *categories is given by:*

- *Its objects are all the small categories.*
- *Given categories $\mathcal{C}$ and $\mathcal{D}$ in* $\mathrm{Ob}(\mathrm{Cat})$*, its class of morphisms* $\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathcal{D})$ *consists of all the functors $F : \mathcal{C} \to \mathcal{D}$.*

Functors can be composed and the identity functors (mapping each object and morphism to itself) play then the role of the identities. In particular:

Lemma 5.13. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an isomorphism if and only if it is a bijection both between objects and between morphisms.*

Proof. F is an isomorphism if and only if there is an inverse functor $G : \mathcal{D} \to \mathcal{C}$, meaning that $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{D}}$. This necessarily implies that F and G induce bijections between objects and between morphisms. This is seen to be sufficient as well, since the set-theoretical inverse of a functor inducing bijections between objects and between morphisms must also preserve identities and compositions, so it is also a functor. □

We can particularise the previous definition:

[1] We need to restrict to small categories in order to avoid Cat being an element of itself (with the paradoxes it entails).

Definition 5.14. *The category* Grpoid *is the subcategory of* Cat *such that:*

- *It objects are all the small groupoids.*
- $\mathrm{Hom}_{\mathrm{Grpoid}}(\mathcal{C}, \mathcal{D}) = \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathcal{D})$ *for any two groupoids* $\mathcal{C}$ *and* $\mathcal{D}$*.*

5.3.2 The fundamental groupoid functor

At the level of objects, the functor Π_1 takes X to $\Pi_1(X)$. We now need to explain what it does at the level of morphisms:

Definition 5.15. *Let* $f : X \to Y$ *be a map of spaces. Its* ***pushforward*** *(at the level of fundamental groupoids) is the functor*

$$f_* : \Pi_1(X) \to \Pi_1(Y)$$

defined by:

- $f_*(x) = f(x) \in Y$ *for every object* $x \in \mathrm{Ob}(\Pi_1(X)) = X$*.*
- $f_*([\gamma]) = [f \circ \gamma] \in \pi_1(Y, f(x), f(y))$ *for every morphism* $\gamma \in \pi_1(X, x, y)$ *and every* $x, y \in X$*.*

Going back to the motto that $\Pi_1(X)$ is a gadget that encodes the points in X and the way in which they are connected, we see that f_* is indeed the natural map induced by f.

Lemma 5.16. *The pushforward is well-defined.*

Proof. One needs to check that the definition $f_*([\gamma]) = [f \circ \gamma]$ does not depend on the choice of representative γ. This follows from the fact that any homotopy γ_t of γ relative endpoints can be pushed forward using f to produce a homotopy $f \circ \gamma_t$ of $f \circ \gamma$ relative endpoints. We leave the details to the reader. □

Remark 5.17: Do note that f_* still encodes f as the map at the level of objects. This means that $f_* : \Pi_1(X) \to \Pi_1(Y)$ depends on f itself and not just on its homotopy class. This is different from the pushforward $f_* : [A, X] \to [A, X]$ associated to the functor $[A, -]$. △

We can then state:

Definition 5.18. *We write* $\Pi_1 : \mathrm{Top} \to \mathrm{Grpoid}$ *for the functor that:*

- *Sends a space* X *to its fundamental groupoid* $\Pi_1(X)$*.*
- *Sends a morphism* $f : X \to Y$ *to its pushforward* $f_* : \Pi_1(X) \to \Pi_1(Y)$*.*

We readily see that $(\mathrm{id}_X)_* = \mathrm{id}_{\Pi_1(X)}$, so identities are preserved. Similarly, $g_* \circ f_* = (g \circ f)_*$ so compositions are preserved. This shows that Π_1 is indeed a functor. Details are left to the reader.

The following is immediate from the fact that functors send isomorphisms to isomorphisms

Corollary 5.19. *If* X *and* Y *are homeomorphic, their fundamental groupoids* $\Pi_1(X)$ *and* $\Pi_1(Y)$ *are isomorphic.*

5.4 The fundamental group functor

Similarly:

Definition 5.20. *Let $f : (X, x) \to (Y, y)$ be a pointed map. Its* ***pushforward*** *(at the level of fundamental groups) is the group homomorphism*

$$f_* : \pi_1(X, x) \to \pi_1(Y, y)$$

defined by $f_*([\gamma]) = [f \circ \gamma]$.

This is well-defined (i.e. does not depend on the choice of representative γ) because the same statement was true for the pushforward of fundamental groupoids.

Definition 5.21. *We write $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$ for the functor that:*

- *Sends a pointed space (X, x) to its fundamental group $\pi_1(X, x)$.*
- *Sends a pointed map $f : (X, x) \to (Y, y)$ to its pushforward $f_* : \pi_1(X, x) \to \pi_1(Y, y)$.*

That this is a functor (i.e. preserves compositions and identities) follows once again from the fact that $\pi_1(X, x)$ sits inside $\Pi_1(X)$ as the isotropy at x.

We can use the following corollary to test whether a subspace is a retract:

Corollary 5.22. *Let $A \subset B$ be a retract; denote the retraction by r and the inclusion by i. Fix a point $a \in A$. Then $r_* : \pi_1(B, a) \to \pi_1(A, a)$ is surjective and i_* is its inverse.*

Proof. In Top_*, the map r is a left-inverse of i. Since π_1 is a functor, it preserves invertibility (Lemma 1.16), so $r_* = \pi_1(r)$ is a left-inverse in Grp of $i_* = \pi_1(i)$. It follows that the former is surjective and the latter is injective. □

5.4.1 The pointed homotopy category

Observe that:

Lemma 5.23. *Suppose that $f, g : (X, x) \to (Y, y)$ are pointed homotopic. Then $f_* = g_*$.*

Proof. A pointed homotopy F from f to g produces a pointed homotopy $F \circ \gamma$ of γ. □

The lemma implies that:

Corollary 5.24. *We can regard π_1 as a functor $\mathrm{hTop}_* \to \mathrm{Grp}$.*

Proof. The functor takes a morphism $[f]$ in hTop_* to f_*, as defined in Definition 5.20. That this does not depend on the representative f was precisely the content of the previous lemma. That this defines a functor is immediate from the fact that we earlier defined π_1 as a functor with domain Top. □

A more formal way of stating the result is that our starting functor $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$ factors through the quotient functor $\mathrm{Top}_* \to \mathrm{hTop}_*$.

Remark 5.25: You can verify that $[(\mathbb{S}^1, 1), -] : \mathrm{hTop}_* \to \mathrm{Set}$ is the result of composing $\pi_1 : \mathrm{hTop}_* \to \mathrm{Grp}$ with the forgetful functor $\mathrm{Grp} \to \mathrm{Set}$. △

Corollary 5.24 implies that:

Corollary 5.26. *If (X, x) and (Y, y) are pointed homotopy equivalent, their fundamental groups $\pi_1(X, x)$ and $\pi_1(Y, y)$ are isomorphic.*

5.5 Exercises

5.5.1 The circle

For the upcoming exercises you can use the following concepts and statements:

- We define $\gamma_k : \mathbb{S}^1 \to \mathbb{S}^1$ as the map $\gamma_k(z) = z^k$. Here we are using complex coordinates in $\mathbb{S}^1 \subset \mathbb{C}$.
- $[\mathbb{S}^1, \mathbb{S}^1] = \{[\gamma_k] \mid k \in \mathbb{Z}\} \cong \mathbb{Z}$, as sets.
- $\pi_1(\mathbb{S}^1, 1) = \{[\gamma_k] \mid k \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$, as groups.
- In particular, $\pi_1(\mathbb{S}^1, 1)$ is generated by the class of $\gamma_1 = \mathrm{id}_{\mathbb{S}^1}$.

Exercise 5.1: Show that the maps

$$(\gamma_k)_* : [\mathbb{S}^1, \mathbb{S}^1] \to [\mathbb{S}^1, \mathbb{S}^1], \qquad (\gamma_k)_* : \pi_1(\mathbb{S}^1, 1) \to \pi_1(\mathbb{S}^1, 1),$$

are surjective if and only if $k = \pm 1$. Compute their images for all k.

Exercise 5.2: Which homotopy classes in $[\mathbb{S}^1, \mathbb{S}^1]$ can be represented by homotopy equivalences?

Exercise 5.3: Let $\gamma_k : \mathbb{S}^1 \to \mathbb{S}^1$ given by $z \mapsto z^k$, in complex coordinates. For which k is

$$(\gamma_k)_* : \Pi_1(\mathbb{S}^1) \to \Pi_1(\mathbb{S}^1)$$

a groupoid isomorphism?

5.5.2 Pushforward of fundamental group(oids)

Exercise 5.4: Find an example of a space X and two points $p, q \in X$ such that $\pi_1(X, p)$ is not isomorphic to $\pi_1(X, q)$.

Exercise 5.5: Fix a space X, a point $x \in X$, and a class $a \in \pi_1(X, x)$. We let $G \subset \pi_1(X, x)$ be the subgroup generated by a. Find a map $f : \mathbb{S}^1 \to X$ such that its pushforward

$$f_* : \pi_1(\mathbb{S}^1, 1) \to \pi_1(X, x)$$

has G as its image.

Exercise 5.6: Find an example in which $f : A \to B$ induces isomorphisms

$$f_* : \pi_1(A, p) \quad \to \quad \pi_1(B, f(p))$$

for all $p \in A$, but

$$f_* : \Pi_1(A) \quad \to \quad \Pi_1(B)$$

is not surjective.

Exercise 5.7: Let $X \subset Y$ and write ι for the inclusion. Show that $\iota_* : \Pi_1(X) \to \Pi(Y)$ may not be injective itself.

Exercise 5.8: Let $f : X \to Y$ surjective. Show that $f_* : \Pi_1(X) \to \Pi(Y)$ may not be surjective itself.

5.5.3 Groupoids

Exercise 5.9: Let G be a group. Consider a set P and a point $p \in P$.

- Prove that there is a connected groupoid $\mathcal{G} \rightrightarrows P$ with $\mathcal{G}_p \cong G$.
- Prove that $\mathcal{G}$ is unique up to isomorphism.

Hint: Recall that all $\mathcal{G}_{x,y}$ are meant to be isomorphic to each other as sets. For both items it is useful to choose a preferred element in $\mathcal{G}_{p,q}$, for each $q \in P$.

Exercise 5.10: Prove that the coproduct in Grpoid is the disjoint union (at the level of both objects and morphisms).

Change of basepoint and simply-connectedness

Lecture 6

After the previous lecture, there are three issues that we need to understand:

a. Π_1 takes values in Grpoid, which seems to be more complicated than Grp.
b. Moreover, Π_1 has Top as its domain, and not hTop. However, we want to use Π_1 to tell spaces apart up to homotopy equivalence.
c. Similarly, π_1 has hTop_* as its domain, and not hTop.

We will address these during the lecture.

- Our main goal is to explain how π_1 changes as we move the basepoint. We call this the *change of basepoint formula* (Section 6.1).
- We will then show that Π_1 and π_1 preserve products (Section 6.2) and deduce various corollaries.
- We will say that a space is *simply-connected* if it has no holes bounded by loops (Section 6.3). We will characterise this notion using Π_1 and π_1.

6.1 The change of basepoint formulas

We discussed conjugation in groupoids in Section 5.2. We now particularise it to the case of $\Pi_1(X)$.

6.1.1 Conjugation in the fundamental groupoid

In $\Pi_1(X)$, conjugation, pushforward and pullback allow us to relate loops and paths based at different points, as long as said points are path-connected to each other. A pictorial depiction can be found in Figure 6.1.

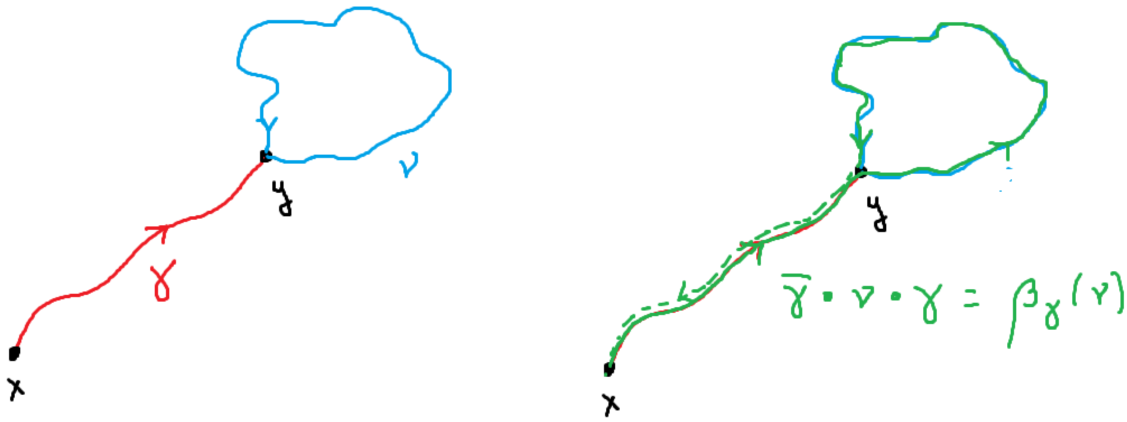


Figure 6.1: Let γ be a path from x to y. We can use the conjugation map $\beta_{[\gamma]} : \pi_1(X, y) \to \pi_1(X, x)$ associated to $[\gamma] \in \pi_1(X, x, y)$ to take classes of loops based at y to classes of loops based at x. On the left we see a loop ν, based at y. On the right we see the conjugated loop $\beta_\gamma(\nu)$, based at x.

Notation 6.1: Let γ be a path with from x to y. Then, the conjugation $\beta_{[\gamma]} : \pi_1(X, y) \to \pi_1(X, x)$ in $\Pi_1(X)$ is often called the **change of basepoint** associated to $[\gamma]$. The same name is also used for the function $\beta_\gamma(\nu) = \bar{\gamma} \bullet \nu \bullet \gamma$ taking loops based at y to loops based at x. △

The crucial corollary in our study of the fundamental group is that:

Corollary 6.2. *Let X be a space. Let $x, y, z, w \in X$ be points in the same path-component. Then:*

- $\pi_1(X, x) \simeq \pi_1(X, y)$ *as groups.*
- $\pi_1(X, x, y) \simeq \pi_1(X, z, w)$ *as sets.*

Proof. Since x and y are in the same path-component there is a path γ from one to the other. I.e. $[\gamma] \in \pi_1(X, x, y)$. It follows that

$$\beta_{[\gamma]} : \pi_1(X, y) \to \pi_1(X, x)$$

is a group isomorphism according to Lemma 5.6. This proves the first claim. The second one is proven similarly using the pullback and the pushforward. □

You should think about it as follows: $\pi_1(X, x)$ is meant to detect the "holes" that are in the path-component of x by looking at loops based at x. If we instead use $\pi_1(X, y)$, this should make no difference, since x and y are isomorphic as objects of $\Pi_1(X)$.

6.1.2 Homotopies of paths not relative to endpoints

We now state the first "change of basepoint formula". It describes how classes relative endpoints vary when we use a homotopy that is itself not relative endpoints.

Theorem 6.3. *Let $F : [0,1]^2 \to X$ be a homotopy of paths (not relative endpoints). Denote $\gamma_s = F(-,s)$ for the path at time s. Write as well $\alpha_s(t) = F(0,ts)$ and $\tau_s(t) = F(1,ts)$.*

Then, $[\gamma_s] \in \pi_1(X, \gamma_s(0), \gamma_s(1))$ relates to $[\gamma_0] \in \pi_1(X, \gamma_0(0), \gamma_0(1))$ by the formula:

$$[\gamma_0] = \overline{[\tau_s]} \bullet [\gamma_s] \bullet [\alpha_s].$$

Proof. We must exhibit a homotopy relative endpoints between γ_0 and $\overline{\tau_s} \bullet (\gamma_s \bullet \alpha_s)$. The idea is shown in Figure 6.2: it amounts to performing a homotopy within $[0,1]^2$.

Consider the paths

$$\beta_0, \beta_1 : [0,1] \to [0,1]^2$$

given by $\beta_0(t) = (t,0)$ and

$$\beta_1 = \overline{(t \mapsto (1,st))} \quad \bullet \quad ((t \mapsto (t,s)) \quad \bullet \quad (t \mapsto (0,st))).$$

These paths are defined so that $\gamma_0 = F \circ \beta_0$ and $\overline{\tau_s} \bullet (\gamma_s \bullet \alpha_s) = F \circ \beta_1$. Furthermore, both paths define classes in $\pi_1([0,1]^2, (0,0), (1,0))$, which consists of a single element, according to Lemma 4.18. This implies that there is a homotopy relative endpoints $G : [0,1]^2 \to [0,1]^2$ with $G(-,0) = \beta_0$ and $G(-,1) = \beta_1$. The argument concludes by setting $F \circ G$ as the desired homotopy relative endpoints. □

We can particularise the theorem to the case of loops:

Corollary 6.4. *Let $F : \mathbb{S}^1 \times [0,1] \to X$ be a homotopy of loops. Denote $\gamma_s = F(-,s)$ for the loop at time s. Write $\alpha_s(t) = F(1,ts)$.*

Then, $[\gamma_s] \in \pi_1(X, \gamma_s(0))$ relates to $[\gamma_0] \in \pi_1(X, \gamma_0(0))$ by the formula:

$$[\gamma_0] = \overline{[\alpha_s]} \bullet [\gamma_s] \bullet [\alpha_s].$$

6.1.3 Change of basepoint for maps

We now use the previous corollary to deduce the second change of basepoint formula:

Theorem 6.5. *Let $F : X \times [0,1] \to Y$ be a homotopy. Denote $f_s : X \to Y$ for the map $f_s := F(-,s)$. Fix a point $x \in X$ and denote $\alpha_s : [0,1] \to X$ for the path $\alpha_s(t) := F(x,st)$. Then:*

$$(f_0)_* = \beta_{[\alpha_s]} \circ (f_s)_* : \pi_1(X,x) \to \pi_1(Y, f_0(x)).$$

Proof. Each map f_s defines its own pushforward $(f_s)_* : \pi_1(X,x) \to \pi_1(Y, f_s(x))$. Since the basepoint $f_s(x)$ in Y moves with s, the pushforwards cannot possibly be the same for all s. However, we will prove that they are related by conjugation.

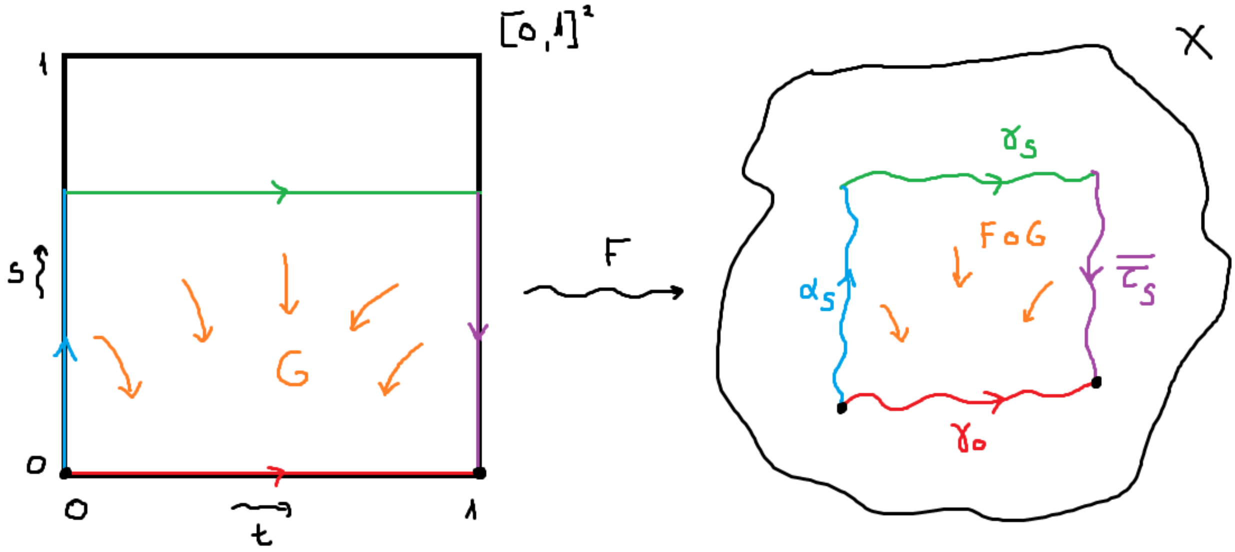


Figure 6.2: The paths described in the proof of Theorem 6.3. Each path (right) is obtained by restricting F to some interval in $[0,1]^2$ (left). The bottom path is homotopic relative endpoints (via $F \circ G$) to the concatenation of the other three, as is apparent from the analogous statement for intervals (via G).

Given any class of loop $[\gamma] \in \pi_1(X, x)$ we can take a representative γ and consider the homotopy of loops $G : \mathbb{S}^1 \times [0,1] \to Y$ defined by $G := F \circ \gamma$. Applying Corollary 6.4 we obtain:

$$\begin{aligned}(f_0)_*([\gamma]) = [f_0 \circ \gamma] &= [\alpha_s] \bullet [f_s \circ \gamma] \bullet [\overline{\alpha_s}] \\ &= \beta_{[\overline{\alpha_s}]} \circ (f_s)_*([\gamma]),\end{aligned}$$

as claimed. □

We can use this to deduce:

Corollary 6.6. *Let $f : X \to Y$ be a homotopy equivalence. Then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism for all $x \in X$.*

Proof. In the upcoming proof it is important that we keep track of the domains and codomains of the pushforwards we compute.

Let $g : Y \to X$ be a homotopy inverse of f. Since $g \circ f$ is homotopic to id_X we deduce (Theorem 6.5) that

$$g_* \circ f_* = (g \circ f)_* : \pi_1(X, x) \to \pi_1(X, g(f(x)))$$

and $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x)}$ are conjugate to each other. In particular, since conjugation is an isomorphism, we deduce that $g_* \circ f_*$ is an isomorphism. It follows that $g_* : \pi_1(Y, f(x)) \to \pi_1(X, g(f(x)))$ is surjective.

We similarly observe that

$$f_* \circ g_* = (f \circ g)_* :: \pi_1(Y, f(x)) \to \pi_1(X, f(g(f(x))))$$

so $g_* : \pi_1(Y, f(x)) \to \pi_1(X, g(f(x)))$ is injective. It follows that it is bijective. In turn, it follows that $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ had to be a bijection as well. ☐

We will use this theorem repeatedly throughout the course. This is the (only?) result that allows us to prove that two spaces X and Y are not homotopy equivalent. Namely, this will be the case if X has a path-component with fundamental group different from the fundamental groups of all path-components of Y.

Moreover, note that this is just a test. If X and Y are homotopy equivalent, you have to prove it by exhibiting a explicit homotopy equivalence.

6.2 The fundamental groupo(oid) preserves products

We now show that Π_1 and π_1 preserve products. Along the way, we discuss products in Grpoid and Top_*.

6.2.1 Product in the category of groupoids

In Grp, the categorical product is the usual set-theoretical product. In groupoids, the same is true:

Proposition 6.7. *Let $\mathcal{G} \rightrightarrows B$ and $\mathcal{G}' \rightrightarrows B'$ be groupoids. Then their product in* Grpoid *is their product as sets, both at the level of objects and morphisms:*

$$\mathcal{G} \times \mathcal{G}' \rightrightarrows B \times B'.$$

Concretely, a morphism in $\mathcal{G} \times \mathcal{G}'$ from $(a, a') \in B \times B'$ to (b, b') is a pair $(f : a \to b, f' : a' \to b')$. Composition is performed componentwise.

Proof. Write π and π' for the functors projecting $\mathcal{G} \times \mathcal{G}'$ to $\mathcal{G}$ and $\mathcal{G}'$, respectively. Consider some other groupoid $\mathcal{H} \rightrightarrows C$ and a pair of functors $F : \mathcal{H} \to \mathcal{G}$ and $F' : \mathcal{H} \to \mathcal{G}'$. We can then define $(F, F') : \mathcal{H} \to \mathcal{G} \times \mathcal{G}'$ to be the (usual, set-theoretical) product map at the level of objects and morphisms. Due to the universal property of the product in Set, (F, F') is the unique function that factorises F and F', meaning $\pi \circ (F, F') = F$ and $\pi' \circ (F, F') = F'$. We just need to check that it is a functor.

First, note that (F, F') defines a function at the object level $C \to B \times B'$ and another at the morphism level. We must show these are compatible in the sense that each morphism $h : x \to y$ in $\mathcal{H}$ must be taken to a morphism $(F, F')(f)$ with source $(F, F')(x)$ and target $(F, F')(y)$. This is readily checked, since $(F, F')(f) = (F(f), F'(f))$ has source $(F(x), F'(x)) = (F, F')(x)$ and target $(F(y), F'(y)) = (F, F')(y)$.

Let h and j be composable morphisms in $\mathcal{H}$. We see that

$$(F, F')(h \circ j) = (F(h \circ j), F'(h \circ j)) = (F(h) \circ F(j), F'(h) \circ F'(j)) = (F, F')(h) \circ (F, F')(j)$$

so (F, F') preserves compositions. Similarly, it preserves identities, so it is a functor. □

Since every group can be regarded as a groupoid, this proposition recovers our claim about the product in Grp.

6.2.2 Fundamental groupoid and products

The following statement is a consequence of the properties of the product topology, as in the proof of Proposition 3.26; details are left to the reader:

Proposition 6.8. *Let $\{X_i\}_{i=1}^n$ be a finite collection of spaces. Then:*

$$\Pi_1(X_i) \simeq \prod_i \Pi_1(X_i).$$

6.2.3 Product in pointed spaces

The following lemma says that Top_* inherits the usual product from Top:

Lemma 6.9. *Let (X, x) and (Y, y) be pointed spaces. Then $(X \times Y, (x, y))$ satisfies the universal property of the product in* Top_*.

Proof. $X \times Y$ is the usual product in Top, so every pair of maps $f : Z \to X$ and $g : Z \to Y$ factors through uniquely via the map $(f, g) : Z \to X \times Y$. It remains to show that this unique factorising map is in fact a morphism in Top_*, i.e. it is pointed. If f and g are pointed, for a choice of basepoint $z \in Z$, $x \in X$, and $y \in Y$, we readily see that (f, g) is also pointed once we set (x, y) to be the basepoint in $X \times Y$. □

6.2.4 Fundamental group and products

The following is then an immediate corollary of Proposition 6.8:

Corollary 6.10. *Let $\{(X_i, x_i)\}_{i=1}^n$ be a finite collection of spaces. Then:*

$$\pi_1(\prod_i (X_i, x_i)) \simeq \prod_i \pi_1(X_i, x_i).$$

We have now come full circle. The theory we have developed allows us to distinguish the tori from one another (compare to Remark 3.28):

Corollary 6.11. *Let $(\mathbb{T}^n, p) = \prod_{i=1}^n (\mathbb{S}^1, 1)$ be the pointed n-dimensional torus. Then:*

- *$\pi_1(\mathbb{T}^n, p) \simeq \mathbb{Z}^n$, as groups.*

- $\mathbb{T}^n$ *and* $\mathbb{T}^m$ *are homotopy equivalent if and only if* $n = m$.

Proof. Note that $(\mathbb{T}^n, p)$ and $(\mathbb{T}^n, q)$ are homeomorphic for any two choices of basepoint p and q. This follows as in Example 4.9: the translations are homeomorphisms and allow us to move p to q. This implies that $\pi_1(\mathbb{T}^n, p) \simeq \pi_1(\mathbb{T}^n, q)$, so we can compute the fundamental group at any point of the torus. The first statement then follows from the fact $\pi_1(\mathbb{S}^1, 1) \simeq \mathbb{Z}$ and Corollary 6.10, by considering the basepoint $(1, 1, \cdots, 1)$.

The second statement follows from the first. Homotopy equivalent spaces have isomorphic fundamental groups (Corollary 6.6) and $\mathbb{Z}^n$ and $\mathbb{Z}^m$ are not isomorphic if $n \neq m$ (Lemma 6.12 below). □

In the proof of the corollary we used two crucial facts. One from Group Theory:

Lemma 6.12. *The additive groups* $\mathbb{Z}^n$ *and* $\mathbb{Z}^m$ *are not isomorphic if* $n \neq m$.

In the next chapter we will look more closely at groups and prove this statement (Lemma 8.4).

And the following group version of Theorem 3.2:

Theorem 6.13. $\pi_1(\mathbb{S}^1, 1) \simeq \mathbb{Z}$, *as groups.*

Which we have stated repeatedly but not proven yet. It will be established in Section 10.3.

6.3 Simply-connectedness

The aim of this course is to use loops to detect holes in topological spaces. However, some spaces have no "holes detectable by loops":

Definition 6.14. *Let X be a space. A space is **simply-connected** if the following conditions hold:*

- *X is path-connected.*
- $\pi_1(X, x) = \{[c_x]\}$ *for some* $x \in X$.

The following lemma provides some alternate characterisations:

Lemma 6.15. *The following are equivalent for a non-empty space X:*

a. *X is simply-connected.*
b. *X is path-connected and* $\pi_1(X, x) = \{[c_x]\}$ *for all* $x \in X$.
c. $\pi_1(X, x, y) \simeq \{.\}$ *for all* $x, y \in X$.
d. $\Pi_1(X)$ *is isomorphic to the pair groupoid* $X \times X \rightrightarrows X$.
e. $[\mathbb{S}^1, X] \simeq \{.\}$.

Proof. (a) implies (b). Indeed, all fundamental groups in a component are isomorphic to one another, so triviality of one of them implies triviality of all. The reverse implication is obvious.

(a) implies (c). Due to path-connectedness, all morphism sets in $\Pi_1(X)$ are isomorphic and thus consist of a single element. For the converse, note that all morphism sets in $\Pi_1(X)$ being non-empty implies that X is path-connected. Morphism sets consisting of a single element gives in particular that the fundamental group at each basepoint is trivial.

(c) implies (d). Indeed, assumption (c) says that all morphism sets in $\Pi_1(X)$ consist of a single element. The unique morphism $[\gamma] \in \pi_1(X, x, y)$ is then identified with the morphism (x, y) in $X \times Y$. This is readily seen to be a bijection $\Pi_1(X) \to X \times X$ which, by uniqueness of morphism sets, automatically preserves composition and inverses and is thus a functor. The converse is im

(b) implies (e). Take a class $[\gamma] \in [\mathbb{S}^1, X]$ and note that γ also represents a class in $\pi_1(X, \gamma(1))$. Item (b) implies that this class is trivial, meaning that γ is nullhomotopic using a homotopy relative endpoints. In particular, it is nullhomotopic without fixing endpoints.

(e) implies (a). Recall that there is an injective map $\pi_0(X) \to [\mathbb{S}^1, X]$ sending $[x]$ to the class of the constant map $[c_x]$. Item (e) then implies $\pi_0(X) \simeq \{.\}$, meaning that X is path-connected. Consider now a class $[\gamma] \in \pi_1(X, x)$ and regard it as a class in $[\mathbb{S}^1, X]$. I.e. we take the loop $\gamma : \mathbb{S}^1 \to X$, which is based at $x \in X$, and now we allow homotopies $F : \mathbb{S}^1 \times [0,1] \to X$ which are not relative endpoints (i.e. $F(0, s)$ may not be x). Consider the path $\alpha : [0,1] \to X$ given by $\alpha(t) = F(1, t)$.

The assumption $[\mathbb{S}^1, X] \simeq \{.\}$ says that there is a homotopy F between γ and a constant loop c_z. Using the change of basepoint formula (Theorem 6.3) we then deduce

$$[\gamma] = \overline{[\alpha]} \bullet [c_z] \bullet [\alpha] = \overline{[\alpha]} \bullet [\alpha] = [c_x].$$

Since this is true for all homotopy classes in $\pi_1(X, x)$, simply-connectedness follows. □

The prototypical examples are the spaces that have no holes at all:

Corollary 6.16. *A contractible space X is simply-connected.*

Proof. According to Lemma 3.1, $[\mathbb{S}^1, X] = \{.\}$, which implies the result thanks to the criteria in Lemma 6.15. □

There are spaces that are simply-connected but not contractible. All the higher spheres $\mathbb{S}^n$, $n > 1$, are examples. We will prove that they are simply-connected in Section 9.4.1) but we will not be able to prove that they are not contractible in this course. The intuition is that each $\mathbb{S}^n$ has higher-dimensional holes not detectable via loops.

6.4 Exercises

6.4.1 Simply-connectedness

Exercise 6.1: Let A be a non-empty topological space. Prove that $A \times \mathbb{S}^1$ is not simply-connected.

Exercise 6.2: Let X be simply-connected space. Let $A \subset X$ be a retract. Prove that A is also simply-connected.

Exercise 6.3: Let $f : X \to Y$ be a map. Suppose that $f(X) \subset A \subset Y$, where A is a subspace that is simply-connected. Deduce that $f_* : \pi_1(X, p) \to \pi_1(Y, f(q))$ is the zero homomorphism (it sends all elements to the identity).

6.4.2 Change of basepoint

Exercise 6.4: Fix a space X and a point $x \in X$. Regard $\mathbb{S}^1$ as the quotient $[0, 1]/(0 \cong 1)$. A loop $\gamma : \mathbb{S}^1 \to X$ satisfying $\gamma(0) = x$ defines a class both in $\pi_1(X, x)$ and in $[\mathbb{S}^1, X]$. Forgetting the condition $\gamma(0) = x$ therefore provides a function $\Psi : \pi_1(X, x) \to [\mathbb{S}^1, X]$. Prove that:

a. Ψ is surjective if and only if X is path-connected.
b. Ψ is injective if and only if $\pi_1(X, x)$ is abelian.

Hint: You may want to use Exercise 2.5.

Groups

Lecture 7

In our previous lecture we studied the change of basepoint formula. It basically told us that $\Pi_1(X)$ contains a lot of redundant information (many isomorphic fundamental groups and morphism sets).

Separately, we proved that π_1 preserves the product, which allowed us to compute the fundamental group of the n-dimensional torus. Trying to compare $\mathbb{Z}^n$ to $\mathbb{Z}^m$, as groups, made us realise that we need to understand groups better. We cannot expect to say very meaningful things about hTop by looking at Grp if we do not understand Grp properly!

In this lecture:

- We will talk about *equivalence of categories*, which will allow us to formalise the claim that $\Pi_1(X)$ contains redundancies (Section 7.1).
- We will discuss *group presentations* (Definition 7.15), which are a very common and handy way of defining and manipulating groups.
- We will introduce the *abelianisation* functor (Definition 7.34), which assigns an abelian group to each group. The idea is that the former are easier to compare than the latter.

When introducing various group theoretical concepts we will be practical: we will sometimes opt for non-intrinsic definitions whenever these are more useful for explicit computations. The interested reader may want to refer to the "extra material" in Appendix 7.6, where things are done intrinsically.

7.1 Equivalence of categories

In a category, we may have many different objects that are isomorphic to one another. If we were to remove some of them, yielding a smaller category, we would not lose information. That is the content of the following definition:

Definition 7.1. *An* ***equivalence of categories*** *is a functor $F : \mathcal{C} \to \mathcal{D}$ that satisfies:*

- ***Essential surjectivity:*** *For every object $d \in \mathcal{D}$ there is $c \in \mathcal{C}$ such that d is isomorphic to $F(c)$.*
- ***Fullness and faithfulness:*** *$F : \mathrm{Hom}_{\mathcal{C}}(c, c') \to \mathrm{Hom}_{\mathcal{D}}(F(c), F(c'))$ is a bijection for all objects $c, c' \in \mathcal{C}$.*

A particularly important example for us will be:

Definition 7.2. *Let $\mathcal{C}$ be a category. A **skeleton** of $\mathcal{C}$ is a subcategory $\mathcal{D}$ satisfying:*

- *The inclusion $\mathcal{D} \to \mathcal{C}$ is an equivalence.*
- *If a and b are isomorphic objects in $\mathcal{D}$, then $a = b$.*

That is, for each isomorphism class of objects in $\mathcal{C}$, $\mathcal{D}$ contains exactly one representative. By taking $\mathcal{D}$, we have removed redundancies in $\mathcal{C}$.

Example 7.3: Let $\mathrm{Vect}_{\mathbb{R}}^{\mathrm{fin}}$ be the category of finite dimensional vector spaces over $\mathbb{R}$, with linear maps as morphisms. Then the subcategory whose objects are $\{\mathbb{R}^n\}_n$ and whose morphisms are all linear maps is a skeleton. This follows from the fact that every finite dimensional vector space is isomorphic to $\mathbb{R}^n$, for some n. △

The previous example hints already at a general statement:

Lemma 7.4. *Every category $\mathcal{C}$ has a skeleton.*

Proof. The proof uses the axiom of choice. The class of objects $\mathrm{Ob}(\mathcal{C})$ can be partitioned into the equivalence classes given by isomorphism of objects. The axiom of choice tells us that we can select one object per equivalence class. The collection of these will be $\mathrm{Ob}(\mathcal{D})$. The morphisms in $\mathcal{D}$ are then all the morphisms in $\mathcal{C}$ involving objects in $\mathrm{Ob}(\mathcal{D})$. □

7.1.1 Skeleta for the fundamental groupoid

The example that is most important to us is the following:

Corollary 7.5. *Let X be a space and $\Pi_1(X)$ its fundamental groupoid. Choose a collection of points $\{x_i \in X\}_i$ such that each path-component of X is represented by exactly one x_i. Then $\coprod_i \pi_1(X, x_i)$ is a skeleton of $\Pi_1(X)$.*

Corollary 7.6. *If X is path-connected, $\Pi_1(X)$ is equivalent to the group $\pi_1(X, x)$, for any point x.*

Then:

Lemma 7.7. *Let $f : X \to Y$ be a homotopy equivalence. Then $f_* : \Pi_1(X) \to \Pi_1(Y)$ is a equivalence of groupoids.*

Proof. The map f provides a bijection $\pi_0(X) \cong \pi_0(Y)$. This implies that the functor f_* induces similarly a bijection of components $\pi_0(\Pi_1(X)) \cong \pi_0(\Pi_1(Y))$. This proves essential surjectivity of f_*.

Moreover, f_* provides an isomorphism at the level of isotropy groups. From this, using conjugation, it follows that f_* also provides a bijection between all morphism sets $\pi_1(X, x, x') \cong \pi_1(Y, f(x), f(x'))$, i.e. fullness and faithfulness. □

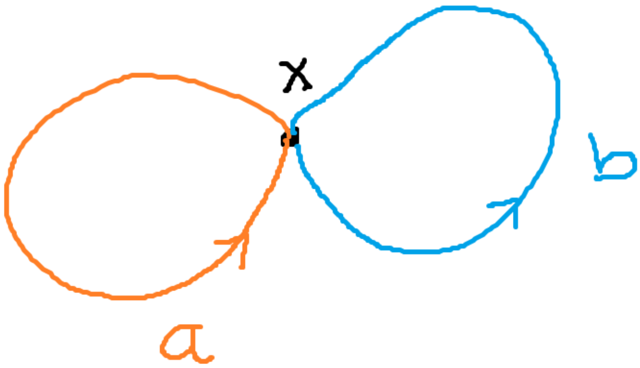


Figure 7.1: The wedge of two circles $(\mathbb{S}^1, 1) \vee (\mathbb{S}^1, 1)$. A general definition of the wedge product will appear in Corollary 8.20.

You should think of this as saying that Π_1 is a functor from hTop to some sort of "homotopy category" of groupoids in which equivalences become isomorphisms.

7.2 Free groups

Now we turn our attention to groups. To motivate the upcoming definitions, consider the following examples:

Example 7.8: We have argued heuristically that $\pi_1(\mathbb{S}^1, 1) \simeq (\mathbb{Z}, +)$, as groups. Concretely, the class of the identity map $a = [\mathrm{id}_{\mathbb{S}^1}] \in \pi_1(\mathbb{S}^1, 1)$ is a generator for the group, meaning that all the elements of $\pi_1(\mathbb{S}^1, 1)$ are of the form a^k, for some $k \in \mathbb{Z}$. The class a^k is represented by the loop $z \mapsto z^k$, that turns k times counterclockwise around the circle. We can write $(\mathbb{Z}, +) \simeq \langle a \mid \rangle$, denoting precisely "the group generated by the symbol a". △

Example 7.9: The space appearing in Figure 7.1 is the wedge of two circles $(X, x) = (\mathbb{S}^1, 1) \vee (\mathbb{S}^1, 1)$. The inclusion of the leftmost circle is a loop, and we denote its class by $a \in \pi_1(X, x)$. The same is true for the rightmost circle, whose class we denote by $b \in \pi_1(X, x)$. We can now start concatenating these two classes. For instance, we could write ba for the class where we perform a first and then b. We could write aba^2 for running a twice, then b, then a again. If we write $a^{-1}a$ we are running a and then its reverse, which is its inverse. $a^{-1}a$ is then the same as the class of the constant loop $[c_x]$, which we often denote by $\emptyset$ (meaning the word that contains no as nor bs). The claim, which will be proven in Corollary 8.30, is that $\pi_1(X, x)$ is the group $\langle a, b \mid \rangle$, the group whose elements are words written with the symbols a and b and whose operation is the concatenation of words. I.e. a concatenated with b is ba. △

With this intuition at hand, we can now introduce:

Definition 7.10. *Let I be a set.*

- *We consider **words** written in the alphabet $\{g, g^{-1}\}_{g\in I}$.*
- *Let ω and ω' be words and let $g \in I$ be a letter. We say that the words $\omega g g^{-1}\omega'$ and $\omega g^{-1} g \omega'$ are related to $\omega\omega'$ by **move I**.*

*Then, the **free group** with I generators is denoted by $*_I\mathbb{Z}$ and is defined by:*

- *Its elements are equivalence classes of words.*
- *We consider the equivalence relation generated by move I.*
- *The group multiplication of two equivalence classes is given by concatenating representatives.*

Lemma 7.11. *The group multiplication in $*_I\mathbb{Z}$ is well-defined. The identity is represented by the empty word. The inverse of an element represented by $g_n \cdots g_1$ is represented by $g_1^{-1} \cdots g_n^{-1}$.*

Proof. Let W be the collection of all words, before quotienting by move I. Concatenation is a well-defined operation in W, with identity given by the empty word. The only element with an inverse is the empty word.

We then have the quotient $W \to *_I\mathbb{Z}$. We just need to verify that the multiplication is compatible with the quotient. Consider then two words ω and ω' with concatenation $\omega\omega'$. Suppose we now replace ω' by ω'' using move I. Then it follows that $\omega\omega'$ and $\omega\omega''$ are similarly related by move I. Reasoning similarly for ω shows that the multiplication descends to the quotient, since move I generates the equivalence relation. We conclude by noting that move I has introduced the claimed inverses. □

From now on, we always write elements of $*_I\mathbb{Z}$ as words, without indicating that we are taking equivalence classes.

Let us revisit Example 7.9:

Example 7.12: Suppose I consists of two elements. Then we write $*_I\mathbb{Z} = \mathbb{Z} * \mathbb{Z}$ for the corresponding free group with two generators. Let us write $g_1 = a$ and $g_2 = b$ for them. Some words using the alphabet $\{a, b\}$ are for instance ab, aba^{-1}, or $aaaaaaaaab$. Something we do in this situation for notational convenience is to use exponents when a generator appears repeatedly. That is, we may write a^9b instead of $aaaaaaaaab$. We can then consider an example of composition:

$$ab^2 \cdot b^{-1}a^3b = ab^2b^{-1}a^3b = aba^3b$$

where in the second identity we cancelled one b with its inverse (move II). △

7.3 Group presentations

Every non-identity element in the free group $*_I\mathbb{Z}$ has infinite order. You can see this simply by writing the element as a word ω and then considering the elements ω^k. The set of all of them forms a subgroup isomorphic to $\mathbb{Z}$. However, you have also encountered groups in which some or even all the elements have finite order. This is certainly the case in finite groups:

Example 7.13: The easiest example is the cyclic group of order n, which we write as

$$\mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n \rangle \simeq_{\mathrm{Set}} \{\mathrm{id}, a, a^2, \cdots, a^{n-1}\}.$$

The notation in the middle says: "we take the group with a single generator a, but we impose that $a^n = \mathrm{id}$ is the identity". △

Example 7.14: Another example would be the dihedral group of order n:

$$\mathbb{D}_n = \langle a, b \mid a^n, b^2, (ab)^2 \rangle \simeq_{\mathrm{Set}} \{\mathrm{id}, a, a^2, \cdots, a^{n-1}, b, ab, a^2b, \cdots, a^{n-1}b\}.$$

This notation says "we take the group with a two generators a and b, but we impose that a^n, b^2, and $(ab)^2$ are the identity". It follows that a has order n, b has order 2, and their product ab has also order 2. △

The general recipe behind these concrete examples is that:

Definition 7.15. *Let I be a set. Let R be a set of words in written in the alphabet $I \cup I^{-1}$. Then we say that:*

- *Two words are related by* move II *if one of them is of the form $\omega r^{\pm 1}\omega'$ and the other of the form $\omega\omega'$, where ω and ω' are arbitrary words and r is a word in R.*

*The group with **generators** I and **relations** R, denoted by*

$$\langle I \mid R \rangle = \langle g \in I \mid r \in R \rangle,$$

is defined by:

- *Its elements are equivalence classes of words.*
- *We consider the equivalence relation generated by moves I and II.*
- *The group multiplication of two equivalence classes is given by concatenating representatives.*

As in the case of the free group: the group multiplication is well defined, the identity is the empty word, and the inverse of $g_n \cdots g_1$ is $g_1^{-1} \cdots g_n^{-1}$.

It turns out that we do not lose any generality by focusing on groups of this form.

Lemma 7.16. *Every group H has a presentation.*

Proof. Let S be the set underlying H. Then we can consider the free group $F = *_S\mathbb{Z}$. There is a surjective group homomorphism $\Psi : F \to H$ that sends each generator of F (i.e. an element of S) to the corresponding element in H. If we write $K \subset F$ for the kernel of Ψ, the first isomorphism theorem tells us that $H \simeq F/K$. It follows that H is presented as $\langle S \mid K \rangle$. □

As seen in the Lemma, presentations could a priori have very large cardinality, but we want to focus on groups that are "not too big":

Definition 7.17. *Let H be a group. We say that:*

- $\langle I \mid R\rangle$ *is a* ***presentation*** *of H if the two are isomorphic.*
- *H is* ***finitely generated*** *if it has a presentation in which I is a finite set.*
- *H is* ***finitely presented*** *if it has a presentation in which I and R are both finite sets.*

Remark 7.18: Any given H has many different presentations (meaning different choices of generators I and relations R). For instance, $\mathbb{Z}$ can be presented as $\langle a \mid \rangle$, which is its usual presentation, but also as $\langle a, b \mid b\rangle$, which has a redundant generator that is then part of the relations. It turns out that this gets us into a computability issue: there is no algorithm taking as input two presentations and producing as output whether they yield isomorphic groups[a]. △

[a] For two concrete presentations we can often manipulate the two and see whether they produce isomorphic groups, but there is no algorithm that works for whatever two inputs you give.

7.4 Group homomorphisms in terms of presentations

Presentations provide for us a handy way of writing group homomorphisms. First note:

Lemma 7.19. *Consider groups $G = \langle I \mid R\rangle$ and H. Then, every group homomorphism $\phi : G \to H$ is uniquely determined by the values of the generators $\{\phi(g)\}_{g\in I}$.*

Proof. Any element in G can be expressed as a word ω written in terms of generators. Replacing every appearance of a generator $g \in I$ by $\phi(g)$ and composing (as elements of H) yields $\phi(\omega)$. □

Conversely:

Corollary 7.20. *Consider groups $G = \langle I \mid \rangle = *_I\mathbb{Z}$ and H. Let $\{v_g \in H\}_{g\in I}$. This uniquely defines a group homomorphism*

$$\Phi : G \to H$$

by setting $\Phi(g) = v_g$.

Proof. Since any word ω in G can be written in terms of generators, reasoning as in the lemma tells us that $\phi(\omega)$ is obtained by replacing each generator g in ω by v_g, and then composing all these elements in H. This defines Φ on words, but we need to show that this does not depend on representatives. This follows from the fact that gg^{-1} is sent to $v_g(v_g)^{-1} = \mathrm{id}_H$. I.e. Φ is invariant under move I and thus well-defined after quotienting by the relation it defines. □

In particular:

Corollary 7.21. *Let $G = \langle I \mid R\rangle$. Then, there is a canonical projection homomorphism $\pi : *_I\mathbb{Z} \to G$.*

Proof. π sends the generators I of the free group to themselves, as elements of G. □

We can use this to deduce a similar statement holds for general groups, as long as we are careful with the relations:

Lemma 7.22. *Consider groups $G = \langle I \mid R \rangle$ and H. Let $\{v_g \in H\}_{g \in I}$ and consider the unique group homomorphism $\Phi : *_I \mathbb{Z} \to H$ that they define. Then, the following are equivalent:*

- *There is a unique group homomorphism $\phi : G \to H$ such that $\phi \circ \pi = \Phi$.*
- *The relations in $R \subset *_I \mathbb{Z}$ are in the kernel of Φ.*

Proof. ϕ is uniquely defined on words from the collection $\{v_g \in H\}_{g \in I}$ (or, equivalently, from Φ). It takes concatenation to composition by construction. It remains to show that it is well-defined on equivalence classes of words. This is equivalent to proving that it is invariant under move II. This is in turn equivalent to R being in the kernel of Φ. □

7.5 Abelianisation

The discussion in Remark 7.18 says that groups can be very complicated and difficult to tell apart. This is not very handy for us, since we want to be able to tell groups apart in order to tell spaces apart (thanks to the π_1 functor). The story simplifies considerably for abelian groups, so our goal now is to produce a functor

$$\mathfrak{Ab} : \mathrm{Grp} \to \mathrm{Ab},$$

the idea being that we may be able to tell groups apart by telling the associated abelian groups apart.

7.5.1 Abelian groups

Recall:

Definition 7.23. *A group H is **abelian** if $ab = ba$ holds, for every two elements $a, b \in H$.*

This equation can be rewritten by grouping all terms together, yielding $b^{-1}a^{-1}ba = \mathrm{id}$. The expression $b^{-1}a^{-1}ba$ is called the **commutator** of a and b and is denoted as $[a, b]$.

Example 7.24: Consider the abelian group $\mathbb{Z}^n$. Its generators are the elements $\{a_i\}$, with a_i the vector that has all entries zero except for the ith one, which is 1. Observe that performing addition is done entry by entry in an independent way so indeed these generators commute. It follows that $\mathbb{Z}^n$ can be presented as

$$\langle a_1, \cdots, a_n \mid [a_i, a_j] \text{ for all } i, j \rangle$$

and under this isomorphism the element $(c_1, \cdots, c_n) \in \mathbb{Z}^n$ can be written as $a_n^{c_n} \cdots a_1^{c_1}$. Do note that we are replacing the "additive" notation for the operation in $\mathbb{Z}^n$ by the "multiplicative" notation using words. △

Example 7.25: The cyclic group $\mathbb{Z}/n\mathbb{Z}$ is abelian. △

The following is a fundamental result in Group Theory, which we will just take as a black box without proof:

Theorem 7.26. *Suppose A is a finitely generated abelian group. Then A is isomorphic to*

$$\mathbb{Z}^{a_0} \oplus (\bigoplus_{i\in\mathbb{N}} (\mathbb{Z}/i\mathbb{Z})^{a_i})$$

for some choice of non-negative integers $\{a_i\}$ (with only finitely many of them non-zero).

7.5.2 Abelianisation

The following is a recipe that produces abelian groups from groups:

Definition 7.27. *Let G be a group with presentation $\langle I \mid R\rangle$. Then, its* ***abelianisation*** *is the group*

$$\mathfrak{Ab}(G) := \langle I \mid R, [g_i, g_j] \text{ for all generators } g_i, g_j \in I \ \rangle.$$

Using Lemma 7.22 we see that there is a quotient homomorphism

$$\pi_{\mathfrak{Ab}} : G \to \mathfrak{Ab}(G)$$

that sends each generator in I to its equivalence class in the abelianisation.

Remark 7.28: Lemma 7.27 is handy, but not elegant. It defines the abelianisation using a presentation as an auxiliary piece of data. Corollary 7.32 shows that the result does not depend on the choice of presentation. In Subsection 7.6.2 we discuss the abelianisation in more intrinsic terms. △

Example 7.29: The abelianisation of the free group $*_I\mathbb{Z}$ is $\mathbb{Z}^I$. To see this, observe that we can order I and then prove that all words in $\mathfrak{Ab}(*_I\mathbb{Z})$ are equivalent (via move II applied to the commutatation relations) to a word in the following standard form: $g_n^{a_n}\cdots g_1^{a_1}$. This word we identify with the tuple $(a_1, \cdots, a_n)$ in $\mathbb{Z}^I$. △

Example 7.30: The abelianisation of the dihedral group D_n is:

$$D_n/[D_n, D_n] = \langle a, b \mid a^n, b^2, (ab)^2, [a, b]\rangle \simeq \langle a, b \mid a^n, a^2, b^2, [a, b]\rangle$$

Indeed, once we introduce the commutator as a relation, it holds that:

$$\mathrm{id} = (ab)^2 = abab = a^2b^2 = a^2$$

where we used that a and b now commute in the third equality. Now there are two situations. If n is even, we have obtained the group

$$\langle a, b \mid a^2, b^2, [a, b]\rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2$$

but if n is odd, we have that $a^n = a^2 = \mathrm{id}$ implies that $a = \mathrm{id}$. This means that we obtain:

$$\langle a, b \mid a, b^2, [a, b]\rangle \simeq \langle b \mid b^2\rangle \simeq \mathbb{Z}/2\mathbb{Z}$$

instead. △

7.5.3 Abelianisation as a functor

The abelianisation process also acts on group homomorphisms, as follows:

Lemma 7.31. *Let $\phi : G = \langle I \mid R \rangle \to H$ be a group homomorphism. Then, there is a unique group homomorphism $\mathfrak{Ab}(\phi) : \mathfrak{Ab}(G) \to \mathfrak{Ab}(H)$ such that the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ \downarrow & & \downarrow \\ \mathfrak{Ab}(G) & \xrightarrow[\mathfrak{Ab}(\phi)]{} & \mathfrak{Ab}(H) \end{array}$$

Proof. We have a homomorphism $\Phi : *_I \mathbb{Z} \to \mathfrak{Ab}(H)$ given by $\pi_{\mathfrak{Ab}} \circ \phi|_I$. This uniquely defines $\mathfrak{Ab}(\phi)$, since it prescribes what it must do on generators. It remains to show that this definition is well-defined.

According to Lemma 7.22, it suffices to check that the relations of $\mathfrak{Ab}(G)$ are in its kernel. For R, this is true by definition. For the commutators $[g_i, g_j]$, this follows from the fact that

$$\phi([g_i, g_j]) = [\phi(g_i), \phi(g_j)] = 0,$$

since ϕ was a homomorphism. The commutativity of the diagram follows by construction. □

Thus:

Corollary 7.32. *The abelianisation of a group is independent of the presentation, up to isomorphism.*

Proof. The identity isomorphism from G to itself can be seen as a homomorphism between the two presentations

$$\phi : \langle I \mid R \rangle \to \langle J \mid S \rangle.$$

By the lemma, it descends to a homomorphism between the abelianisations. So does the inverse, showing that the abelianisations are isomorphic. □

If we assume that H is abelian, we also obtain the following factorisation property:

Corollary 7.33. *Let $\phi : G \to H$ be a group homomorphism, with H abelian. Write $\psi : G \to \mathfrak{Ab}(G)$ for the canonical homomorphism. Then, there is a unique map*

$$\mathfrak{Ab}(\phi) : \mathfrak{Ab}(G) \to H$$

such that $\phi = \mathfrak{Ab}(\phi) \circ \psi$.

This is the so-called *universal property of the abelianisation.* You can think of it as follows: The functors $\mathrm{Hom}_{\mathrm{Grp}}(G, -)$ and $\mathrm{Hom}_{\mathrm{Grp}}(\mathfrak{Ab}(G), -)$ are equivalent when evaluated on Ab; i.e. the abelian groups cannot tell G and $\mathfrak{Ab}(G)$ apart by looking at homomorphisms coming from G and from $\mathfrak{Ab}(G)$.

Definition 7.34. *The* ***abelianisation functor*** $\mathfrak{Ab} : \mathrm{Grp} \to \mathrm{Ab}$ *is defined as:*

- *Given a group G, its image is its abelianisation $\mathfrak{Ab}(G)$.*
- *Given a group homomorphism $\phi : G \to H$, its image is $\mathfrak{Ab}(\phi) : \mathfrak{Ab}(G) \to \mathfrak{Ab}(H)$.*

The uniqueness in Lemma 7.31 shows that identities are indeed mapped to identities by $\mathfrak{Ab}$ and similarly for compositions. I.e. we have indeed defined a functor[1].

7.5.4 Conjugation

Recall that the conjugation in a group G by a group element g is the group isomorphism $\beta_g : G \to G$ given by $h \mapsto g^{-1}hg$. The following says that conjugation is trivial in the abelian setting:

Lemma 7.35. *The abelianisation $\mathfrak{Ab}(\beta_g) : \mathfrak{Ab}(G) \to \mathfrak{Ab}(G)$ is the identity.*

Proof. We compute:

$$(\mathfrak{Ab}(\beta_g))([h]) = [g^{-1}][h][g] = [g^{-1}][g][h] = [h],$$

where $[h] \in \mathfrak{Ab}(G)$ denotes the class of $h \in G$ in the abelianisation. In the second step we used that classes in the abelianisation commute. Since the statement holds for all $[h] \in \mathfrak{Ab}(G)$, the claim follows. □

7.6 Appendix: Extra details

We now revisit the previous concepts from a more "intrinsic" lens.

The upcoming contents will not be necessary for the remainder of this course.

7.6.1 Presentations as quotients of the free group

Given a group presentation $G = \langle I \mid R \rangle$, all the words related to the empty word by moves I and II become trivial. These words form a subgroup of the free group, as we now explain.

Definition 7.36. *Let H be a group and let $R \subset H$ be a collection of elements. Then, the* ***normal subgroup*** *$N_R \subset H$ generated by R consists of those elements in H of the form*

$$h_n r_n^{\pm 1} h_n^{-1} \cdots h_1 r_1^{\pm 1} h_1^{-1}$$

for some n, with r_i in R and h_i in H.

[1] Do observe that this functor is somewhat different than the previous ones we have encountered. It takes the category Grp to the subcategory Ab and, in doing so, we also obtain a canonical morphism from G to its image $\mathfrak{Ab}(G)$. This is a particular case of a general phenomenon called **localisation/reflection**. Concretely, Ab is said to be a reflective subcategory of Grp and $\mathfrak{Ab}$ is said to be a localisation.

We leave it to the reader to verify that this is a subgroup and that it is normal.

Lemma 7.37. *$N_R \subset H$ is the smallest normal subgroup containing R.*

Proof. We have to show that N_R is contained in any normal subgroup N that contains R. Being a subgroup, N contains r and r^{-1}, for each $r \in R$. By normality, N then contains any conjugate $hr^{\pm 1}h^{-1}$. By the subgroup condition, it contains all compositions of conjugates. I.e. it contains N_R. □

Then:

Lemma 7.38. *The group $G = \langle I \mid R \rangle$ is the quotient of the free group $F = \langle I \mid \rangle$ by N_R.*

Proof. Observe that we have a group homomorphism $\phi : F \to G$, taking each generator in F to the corresponding generator in G. Let $K \subset F$ be the kernel. Each word in R is, by construction, contained in K. Since the kernel is always normal, we deduce that the smallest normal subgroup containing R is contained in K. This shows $N_R \subset K$.

For the converse, let ω be a word in K. We have to show that $\omega \in N_R$, i.e. it can be written as a product of conjugates s_i of elements in $R \cup R^{-1}$. Since $\phi(\omega)$ is the identity in G, we deduce that it is related to the empty word by a sequence of moves I and II. It is thus enough if we prove that $\omega \in N_R$ if and only if $\omega' \in N_R$, whenever ω' and ω are related by a single move.

For move I there is nothing to check: if ω and ω' relate by move I, they are the same element in F.

For move II, assume without loss of generality that $\omega' \in N_R$. We want to prove $\omega \in N_R$. Let us argue first in the concrete case in which $\omega' = s_1 = \eta_1 r_1^{-1} \eta_1^{-1}$ consists of a single conjugate. Decompose $\eta_1 = \eta'\eta''$ and suppose some $r^{\pm 1}$ is inserted between η' and η'' (the other case is similar). Then we can write:

$$\begin{aligned}\omega = \eta' r^{\pm 1} \eta'' r_1^{-1} \eta^{-1} 1 &= \eta' r^{\pm 1} (\eta'^{-1}\eta')\eta'' r_1^{-1} \eta_1^{-1} \\ &= (\eta' r^{\pm 1} \eta'^{-1})(\eta'\eta'' r_1^{-1} \eta_1^{-1}) = (\eta' r^{\pm 1} \eta'^{-1})\omega'\end{aligned}$$

so ω is a product of conjugates and thus an element of N_R.

For the general case write ω' as a product of conjugates $s_i = \eta_i r_i^{\pm 1} \eta_i^{-1}$. Move II means that $\omega' = \beta\alpha$ and $\omega = \beta r^{\pm 1} \alpha$. If $\alpha = s_l \cdots s_1$ for some l, we are done, because it is then an element of N_R, so is β, and thus so is ω. Assuming otherwise means that $r^{\pm 1}$ has been inserted in the middle of some s_i. This reduces the proof to the particular case in which we apply move II to a single term, as done above. □

7.6.2 Abelianisation

We now explain how to define the abelianisation without the use of presentations.

Definition 7.39. *The **commutator subgroup** $[G, G]$ of G is the normal subgroup generated by all the commutators.*

You may want to prove that an element in $[G, G]$ is a product of commutators. This follows from the fact that the conjugation of a commutator is the commutator of the conjugations. As a follow-up exercise, you should find an example in which $[G, G]$ contains an element that is not a commutator itself.

Definition 7.40. *Let G be a group. Its **abelianisation** is the group $\mathfrak{Ab}(G) := G/[G, G]$.*

Do note that the abelianisation is indeed abelian, since we have quotiented all commutators.

Definition 7.41. *The two definitions of abelianisation we have given are equivalent.*

Proof. Write $G = \langle I \mid R\rangle$ and denote the elements of I by g_i. The commutator subgroup of G is, by definition, generated by all commutators. In particular, it contains the elements $c_{i,j} = [g_i, g_j]$. We claim that these generate $[G, G]$. I.e. any other commutator $[w, z]$, where w and z are words, is a composition of elements in $\{c_{i,j}\}_{i,j}$. We can equivalently claim that $[G, G]/C$ is trivial, where C is the normal subgroup of $[G, G]$ generated by the $\{c_{i,j}\}_{i,j}$.

Indeed, consider $[w, z] \in [H, H]/C$[2]. Both w and z are words in the alphabet I. Since $c_{1,j}$ has been quotiented out, we can move all appearances of g_1 to the left of the word $[w, z]$. However, since $[w, z] = z^{-1}w^{-1}zw$, each appearance of g_1 in w gets compensated by an appearance of g_1^{-1} in w^{-1}. The same reasoning applies to z and z^{-1}. This implies that $[w, z] \in [G, G]/C$ can be written as a word with no appearances of g_1. We can then do induction on the g_i and deduce that $[w, z] \in [G, G]/C$ is the trivial word. This means that $[G, G] = C$.

This reasoning implies that $G/[G, G]$ has I as generators, still contains R as relations, and additionally has the elements in C as new relations. □

7.7 Exercises

7.7.1 Skeleta of categories

Exercise 7.1: Consider the category of opens of a space X (Exercise 1.5). Prove that there is no (strictly smaller) subcategory equivalent to it.

7.7.2 Abelianisation

Exercise 7.2: Show that the following properties are equivalent for a group G:

- The abelianisation of G is the trivial group.
- For all abelian groups H, the only group homomorphism $G \to H$ is the trivial homomorphism.

Exercise 7.3: Consider the group $A_n := \{a_1, b_1, \cdots, a_n, b_n \mid \prod_i [a_i, b_i]\}$. Compute its abelianisation.

[2]We should write $[[w, z]]$ to indicate "the class of $[w, z]$ in the quotient $[H, H]/C$", but this seems unnecessarily confusing.

Exercise 7.4: Consider the group $B_n := \{a_1, \cdots, a_n, \mid \prod_i a_i^2\}$. Compute its abelianisation.

Exercise 7.5: Let $G = \langle a, b \mid a^2, b^2 \rangle$.

- Find a subgroup $H \subset G$ isomorphic to $\mathbb{Z}$ and of index 2.
- Compute the abelianisation of G and H.

Homologies and easy van Kampen

Lecture 8

We have learnt how to associate an abelian group to each group, yielding a functor Grp $\to$ Ab. The idea was that comparing $\pi_1(X, x)$ with $\pi_1(Y, y)$ can be quite difficult, so it is sometimes easier to compare abelianisations. We will now wrap up this line of ideas by:

- Introducing the *tensor with* $\mathbb{R}$ functor (Definition 8.3). It will assign an $\mathbb{R}$-vector space to each abelian group. Vector spaces are even easier to compare!
- Introducing the *first homology with integral coefficients* (Definition 8.5). It combines the abelianisations of the fundamental groups of all path-components and combines them into one abelian group.
- Introducing the *first homology with real coefficients* (Definition 8.15), which is the tensor with $\mathbb{R}$ of the integral one.

These invariants are a trade-off: Using them we lose some of the interesting information encoded in π_1, but the information that is left is sometimes enough to tell X and Y apart.

The second goal of the lecture is to state the theorem of van Kampen. This is the main result of the course but, for now, we content ourselves with a simpler version:

- The easy version (Section 8.5) says that π_1 takes the coproduct in Top_* to the coproduct in Grp, at least for well-behaved spaces.
- We will describe explicitly the coproduct in Top_*, which is called the *wedge* (Section 8.3).
- We will also describe the coproduct in Grp, which we call the *free product* (Section 8.4).

Lastly, we will also:

- Introduce *pushouts* (Section 8.6), which generalise the concept of "(not necessarily disjoint) unions" to any category.
- This will require that we define commutative diagrams in a category (Definition 8.38).

The general version of van Kampen says that π_1 and Π_1 send (nice) pushouts to pushouts.

8.1 Tensoring with $\mathbb{R}$

As we stated in Theorem 7.26, every finitely generated abelian group A is isomorphic to a group of the form:

$$\mathbb{Z}^{a_0} \oplus (\bigoplus_{i \in \mathbb{N}} (\mathbb{Z}/i\mathbb{Z})^{a_i}).$$

I.e. a product of various copies of $\mathbb{Z}$ and various copies of the cyclic groups.

Definition 8.1. *Given an A presented in this manner, we can associate to it the vector space:*

$$A \otimes_{\mathbb{Z}} \mathbb{R} := \mathbb{R}^{a_0}.$$

Elements in $\mathbb{Z}^{a_0}$ can be written as tuples $(x_1, \cdots, x_{a_0})$, where $x_i \in \mathbb{Z}$. Elements in $\mathbb{R}^{a_0}$ are then tuples in which the coefficients are real instead.

Furthermore, given B similarly presented as

$$\mathbb{Z}^{b_0} \oplus (\bigoplus_{i \in \mathbb{N}} (\mathbb{Z}/i\mathbb{Z})^{b_i}),$$

and a group homomorphism $\phi : A \to B$, we can consider the map

$$\psi := \pi \circ \phi|_{\mathbb{Z}^{a_0}} : \mathbb{Z}^{a_0} \to \mathbb{Z}^{b_0}$$

where $\pi : B \to \mathbb{Z}^{b_0}$ is the standard projection. The map ψ is basically a matrix with integer entries.

Definition 8.2. *Given ϕ as above, we associate to it the linear map*

$$\psi \otimes_{\mathbb{Z}} \mathbb{R} : \mathbb{R}^{a_0} \to \mathbb{R}^{b_0},$$

which amounts to taking the matrix ψ, which has integer coefficients, and regarding it as a linear map between $\mathbb{R}$-vector spaces.

The constructions we have presented are not intrinsic, since they depend on auxiliary isomorphisms to put abelian groups in normal form. Nonetheless:

Definition 8.3. *We write*

$$\otimes_{\mathbb{Z}}\mathbb{R} : \mathrm{Ab} \to \mathrm{Vect}_{\mathbb{R}}$$

for the functor defined as:

- *Given an abelian group A presented as above, its image is $\mathbb{R}^{a_0}$.*
- *Given a group homomorphism $\phi : A \to B$, its image is $\psi \otimes_{\mathbb{Z}} \mathbb{R}$.*

These ideas are presented more formally in Section 8.7. There we show that $\otimes_{\mathbb{Z}}\mathbb{R}$ is a well-defined functor.

We know that two $\mathbb{R}$-vector spaces are isomorphic if and only if they have the same dimension. We deduce:

Lemma 8.4. *Let A and B be abelian groups. Suppose that $A \otimes_{\mathbb{Z}} \mathbb{R}$ and $B \otimes_{\mathbb{Z}} \mathbb{R}$ have different dimensions. Then A and B are not isomorphic as abelian groups.*

In particular, $\mathbb{Z}^m$ is isomorphic to $\mathbb{Z}^n$ if and only if $m = n$.

Proof. Suppose that A and B are isomorphic as abelian groups. Since $\otimes_{\mathbb{Z}} \mathbb{R}$ is a functor we deduce that $A \otimes_{\mathbb{Z}} \mathbb{R}$ and $B \otimes_{\mathbb{Z}} \mathbb{R}$ are isomorphic, meaning that they have the same dimension. □

8.2 First homology

We can now use these tools in order to produce invariants with values in Ab and $\mathrm{Vect}_{\mathbb{R}}$, starting from our favourite invariant with values in Grp.

8.2.1 First homology with integer coefficients

Our goal is to have an invariant hTop $\to$ Ab. We define:

Definition 8.5. *Let X be a space. Fix a collection of points $\{x_i\} \subset X$, one per path-connected component of X. Then, the **first homology with integral coefficients**[1] is*

$$H_1(X, \mathbb{Z}) := \bigoplus_i \mathfrak{Ab}(\pi_1(X, x_i)).$$

I.e. $H_1(-, \mathbb{Z})$ is meant to capture (some of) the behaviour of loops coming from different path-components at once.

Lemma 8.6. *$H_1(X, \mathbb{Z})$ does not depend on the concrete choice of $\{x_i\} \subset X$.*

Proof. Given any two collections, we put them in bijection using their bijection with $\pi_0(X)$. We thus write $\{x_i\}$ and $\{y_i\}$, with the same index set. $\pi_1(X, x_i)$ and $\pi_1(X, y_i)$ are then isomorphic, but not canonically. Namely, for every class $a \in \pi_1(X, x_i, y_i)$ we have an isomorphism $\beta_a : \pi_1(X, y_i) \to \pi_1(X, x_i)$ given by conjugation. The claim is that the isomorphism

$$\mathfrak{Ab}(\beta_a) : \mathfrak{Ab}(\pi_1(X, y_i)) \to \mathfrak{Ab}(\pi_1(X, x_i))$$

is independent of the class a. I.e. all the different conjugations become equivalent once we abelianise.

This claim is in turn equivalent to the claim that

$$\mathfrak{Ab}(\beta_b \circ \beta_a) : \mathfrak{Ab}(\pi_1(X, y_i)) \to \mathfrak{Ab}(\pi_1(X, y_i))$$

is the identity for every $a \in \pi_1(X, x_i, y_i)$ and every $b \in \pi_1(X, y_i, x_i)$. Indeed: since $ba \in \pi_1(X, y_i)$, we have that $\beta_b \circ \beta_a = \beta_{ba}$ is a conjugation by an element in the group $\pi_1(X, y_i)$. The proof is then complete as a consequence of Lemma 7.35, which says that $\mathfrak{Ab}(\beta_{ab}) = \mathrm{id}$. □

[1] In later courses you will see this invariant defined in other ways. Proving that these other definitions agree with the abelianisation of the fundamental group is then a theorem.

Example 8.7: Consider the circle $\mathbb{S}^1$. Its fundamental group $\pi_1(\mathbb{S}^1, 1)$ is isomorphic to $\mathbb{Z}$, which is already abelian. It follows that $H_1(\mathbb{S}^1, \mathbb{Z}) \simeq \mathbb{Z}$. △

Example 8.8: Let $X = \mathbb{S}^1 \coprod \mathbb{S}^1$. Then we have two components, each with first homology isomorphic to $\mathbb{Z}$. It follows that $H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^2$. △

Example 8.9: Consider the torus $\mathbb{T}^2$. Once again the fundamental group $\pi_1(\mathbb{T}^2, p) \simeq \mathbb{Z}^2$ is abelian so $H_1(\mathbb{T}^2, \mathbb{Z}) \simeq \mathbb{Z}^2$. This means that we cannot distinguish it from $\mathbb{S}^1 \coprod \mathbb{S}^1$ by looking at homology! (But we can certainly tell them apart by looking at π_0 or π_1). △

8.2.2 First homology with integer coefficients as a functor

Our ultimate goal is to define a functor $H_1(-, \mathbb{Z})$, so we need to explain how it acts on morphisms:

Definition 8.10. *Let $f : X \to Y$ be a map. Its **pushforward***

$$f_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$$

at the level of first homology is defined as follows:

- *Decompose $X = \coprod_i X_i$ into path-components and write $f_i = f|_{X_i}$ for the restriction to X_i.*
- *Let $(f_i)_* : H_1(X_i, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$ be the abelianisation of the pushforward $\pi_1(X_i, x_i) \to \pi_1(Y, f(x_i))$ between fundamental groups.*
- *Set $f_* := +_i (f_i)_* : H_1(X, \mathbb{Z}) = \bigoplus_i H_1(X_i, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$.*

Concretely, given elements $a_i \in H_1(X_i, \mathbb{Z})$ and thus an element $(a_1, \cdots, a_n) \in H_1(X, \mathbb{Z})$, we set $f_*(a_1, \cdots, a_n)$ to be the sum $(f_1)_*(a_1) + \cdots + (f_n)_*(a_n) \in H_1(Y, \mathbb{Z})$.

That is, on each path-component we consider the abelianisation of the pushforward between fundamental groups, and then we use the fact that $H_1(Y, \mathbb{Z})$ is an abelian group to add all of these together.

Lemma 8.11. *$f_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$ is well-defined and only depends on the homotopy class of $f : X \to Y$.*

Proof. Write X as an union of path-components $\coprod_i X_i$. Our definition of f_* involved making some choices; namely, fixing basepoints $x_i \in X_i$ in each path-component. However, we already proved in Lemma 8.6 that all these different choices do not matter, up to canonical identification. This proves that f_* is well-defined.

For the second claim we consider a homotopy $f_s : X \to Y$ of f. This homotopy is equivalent to a homotopy $(f_i)_s$ of each individual $f_i = f|_{X_i}$. By the change of basepoint formula (Theorem 6.3), the pushforwards at the level of fundamental groups differ from one another by a conjugation, as s varies. This implies, using once again Lemma 8.6, that they are the same once we abelianise. □

Which allows us to conclude:

Corollary 8.12. $H_1(-,\mathbb{Z}) : \mathrm{hTop} \to \mathrm{Ab}$ *is a functor. It agrees with the composition* $\mathfrak{Ab} \circ \pi_1$ *when evaluated on path-connected spaces.*

Corollary 8.13. *Homotopy equivalent spaces have isomorphic first homologies with integral coefficients.*

Example 8.14: Let $X = \mathbb{S}^1 \coprod \mathbb{S}^1$ and let $Y = \mathbb{T}^2$. The first homology $H_1(X,\mathbb{Z}) \simeq \mathbb{Z}^2$ has two generators $\{[a],[b]\}$. The class $[a]$ is represented by the map $a : \mathbb{S}^1 \to X$ that is the identity onto the first path-component. Similarly, $b : \mathbb{S}^1 \to X$ is the identity onto the second path-component. We can also find generators $\{[c],[d]\}$ for $H_1(Y,\mathbb{Z}) \simeq \mathbb{Z}^2$. We let $c, d : \mathbb{S}^1 \to Y$ be the classes of the maps $z \mapsto (z,1)$ and $z \mapsto (1,z)$, respectively.

We can now map $f : X \to Y$ by setting $f(z) = (z^2, 1)$ if z is in the first path-component of X and $f(z) = (z,z)$ otherwise. We see that $f \circ a(z) = (z^2,1)$ and $f \circ b(z) = (z,z)$. This means that $f_* : H_1(X,\mathbb{Z}) \to H_1(Y,\mathbb{Z})$ is given by $f_*([a]) = [c]^2$ and $f_*([b]) = [c][d] = [d][c]$. In particular, we see that the element $[c] \in H_1(Y,\mathbb{Z})$ is not in the image of f_*. △

8.2.3 First homology with real coefficients

We can now tensor the first homology with integral coefficients with $\mathbb{R}$:

Definition 8.15. *The* ***first homology with real coefficients*** *is the functor*

$$H_1(X,\mathbb{R}) := (\otimes_{\mathbb{Z}}\mathbb{R}) \circ H_1(-,\mathbb{Z}) : \mathrm{hTop} \to \mathrm{Vect}_{\mathbb{R}}.$$

Corollary 8.16. *Homotopy equivalent spaces have isomorphic first homologies with real coefficients.*

Example 8.17: $H_1(\mathbb{S}^1,\mathbb{R}) \simeq \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}$. △

Example 8.18: $H_1(\mathbb{S}^1 \coprod \mathbb{S}^1,\mathbb{R}) \simeq \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^2$. △

Example 8.19: $H_1(\mathbb{T}^2,\mathbb{R}) \simeq \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^2$. △

In upcoming chapters we will encounter spaces X for which $H_1(X,\mathbb{Z})$ has elements of finite order. These will then be sent to zero when we consider $H_1(X,\mathbb{R})$.

8.3 The coproduct for pointed spaces

Our next goal is to state a first simple version of the Theorem of van Kampen. To this end, we introduce:

Definition 8.20. *Let (A,a) and (B,b) be pointed spaces. Their* ***wedge product*** *is defined as:*

$$(A,a) \vee (B,b) := \left(\frac{A \coprod B}{a \simeq b}, [a] = [b] \right).$$

That is, we take A and B and we glue their basepoints. Observe that there are canonical inclusions ι_A and ι_B of (A,a) and (B,b) into their wedge.

Lemma 8.21. *The coproduct in* Top_* *is the wedge product.*

Proof. We must show that the wedge satisfies the universal property. Given any morphisms $(A,a) \to_f (C,C) \leftarrow_g (B,b)$ we must show that there is a unique pointed map $h : (A,a) \vee (B,b) \to (C,c)$ such that $f = h \circ \iota_A$ and $g = h \circ \iota_B$. Since the images of ι_A and ι_B cover the wedge, these identities uniquely define h as a morphism in Set. Moreover, since f and g are pointed, $h([a]) = f(a) = g(b) = h([b])$ is well-defined and h is thus pointed. It remains to observe that h is continuous by the properties of the quotient topology. □

8.4 The coproduct for groups

We also need:

Definition 8.22. *Let G and H be groups. We consider* ***words*** *written in the alphabet $G \coprod_{\mathrm{Set}} H$. We also consider the following* ***moves****:*

- *I. Let e_G and e_H be the identities of G and H, respectively. Then, the words e_G and e_H are equivalent to the empty word.*
- *II. Given $g_1, g_2 \in G$ we can consider the word g_1g_2 with two letters. It is equivalent to the word $(g_1.g_2)$ that has a single letter (the dot denotes taking the product in G).*
- *II'. Fix letters $h_1, h_2 \in H$. The word h_1h_2 is equivalent to the word $(h_1.h_2)$ given by composing the two letters in H.*

In general, these moves can be applied to subwords of any given word.

The ***free product*** *$G * H$ of G and H is defined as:*

- *Its elements are equivalence classes of words.*
- *We consider the equivalence relation generated by moves I and II.*
- *The group multiplication of two equivalence classes is given by concatenating representatives.*

Observe that elements in G (or respectively H) can be regarded as words in $G * H$ consisting of a single letter. It follows that there are injective group homomorphisms $\iota_G : G \to G * H$ and $\iota_H : H \to G * H$.

In applications, we will often work in the following setup:

Lemma 8.23. *Consider the groups $G = \langle I_0 \mid R_0 \rangle$ and $H = \langle I_1 \mid R_1 \rangle$. Then their free product is isomorphic to*

$$\langle I_0, I_1 \mid R_0, R_1 \rangle.$$

Proof. Let us denote $A = \langle I_0, I_1 \mid R_0, R_1 \rangle$. Observe that A consists of words written in the alphabet $I_0 \coprod I_1$ (and their inverses), whereas the free product consists of more general words written using arbitrary elements of G and H. It follows that words in the former can be regarded tautologically as words in the latter, yielding a group homomorphism $\psi : A \to G*H$. In particular, ψ is injective. To prove surjectivity, express each element of G in terms of generators and similarly for H. In doing so we are representing every element in $G * H$ as a word in A. □

Example 8.24: If I is a set, the free product of I copies of $\mathbb{Z}$ is precisely the free group $*_I\mathbb{Z}$ (as the notation already suggested). △

Example 8.25: The free product $\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ is presented as $\langle a, b \mid a^p, b^q \rangle$. Here we think of a as a generator of $\mathbb{Z}/p\mathbb{Z}$ and b as a generator of $\mathbb{Z}/q\mathbb{Z}$. Abelianising $\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ yields then the group $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$. △

Lastly:

Lemma 8.26. *The coproduct in* Grp *is the free product.*

Proof. Suppose we are given groups G and H and some auxiliary group O with maps $\phi_G : G \to O$ and $\phi_H : H \to O$. We claim that there is a unique group homomorphism $\psi : G * H \to O$ factoring both as $\phi_G = \psi \circ \iota_G$ and $\phi_H = \psi \circ \iota_H$. Indeed, being a group homomorphism implies that each word $c_n \cdots c_1 \in G * H$ must be taken to

$$\psi(c_n \cdots c_1) := \phi(c_n) \cdots \phi(c_1) \in O,$$

where ϕ is either ϕ_G or ϕ_H, depending on whether c_i is an element of G or H. This proves uniqueness of ψ and by construction it is a group homomorphism.

It remains to observe that it is well-defined. To do so, we go through the moves, verifying that ψ invariant under them. For move (I) we observe that $\psi(e_G) = \psi(e_H) = e_O$, so it is well-defined at the identity element. For move (II) we verify that

$$\psi(g_1 g_2) = \psi(g_1)\psi(g_2) = \phi_G(g_1).\phi_G(g_2) = \phi_G(g_1 g_2) = \psi(g_1.g_2),$$

for every two elements $g_1, g_2 \in G$. An analogous computation goes through for move (II'), concluding the proof. □

8.5 Van Kampen for wedges

Having introduced the coproduct in Top_* and Grp, we are ready to tackle the easiest case of van Kampen. Unfortunately, it is not true that π_1 sends the coproduct to the coproduct. We need our pointed spaces to be nice enough:

Definition 8.27. *Let (A, a) be a pointed space. We say that it is **well-pointed** if there is a neighbourhood $U \ni a$ such that U deformation retracts to a.*

This condition is stronger than what one normally requires for being well-pointed, but for our purposes it is general enough. It says that a has neighbourhoods that are as simple as $\{a\}$ itself, from a homotopical viewpoint.

Thus: the functor π_1 sends (nice) coproducts in Top_* to coproducts in Grp:

Proposition 8.28. *Let (A, a) and (B, b) be well-pointed. Then*

$$\pi_1((A, a) \vee (B, b)) \simeq \pi_1(A, a) * \pi_1(B, b).$$

We prove it in Section 9.4.2. The conclusion is not necessarily true if we drop well-pointedness; a concrete instance is shown in Lemma 8.36 below.

Corollary 8.29. *Let (A, a) and (B, b) be well-pointed. Denote $(X, x) := (A, a) \vee (B, b)$:*

- $H_1(X, \mathbb{Z}) \simeq H_1(A, \mathbb{Z}) \oplus H_1(B, \mathbb{R})$.
- $H_1(X, \mathbb{R}) \simeq H_1(A, \mathbb{R}) \oplus H_1(B, \mathbb{R})$.

The following can be proven inductively using Proposition 8.28 and Theorem 10.13:

Corollary 8.30. *Let I be a finite set. Then:*

$$\begin{aligned} \pi_1(\vee_I(\mathbb{S}^1, 1)) &\simeq *_I\mathbb{Z}, \\ H_1(\vee_I\mathbb{S}^1, \mathbb{Z}) &\simeq \mathbb{Z}^I, \\ H_1(\vee_I\mathbb{S}^1, \mathbb{R}) &\simeq \mathbb{R}^I. \end{aligned}$$

Proof. We just need to verify that $(\mathbb{S}^1, 1)$ is well-pointed. This follows from the fact that a neighbourhood of $1 \in \mathbb{S}^1$ is an interval and it thus deformation retracts to any of its points. □

Corollary 8.31. *Let I and J be finite sets with $|I| \neq |J|$. Then $\vee_I(\mathbb{S}^1, 1)$ and $\vee_J(\mathbb{S}^1, 1)$ are not homotopy equivalent.*

The last two statements are true even if I has infinite cardinality, but one has to adapt the proof of van Kampen to deal with such a situation (Section 11.4).

8.5.1 The Hawaiian earring

In this course we will almost solely focus on spaces that are well-behaved locally (i.e. locally Hausdorff, compact, contractible). For completeness, here is an example that is not so nice:

Definition 8.32. *The **Hawaiian earring** $H \subset \mathbb{R}^2$ is the union $\cup_{n=1}^{\infty} C_n$, where C_n is the circle $\mathbb{S}^1_{1/n}(1/n, 0)$ of radius $1/n$ centered at $(1/n, 0)$.*

If we consider a point $p \in H$ different from 0, the local behaviour of H is good. Every such p has a neighbourhood homeomorphic to an interval, which is thus Hausdorff and contractible. However, the local structure of H around 0 is complicated.

First we observe that:

Lemma 8.33. *The fundamental group $\pi_1(H, 0)$ is non-trivial.*

Proof. There is a retraction $r_n : H \to C_n$ given by sending every other circle to $0 \in H$. This implies (Corollary 5.22) that

$$r_* : \pi_1(H, 0) \to \pi_1(C_n, 0) \simeq \mathbb{Z}$$

is surjective, proving that $\pi_1(H, 0)$ non-zero. □

With a little more care:

Corollary 8.34. *Every neighbourhood U of 0 is not simply-connected (Definition 6.14). In particular, H is not well-pointed.*

Proof. Find n large enough so that $C_n \subset U$. This allows us to define a retraction $r : U \to C_n$ as in Lemma 8.33, implying that $\pi_1(U, 0)$ is non-trivial. □

We can do much better and produce larger subgroups of $\pi_1(H, 0)$. Given any finite subset $I \subset \mathbb{Z}^+$, consider the inclusion $\iota_I : \cup_{n \in I} C_n \to H$:

Lemma 8.35. $(\iota_I)_* : \pi_1(\cup_{n \in I} C_n, 0) \simeq *_I \mathbb{Z} \to \pi_1(H, 0)$ *is injective.*

Proof. ι_I if the right-inverse of the retraction r_I that keeps $\cup_{n \in I} C_n$ fixed and sends the other circles to 0. It follows that $(\iota_I)_*$ is injective and $(r_I)_*$ is surjective. The claim about the isomorphism with $*_I \mathbb{Z}$ follows from the fact that $\cup_{n \in I} C_n$ is homeomorphic to a wedge of $|I|$ circles, as can be shown explicitly. □

In fact, all these maps $(\iota_I)_*$ are coherent. I.e. if we have two collections $I \subset I'$, the map $(\iota_I)_*$ factors via the map $(\iota_{I'})_*$. Taking all of them together implies that there is a monomorphism $*_{\mathbb{Z}^+} \mathbb{Z} \to \pi_1(H, 0)$. This is precisely the pushforward of the continuous map $\vee_{\mathbb{Z}^+} \mathbb{S}^1 \to H$.

Lemma 8.36. *The canonical map*

$$\Psi : \pi_1(H, 0) * \pi_1(H, 0) \to \pi_1((H, 0) \vee (H, 0))$$

is not an isomorphism.

Proof. First observe that each copy of $(H, 0)$ includes into the wedge $(H, 0) \vee (H, 0)$. Denote the inclusion maps by ι, ι'. Their pushforwards are thus maps $\pi_1(H, 0) \to \pi_1((H, 0) \vee (H, 0))$. We can apply the universal property of the coproduct to them, yielding the claimed group homomorphism Ψ.

Write now C_n for the nth circle in the first copy of $(H, 0)$ and C'_n for the nth circle in the second copy. We can assemble a loop $\gamma : (\mathbb{S}^1, 1) \to (H, 0) \vee (H, 0)$ that runs around C_1 during the interval $[0, 1/2]$, then C'_2 over $[1/2, 1/4]$, then C_3 over $[1/4, 1/8]$ and so on. The claim is that $[\gamma]$ is not in the image of Ψ. The idea is that, morally speaking, $[\gamma]$ has been produced by concatenating infinitely classes of loops, alternating between the two copies. Elements in the image of Ψ cannot be of this form since they are represented by words with finitely many letters. □

8.6 The pushout

We now begin working towards the Theorem of van Kampen in full generality. This requires that we introduce the notion of pushout. Before we provide an actual definition, let us explain a bit what the intuition is:

Example 8.37: Consider the category Set. Suppose U is a set, presented as a (not necessarily disjoint) union $U = A \cup B$ of subsets A and B. Given some other set Y and a function $h : U \to Y$, we can consider the restrictions

$$g_A := h|_A : A \to Y \quad \text{and} \quad g_B := h|_B : B \to Y.$$

By construction, these agree on the overlap $I = A \cap B$, i.e. $g_A|_I = g_B|_I$.

Conversely, if we have functions $g_A : A \to Y$ and $g_B : B \to Y$ such that $g_A|_I = g_B|_I$, then there exists a unique function $h : U \to Y$ such that $h|_A = g_A$ and $h|_B = g_B$. That is, discussing functions $U \to Y$ is equivalent to discussing pairs of functions $(A \to Y, B \to Y)$ that agree over $I = A \cap B$. △

Before we go on, we need the following concept:

Definition 8.38. *Let $\mathcal{C}$ be a category. A **diagram** in $\mathcal{C}$ is a collection of objects in $\mathcal{C}$ and morphisms between them. A diagram is **commutative** if for any two objects x and y in the diagram, and any two sequences of composable morphisms $f_n, \cdots, f_1$ and $g_m, \cdots, g_1$, both starting at x and finishing at y, it holds that their compositions agree.*

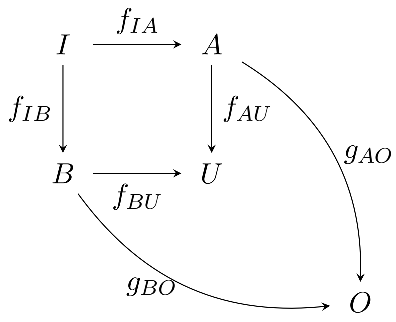
Identically: Given a diagram we can consider the subcategory $\mathcal{D}$ of $\mathcal{C}$ whose objects are the objects of the diagram and whose morphisms are all possible compositions (in $\mathcal{C}$) of morphisms in the diagram. Then, the diagram being commutative means that each $\mathrm{Hom}_{\mathcal{D}}(x, y)$ contains at most one element.

Mimicking the discussion in Example 8.37:

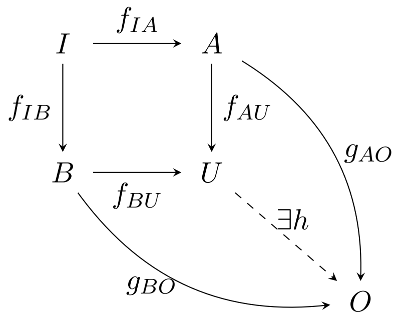
Definition 8.39. *Let $\mathcal{C}$ be a category. A commutative diagram of the form*

$$\begin{array}{ccc} I & \xrightarrow{f_{IA}} & A \\ {\scriptstyle f_{IB}}\downarrow & & \downarrow{\scriptstyle f_{AU}} \\ B & \xrightarrow[f_{BU}]{} & U \end{array}$$

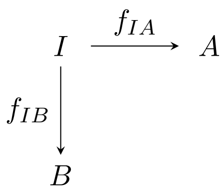
*is said to be a **pushout diagram** if, given any extended commutative diagram of the form*



there exists a unique morphism $h : U \to O$ completing the commutative diagram:



*Alternatively, we say that U, together with the maps f_{AU} and f_{BU}, satisfies the **universal property of the pushout**, or that U is the pushout of the diagram*



Concretely: Commutativity of the first diagram means that

$$f_{AU} \circ f_{IA} = f_{BU} \circ f_{IB} : I \to U.$$

The extended diagram commutes if moreover

$$f_{AO} \circ f_{IA} = f_{BO} \circ f_{IB} : I \to O.$$

And the last completed diagram commuting means that additionally:

$$f_{AO} = h \circ f_{AU} \text{ and } f_{BO} = h \circ f_{BU}$$

Remark 8.40: Following the discussion in Example 8.37, the insight behind this definition is that two mappings from A and B, to some other object O, that are compatible over I, are the exact same data as a map from U to O. △

Do note that we do *not* require the maps $I \to A$ and $I \to B$ to be inclusions (or maps of any particular kind). A pushout is thus a more general concept than a "union".

8.6.1 Uniqueness of the pushout

We mentioned earlier that products and coproducts are unique up to isomorphism. We will now verify that this is the case as well for the pushout. We use the same notation as in Definition 8.39:

Lemma 8.41. *Let (U, f_{AU}, f_{BU}) and $(U', f_{AU'}, f_{BU'})$ be two triples satisfying the universal property of the pushout with respect to the tuple $(A, B, I, f_{IA}, f_{IB})$. Then, the two triples are isomorphic. I.e. there are isomorphisms $h : U \to U'$ and $g : U' \to U$ that commute with all other maps.*

Proof. $(A, B, I, f_{IA}, f_{IB})$ and $(U', f_{AU'}, f_{BU'})$ form together an extended diagram satisfying the hypothesis of Definition 8.39 (i.e. we are thinking of U' being O). This produces a map $h : U \to U'$ for us. Dually, we can think of U being O and obtain a map $g : U' \to U$. We can then look at the composition $g \circ h : U \to U$. This is a morphism from U to itself that commutes with all other morphisms in the diagram. There is another such morphism $U \to U$; namely, the identity id_U. The uniqueness stated in the universal property of the pushout implies that $g \circ h = \mathrm{id}_U$. We then reason dually with $h \circ g$ to deduce that g and h are isomorphisms. □

As you see, the reasoning used in the proof relies only on the uniqueness claimed in the universal property. It follows that the exact same reasoning adapts to the case of products and coproducts (and whatever other universal property you encounter).

8.7 Appendix: Extra details about tensoring

We now formalise the idea of tensoring.

This will not be used elsewhere in the course.

8.7.1 Modules over a ring

Recall:

Definition 8.42. *Let R be a commutative ring with unit e. An R-**module** is an abelian group $(M, +)$ together with a* product by scalars $\cdot : R \times M \to M$ *satisfying:*

- *(Identity) $e.m = m$ for every $m \in M$.*
- *(Associativity) $(gh).m = g.(h.m)$ for every $m \in M$ and every $g, h \in R$.*
- *(Distributivity with the ring addition) $(g + h).m = g.m + h.m$ for every $m \in M$ and every $g, h \in R$.*

- *(Distributivity with the module addition)* $g.(m+n) = g.m + g.n$ *for every* $m, n \in M$ *and very* $g \in R$.

You should think of a module as a "vector space" over a ring. Indeed, you should verify that if R is a field, this is the usual definition of a vector space. We can consider the category Mod_R whose objects are R-modules and whose morphisms are group homomorphisms $\phi : M \to N$ that are compatible with the product by scalars in the sense that $\phi(g.m) = g.\phi(m)$ for every $m \in M$ and every $g \in R$.

The concrete case that is of interest to us:

Lemma 8.43. *The category* Ab *of abelian groups is isomorphic to the category* $\mathrm{Mod}_{\mathbb{Z}}$ *of* $\mathbb{Z}$*-modules.*

Proof. What the statement says, intuitively, is that each abelian group can be seen, in a unique and natural manner, as a $\mathbb{Z}$-module. Furthermore, every group homomorphism between abelian groups is compatible with the product by scalars.

For the first claim observe that, given an abelian group A we can define $\cdot : \mathbb{Z} \times A \to A$ by $(n, a) \mapsto n.a := a + \cdots + a$, using additive notation for the group operation. By construction this is indeed a product by scalars. Furthermore, given any morphism of abelian groups $\psi : A \to B$ it holds that

$$\psi(n.a) = \psi(a + \cdots + a) = \psi(a) + \cdots + \psi(a) = n.\psi(a),$$

proving compatibility with the multiplication by scalars. □

That is to say: abelian groups are "vector spaces" over the integers.

8.7.2 Tensoring

The idea now is that we can "change the coefficients" of the module. Concretely, we can replace integer coefficients (i.e. looking at $\mathbb{Z}$-modules) by real coefficients (i.e. looking at $\mathbb{R}$-vector spaces).

Definition 8.44. *Let* M *be a* $\mathbb{Z}$*-module. Let* R *be a commutative ring with unit* e*. Define* $I = \{m \otimes r \mid m \in M, r \in R\}$.

The R*-module* $M \otimes_{\mathbb{Z}} R$ *is defined, as an abelian group, as the quotient of* $\mathbb{Z}^I$ *by the relations*

- $(m_1 + m_2) \otimes r \simeq m_1 \otimes r + m_2 \otimes r$ *for every* $m_1, m_2 \in M$*, and* $r \in R$.
- $m \otimes (r_1 + r_2) \simeq m \otimes r_1 + m \otimes r_2$ *for every* $m \in M$*, and* $r_1, r_2 \in R$.
- $m \otimes (a.r) \simeq (a.m) \otimes r$ *for every* $a \in Z$*,* $m \in M$*, and* $r \in R$.

Its multiplication by scalars (in R*) is defined as* $(r, m \otimes s) \mapsto m \otimes (r.s)$.

That is, elements in $M \otimes_{\mathbb{Z}} R$ are sums of the form $\sum_i m_i \otimes r_i$. The first property says that we can add the m_i if the corresponding r_i agree. The second property is dual to it. The third one says that multiplying by integers in the R factor is equivalent to multiplying by integers in the M factor.

We leave the following statement to the reader:

Lemma 8.45. *Let M be an $\mathbb{Z}$-module. Let R be a commutative ring with unit e. Then, there is a functor:*

$$\otimes_{\mathbb{Z}} R : \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_R$$

defined by:

- *M is taken to $M \otimes_{\mathbb{Z}} R$.*
- *$\psi : M \to N$ is taken to the map $m \otimes r \mapsto \psi(m) \otimes r$.*

Example 8.46: Consider $\mathbb{Z}$ as a $\mathbb{Z}$-module. Then $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}$. Indeed, each element $a \in \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}$ is of the form $\sum_i m_i \otimes r_i$. If $m_i = 0$ we have that $m_i \otimes r_i = 0$ so we can assume that all m_i are non-zero and write

$$a = \sum_i 1 \otimes (m_i.r_i) = 1 \otimes \left(\sum_i m_i.r_i \right).$$

This allows to identify a with the real number $\sum_i m_i.r_i$. This is the desired isomorphism as $\mathbb{R}$-modules.

One can work similarly on $\mathbb{Z}^n$, entry by entry, and verify that $\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$. △

Example 8.47: Consider $Z/p\mathbb{Z}$ as a $\mathbb{Z}$-module. We claim that $M = \mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} \simeq 0$. Indeed:

$$m \otimes r = m \otimes (p.(r/p)) = (p.m) \otimes (r/p) = 0.$$

I.e. we have used the fact that we can divide by p in $\mathbb{R}$ in order to "pull out" a p factor and pass it to the side of $\mathbb{Z}/p\mathbb{Z}$, yielding zero upon multiplying by m.

Putting these computations together we deduce that

$$\left(\mathbb{Z}^{a_0} \oplus (\bigoplus_{i \in \mathbb{N}} (\mathbb{Z}/i\mathbb{Z})^{a_i}) \right) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^{a_0},$$

as claimed earlier. △

Example 8.48: You do not need to tensor with $\mathbb{R}$. You can use other rings like $\mathbb{Q}$, $\mathbb{C}$ or $\mathbb{Z}/p\mathbb{Z}$. You can verify that for $\mathbb{C}$ it holds that:

$$\left(\mathbb{Z}^{a_0} \oplus (\bigoplus_{i \in \mathbb{N}} (\mathbb{Z}/i\mathbb{Z})^{a_i}) \right) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}^{a_0};$$

the statement is analogous for $\mathbb{Q}$. The case of $\mathbb{Z}/p\mathbb{Z}$ is more interesting, as $\mathbb{Z}/q\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ need not be trivial (e.g. if $p = q$ then we obtain simply $\mathbb{Z}/p\mathbb{Z}$). △

8.8 Exercises

8.8.1 Free product of groups

Exercise 8.1: Let G_1 and G_2 be abelian groups. Prove that the abelianisation of $G_1 * G_2$ is the direct sum.

Exercise 8.2: Prove that the coproduct in Ab is the direct sum.

Exercise 8.3: Prove that the abelianisation functor $\mathfrak{Ab}$: Grp $\to$ Ab sends the coproduct (the free product) to the coproduct (the direct sum).

8.8.2 Applications of easy van Kampen

Exercise 8.4: Let $(W, w) = (\mathbb{S}^1, 1) \vee (\mathbb{S}^1, 1)$. Let us write $\mathbb{S}^1 \subset W$ for the left circle. Find two retractions $r_1, r_1 : W \to \mathbb{S}^1$ that are not homotopic to each other.

Exercise 8.5: Let $A = \mathbb{R}^2 \setminus \{(0, 0), (1, 0)\}$.

- Prove that A is homotopy equivalent to $(\mathbb{S}^1, 1) \vee (\mathbb{S}^1, 1)$.
- Use this to find three non-nullhomotopic maps $f, g, h : \mathbb{S}^1 \to A$ that are not homotopic to each other.

Exercise 8.6: Denote $(A, a) = (\mathbb{S}^1, 1) \vee (\mathbb{S}^1, 1)$ and $B = \mathbb{R}^2 \setminus \{(0, 0), (1, 0)\}$. Find inclusions $\iota_0, \iota_1 : A \to B$ such that:

- $\iota_0(A)$ is a retract.
- $\iota_1(A)$ is not a retract.

The theorem of van Kampen for π_1

Lecture 9

Our overall goal at this point is to state and prove the fundamental group version of van Kampen's theorem (Theorem 9.11). It explains how to compute $\pi_1(X, x)$ when X is presented as a (reasonable) union of subspaces. This result will be our main tool to compute fundamental groups.

In this lecture:

- We will discuss what pushouts are in various categories. In Set and Top, pushouts correspond to unions (Sections 9.1 and 9.2).
- In Grp they correspond to the so-called *amalgamated product* (Definition 9.8).
- Finally, we will state the Theorem of van Kampen (Section 9.4) and prove various corollaries.
- We will show that $\mathbb{S}^n$ is simply-connected if $n > 1$.
- We will prove the simple version of van Kampen that we stated for the wedges (Theorem 8.5).
- We will work out a concrete example in detail, explaining how to use van Kampen in practice (Section 9.5).

9.1 Pushouts in Set

We now consider a pushout diagram in Set. We will use the exact same notation as in Definition 8.39, regarding the objects as sets and the arrows as functions. We claim that, up to isomorphism in Set, the pushout (U, f_{AU}, f_{BU}) of the diagram $(A, B, I, f_{IA}, f_{IB})$ can be constructed as follows:

- U is the quotient of $A \coprod B$ by the relation generated by: $A \ni a \simeq b \ni B$ if there is $i \in I$ such that $f_{IA}(i) = a$ and $f_{IB}(i) = b$.
- f_{AU} is the composition $A \to A \coprod B \to U$.
- f_{BU} is the composition $B \to A \coprod B \to U$.

Remark 9.1: Observe that the following may happen: a can be related to b because there is $i \in I$ such that $f_{IA}(i) = a$ and $f_{IB}(i) = b$. In turn, b may relate to a' because there is $i' \in I$ such that $f_{IA}(i') = a'$ and $f_{IB}(i) = b'$. It then follows that $a, a' \in A$ are related to each other. That is: it is not just that we glue A to B, but we may be forced to identify points in A with each other.

A concrete example is $A = \{a, a'\}$, $B = \{b\}$ and $I = A$, with $I \to A$ the identity and $I \to B$ the unique constant map. Then $U = \{[a] = [a'] = [b]\}$ is the singleton set. △

As promised:

Lemma 9.2. *The triple (U, f_{AU}, f_{BU}) defined as above satisfies the universal property of the pushout with respect to the tuple $(A, B, I, f_{IA}, f_{IB})$.*

Proof. What we must do is prove that, for every triple (O, g_{AO}, g_{BO}) as in the diagram, we can find a unique map $h : U \to O$ making the diagram commute. Indeed, we can define h as follows. Since U is a quotient of $A \coprod B$, we set $h([a]) = g_{AO}(a)$ and $h([b]) = g_{BO}(b)$, where $[a], [b] \in U$ are equivalence classes of elements coming from A or B, respectively. We have in particular established uniqueness: the commutativity of the diagram forced us to choose h in this way.

We must show that h is well-defined. To see this we note that the elements $f_{IA}(i) \in A$ and $f_{IB}(i) \in B$ satisfy $[f_{IA}(i)] = [f_{IB}(i)] \in U$ but

$$h([f_{IA}(i)]) = g_{AO} \circ f_{IA}(i) = g_{BO} \circ f_{IB}(i) = h([f_{IB}(i)])$$

thanks to the commutativity we have by assumption. This proves that h is well-defined, since the equivalence relation defining U is generated by the identities $[f_{IA}(i)] = [f_{IB}(i)]$.

It remains to prove that the resulting diagram is commutative. This follows from:

$$h \circ f_{AU}(a) = h([a]) = g_{AO}(a), \qquad h \circ f_{BU}(b) = h([b]) = g_{BO}(b)$$

which holds for every $a \in A$ and every $b \in B$. □

As a concrete case:

Remark 9.3: The coproduct in Set (the disjoint union) is a concrete case of pushout. Namely, it is the pushout of the diagram

$$\begin{array}{ccc} \emptyset & \xrightarrow{f_{IA}} & A \\ {\scriptstyle f_{IB}}\downarrow & & \\ B & & \end{array}$$

Lemma 9.2 then shows that this pushout is $A \coprod B$, proving that the disjoint union is indeed the coproduct in Set. △

9.2 Pushouts of topological spaces

We can particularise the diagrams appearing in Definition 8.39 to Top, regarding the objects appearing there as topological spaces and the arrows as continuous maps. Then:

Lemma 9.4. *The set-theoretical pushout (U, f_{AU}, f_{BU}) (described in Section 9.1), with the quotient topology inherited from $A \coprod B$, satisfies the universal property of the pushout in* Top.

Proof. With the same setup as previously, we have a unique and well-defined function $h : U \to O$ that makes the diagram commute set-theoretically.

It remains to show that h is indeed continuous. Consider an open subset $V \subset O$ and its preimage $h^{-1}(V) \subset U$. Let $\pi : A \coprod B \to U$ be the quotient map. Then the subsets

$$A \cap \pi^{-1}(h^{-1}(V)) = g_{AO}^{-1}(V) \quad \textit{and} \quad B \cap \pi^{-1}(h^{-1}(V)) = g_{BO}^{-1}(V)$$

are both open thanks to the continuity of g_{AO} and g_{BO}. This in turn implies that their union $\pi^{-1}(h^{-1}(V))$ is open as well, because A and B are themselves open subsets of $A \coprod B$. □

This claim implies once again (by taking $I = \emptyset$) that the disjoint union is also the coproduct in Top.

Remark 9.5: Suppose that X is a space, which we write as $X = A \cup B$. The results in Section 9.1 tell us that X, as a set, is the pushout U of the diagram of inclusions $A \leftarrow A \cap B \to B$. However, U and X may not be homeomorphic. I.e. the topology we obtain in U as a quotient of $A \coprod B$ may simply not be the original topology of X. Nonetheless, the diagram of inclusions

$$\begin{array}{ccc} A \cap B & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & X \end{array}$$

still commutes. This implies that we can use the universal property of the pushout and deduce that there is a unique continuous map $U \to X$ that makes the diagram commute. I.e. the topology in U is finer than the one in X.

As a concrete example, you can take $X = \mathbb{R}$, $A = \mathbb{Q}$ and B the irrationals. Then $A \cap B = \emptyset$ and the pushout of $A \leftarrow \emptyset \to B$ is just the coproduct of A and B, which is their disjoint union (clearly not homeomorphic to $\mathbb{R}$!). △

However:

Example 9.6: Suppose X is written as a union $A \cup B$ with both A and B open or both closed. Then the diagram of inclusions above is indeed a pushout. This follows from the Pasting Lemma 1.25. △

9.2.1 Pushouts of pointed spaces

The story for pointed spaces is completely analogous to what we have just done. In fact, you can prove it from the result in Top via the forgetful functor $\mathrm{Top}_* \to \mathrm{Top}$. The precise statement is:

Lemma 9.7. *The pushout in* Top$_*$ *of* $(A, a) \leftarrow (I, i) \to (B, b)$ *is the pointed space* (U, u)*, where:*

- *U is the pushout in* Top *of* $A \leftarrow I \to B$*,*
- *$u \in U$ is the class of* $[a] = [b]$*.*

Proof. Since $f_{IA}(i) = a$ and $f_{IB}(i) = b$, we deduce that $[a] = [b] \in U$, we denote it by u.

Suppose that we have a triple $((O, o), g_{AO}, g_{BO})$ consisting of a pointed space and two pointed maps from (A, a) and (B, b), respectively. Since U is the pushout in Top and pointed maps are in particular continuous we get a unique continuous map $h : U \to O$ commuting with the rest. It remains to show that h is pointed, which follows from the fact that $h(u) = g_{AO}(a) = o$ by definition of h. □

9.3 Pushouts of groups

Definition 9.8. *Consider the following diagram in* Grp*:*

$$\begin{array}{ccc} I & \xrightarrow{f_{IG}} & G \\ {\scriptstyle f_{IH}}\big\downarrow & & \\ H & & \end{array}$$

Then, the ***amalgamated product*** *of G and H along I (with respect to the group homomorphisms f_{IG} and f_{IH}) is defined as the quotient:*

$$G *_I H := \frac{G * H}{f_{IG}(i) \simeq f_{IH}(i) \text{ for every } i \in I}$$

In detail: elements of $G *_I H$ are equivalence classes of words using the alphabet $G \coprod_{\mathrm{Set}} H$. Two words are equivalent if they are related by the moves:

- *Moves (I), (II), and (II') as in the definition of free product (Definition 8.22).*

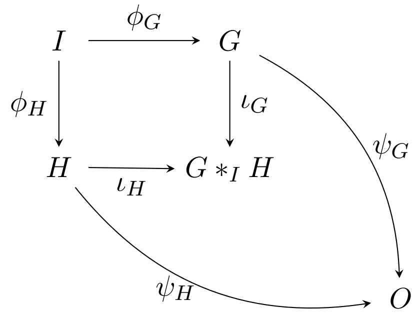
III. The letter $f_{IG}(i)$ is equivalent to the letter $f_{IH}(i)$ for all $i \in I$.

Composition is given by concatenation of words.

Do observe that the amalgamated product depends on the morphisms f_{IG} and f_{IH}, but this is not reflected in the notation $G *_I H$. Furthermore, do observe that there are canonical maps $\iota_G : G \to G *_I H$ and $\iota_H : H \to G *_I H$ which need not be injective.

Lemma 9.9. *The pushout in* Grp *is the amalgamated product.*

Proof. We consider a commutative diagram



where ι_G and ι_H are the canonical maps from G and H into the amalgamated product. We must show that there is a unique $\alpha : G *_I H \to O$ making the diagram commute.

Commutativity of the diagram forces us to set $\alpha(g) := \psi_G(g)$ if g is a letter from G and $\alpha(h) := \psi_H(h)$ if h is a letter from H. Since α should be a group homomorphism, we are forced to defined it on words as follows: we apply α to every individual letter, yielding a word on elements of O, which we then compose. In this manner we have a well-defined map $\tilde{\alpha} : W \to O$, where W is the set of words; by construction $\tilde{\alpha}$ takes concatenation to composition.

It remains to show that α itself is well-defined. This amounts to showing that $\tilde{\alpha}$ is invariant under moves (since this implies that it descends to the quotient $G *_I H$). This we check one move at a time. For moves (I), (II) and (II') we proceed as in the coproduct case; this is left to the reader. For move (III) we verify that:

$$\begin{aligned}\alpha(\omega_1 f_{IG}(i)\omega_2) &= \alpha(\omega_1)\alpha(f_{IG}(i))\alpha(\omega_2) = \alpha(\omega_1)\psi_G(f_{IG}(i))\alpha(\omega_2)\\ &= \alpha(\omega_1)\psi_H(f_{IH}(i))\alpha(\omega_2) = \alpha(\omega_1)\alpha(f_{IH}(i))\alpha(\omega_2)\\ &= \alpha(\omega_1 f_{IH}(i)\omega_2)\end{aligned}$$

which concludes the proof. □

Once again, we can phrase this in terms of group presentations.

Lemma 9.10. *Consider groups $G = \langle A_0 \mid R_0\rangle$, $H = \langle A_1 \mid R_1\rangle$, and $I = \langle A_2 \mid R_2\rangle$. Fix morphisms $f_{IG} : I \to G$ and $f_{IH} : I \to H$. Then:*

$$G *_I H = \langle A_0, A_1 \mid R_0, R_1, \{f_{IG}(i)f_{IH}(i)^{-1}\}_{i\in A_2}\rangle.$$

Proof. One can prove this mimicking the proof of Lemma 8.23; we leave this to the reader. An alternative approach amounts to establishing that the right hand side also satisfies the universal property of the pushout and invoking the uniqueness from Lemma 8.41. □

Do observe that the *generators* of I enter the relations of $G *_I H$ but the relations do not play any role.

9.4 Van Kampen for π_1

We now state the group version of van Kampen's theorem:

Theorem 9.11. *Let X be a space presented as an union $X = A \cup B$ of subsets. Write $I = A \cap B$. Let $j_A : I \to A$, $j_B : I \to B$, $i_A : A \to X$, and $i_B : B \to X$ be the inclusions. We work under the assumption that:*

- *A and B are open.*
- *$I = A \cap B$ is path-connected and we fix a point $p \in I$.*

Then:

$$\begin{array}{ccc} \pi_1(I,p) & \xrightarrow{(j_A)_*} & \pi_1(A,p) \\ {\scriptstyle (j_B)_*}\big\downarrow & & \big\downarrow{\scriptstyle (i_A)_*} \\ \pi_1(B,p) & \xrightarrow[(i_B)_*]{} & \pi_1(X,p) \end{array}$$

is a pushout. In particular:

$$\pi_1(X,p) \simeq \pi_1(A,p) *_{\pi_1(I,p)} \pi_1(B,p).$$

That is, under suitable assumptions, π_1 takes pushouts to pushouts. This result is a consequence of the upcoming Theorem 10.11; see Remark 10.12.

9.4.1 The higher-dimensional spheres

Corollary 9.12. *Let X be a space presented as an union $X = A \cup B$ of simply-connected open subsets. Let $I = A \cap B$ be path-connected. Then X is simply-connected.*

Proof. X is path-connected since it is a union of path-connected subspaces. Moreover, according to the theorem, we have that:

$$\pi_1(X,p) \simeq \pi_1(A,p) *_{\pi_1(I,p)} \pi_1(B,p) \simeq 0 *_{\pi_1(I,p)} 0 \simeq 0$$

for any $p \in I$. □

A particularly interesting example is:

Corollary 9.13. *Let $n > 1$. Then $\pi_1(\mathbb{S}^n, p) \simeq 0$ for all p. The first homology groups also vanish.*

Proof. For all $n > 0$ the sphere is path-connected, so indeed it does not matter which p we pick. We can then choose A to be a neighbourhood of the northern hemisphere and B a neighbourhood of the southern hemisphere. Both can be assumed to be homeomorphic to balls and thus contractible. Then we have that $I = A \cap B$ is a neighbourhood of the equator $\mathbb{S}^{n-1}$ and it deformation retracts to it.

We pick $p \in \mathbb{S}^{n-1}$. The pieces A and B are path-connected because they are contractible. The intersection is path-connected because $\mathbb{S}^{n-1}$ is path-connected for $n > 1$. It follows that we can apply the corollary to yield the conclusion. □

This shows that the higher spheres do not have holes detectable by loops.

9.4.2 Wedges again

Another interesting case is:

Corollary 9.14. *Let X be a space presented as an union $X = A \cup B$ of open subsets. Let $I = A \cap B$ be simply-connected and fix a point $p \in I$. Then:*

$$\pi_1(X, p) \simeq \pi_1(A, p) * \pi_1(B, p).$$

It can be used to establish the theorem of van Kampen for wedges (that we saw in the previous lecture):

Proof of Theorem 8.28. Let (A, a) and (B, b) be well-pointed. Fix an open $U \subset A$ deformation retracting to a and, similarly, an open $V \subset B$ deformation retracting to b.

We can cover $(A, a) \vee (B, b)$ with the opens $V' = A \cup V$ and $U' = B \cup U$. Using the Pasting Lemma we can produce a deformation retraction of V' to A (just deformation retract V to the wedge point $[a] = [b]$). There is a similar deformation retraction of U' to B and another deformation retraction of $I = V' \cap U' = U \cup V$ to the wedge point. In particular, I is contractible. Applying Theorem 9.11 we deduce:

$$\begin{aligned}\pi_1((A, a) \vee (B, b)) &\simeq \pi_1(V', [a]) *_{\pi_1(I,[a])} \pi_1(U', [a]) \simeq \pi_1(A, a) *_0 \pi_1(B, b) \\ &\simeq \pi_1(A, a) * \pi_1(B, b),\end{aligned}$$

as claimed. □

9.5 Worked out example: Using van Kampen

The following explains how van Kampen is used in concrete computations. When solving exercises you should argue similarly.

Lemma 9.15. *Let $X :=$ pushout$(\mathbb{S}^2 \leftarrow \{0, 1\} \rightarrow [0, 1])$. The arrow on the left takes 0 to the north pole N and 1 to the south pole S. The arrow on the right is the usual inclusion. Then:*

$$\pi_1(X, x) \simeq \mathbb{Z}.$$

Proof. First step: choosing a cover. We first cover X by two opens A and B, whose intersection $I = A \cap B$ is path-connected. For this, we let $\gamma \subset \mathbb{S}^2$ be the meridian $\{x = 0, y \geq 0\}$. Let $U \subset \mathbb{S}^2$ be an open containing γ, to be defined precisely in a moment. We then let $A = U \cup [0,1]$ and $B = \mathbb{S}^2 \setminus \gamma$.

Second step: verifying openness. We see that A is open because we can explicitly find neighbourhoods in A of any of its points. For points in $(0,1)$ it suffices to take a small interval. For points in $U \setminus \{N, S\}$ we take $U \setminus \{N, S\}$, which is an open in $\mathbb{S}^2$. For N and S we take the union of a small interval in $[0,1]$ (at 0 or 1) and U. That B is open follows from the fact that γ was closed, so B is an open in $\mathbb{S}^2$ that is disjoint from the poles.

Third step: verifying that the intersection is path-connected. Up to here the argument worked for any U, but now we need to be a bit more precise. We require that U is homeomorphic to an open 2-disc. To do this, consider the point $R = (0, -1, 0) \in \mathbb{S}^2$ and the stereographic projection $\phi : \mathbb{S}^2 \setminus \{R\} \to \mathbb{R}^2$. Recall that ϕ is a homeomorphism. Since γ is contained in the complement of R, $\phi(\gamma)$ is a curve in $\mathbb{R}^2$. Since it is compact, it is contained in any sufficiently big open disc $U' \subset \mathbb{R}^2$. We then let $U = \phi^{-1}(U')$.

Consider now $I = U \setminus \gamma$. This is homeomorphic to $U' \setminus \phi(\gamma)$, via ϕ. This is a big open disc minus a curve. In fact, we can be precise and recall what the stereographic projection does. If you do this you will see that $\phi(\gamma)$ is a closed interval within the X-axis. It then follows that $U' \setminus \phi(\gamma)$ deformation retracts, pushing radially, to some sufficiently big circle $S \subset U'$. In particular, $U \setminus \gamma$ is homotopy equivalent to $S \simeq \mathbb{S}^1$, so it is path-connected.

Third step: computing the fundamental group of the pieces. The previous computation shows that $\pi_1(I, i) \simeq \mathbb{Z}$ for any i. We now look at A and B.

We can see that B is contractible. Indeed, performing stereographic projection, now from $(0, 1, 0)$, we can identify B with a subset of $\mathbb{R}^2$ that is star-shaped. It follows that $\pi_1(B, b) \simeq 0$ for any b.

Lastly, we can show that A deformation retracts to $\gamma \cup [0,1]$. This deformation retraction we define to be fixed over $[0,1]$. By the pasting lemma, it suffices that we explain how U deformation retracts to γ. This is equivalent to proving that U' deformation retracts to $\phi(\gamma)$. This you can do by first contracting U' linearly to the X-axis and then contracting the X-axis to the interval $\phi(\gamma)$. It follows that $\pi_1(A, a) \simeq \mathbb{Z}$ for any a.

Do note that all the pieces are path-connected, which is why we did not have to be careful with basepoints. It follows that X is also path-connected, so we can also compute its π_1 at any point.

Fourth step: explicit generators. We now need to identify the curves whose homotopy classes generate the fundamental groups we have computed. This is required in order to compute pushforwards.

We need to fix a basepoint $i \in I$. We choose $i \in \phi^{-1}(S)$, where $S \subset U'$ is the big circle we chose earlier. We saw that I deformation retracts to $\phi^{-1}(S)$. It is a circle that can be parametrised to yield a generator β of $\pi_1(I, i) \simeq \langle \beta \mid \rangle$.

We deformation retracted A to $\gamma \cup [0,1]$. It follows that this curve (once we choose a parametrisation) precisely represents the generator $\alpha \in \pi_1(A, N) \simeq \langle \alpha \mid \rangle$, with N the north pole. We then use change of basepoint to establish an isomorphism $\pi_1(A, N) \simeq \pi_1(A, i)$. We still denote the generator in $\pi_1(A, i)$ by α.

Fifth step: the pushforwards. Let us write $\iota_A : I \to A$ and $\iota_B : I \to B$ for the inclusions. Observe that

$$(\iota_B)_* : \pi_1(I, i) \to \pi_1(B, i)$$

is identically zero, since the target group is zero. We claim that

$$(\iota_A)_* : \pi_1(I, i) \to \pi_1(A, i)$$

is also zero, even though both groups are isomorphic to $\mathbb{Z}$. The reason is that the curve $\phi^{-1}(S)$ is fully contained in U, which is contractible, meaning that all loops within are nullhomotopic. In particular:

$$(\iota_A)_*(\beta) \in \pi_1(U, i) = \{[c_i]\} \subset \pi_1(A, i).$$

Sixth step: the conclusion. We have checked that A and B are open and that I is path-connected, so we are in the hypothesis of the theorem of van Kampen. We have also deduced that X is path-connected, so we can compute π_1 at any basepoint. Then:

$$\pi_1(X, i) \simeq \pi_1(A, i) *_{\pi_1(I,i)} \pi_1(B, i) \simeq \mathbb{Z} *_{\mathbb{Z}} 0 \simeq \langle \alpha \mid \rangle,$$

where in the last step we have used that $(\iota_A)_*$ and $(\iota_B)_*$ are zero and thus introduce no relations. □

9.6 Exercises

9.6.1 Pushouts

Exercise 9.1: Consider groups $G = \langle a \mid a^m \rangle$ and $H = \langle b \mid b^l \rangle$. Suppose that m and l are coprime. Let $I = \langle c \mid \rangle$ and consider the homomorphisms $\iota_G(c) = a$ and $\iota_H(c) = b^{-1}$. Prove that $G *_I H$ is isomorphic to the trivial group.

9.6.2 Applications of van Kampen

Exercise 9.2: Let $\mathbb{S}^1 \subset \mathbb{S}^2$ be the usual inclusion as the equator. Prove that $\mathbb{S}^1$ is not a retract.

Exercise 9.3: Fix k and l positive integers. Compute the fundamental group of the wedge $(\mathbb{S}^k, p) \vee (\mathbb{S}^l, q)$.

Exercise 9.4: Let X_k be the quotient of $\mathbb{D}^2 \coprod \mathbb{S}^1$ under the identification $\phi : \partial\mathbb{D}^2 \to \mathbb{S}^1$ given by $\phi(z) = z^k$.

- Compute the fundamental group of X_k.
- Compute the first homology with $\mathbb{Z}$ and $\mathbb{R}$ coefficients.

Exercise 9.5: Let $X = \mathbb{R}^2 \setminus \{p, q\}$ with p, q distinct from each other and from the origin. Prove that $\pi_1(X, 0) \cong \mathbb{Z} * \mathbb{Z}$ in the following two ways:

- Compare it to the wedge of two circles.
- Apply van Kampen directly.

Exercise 9.6: Let A, B be two copies of the torus $T^2 := \mathbb{S}^1 \times \mathbb{S}^1$. Compute the fundamental group of

$$C := \frac{A \coprod B}{A \ni (z,1) \cong (z,z) \in B \text{ for every } z \in \mathbb{S}^1}.$$

Exercise 9.7: Let X be $\mathbb{R}^n \setminus \{p_1, \cdots, p_m\}$, where the p_i are distinct from each other and the origin. Compute $\pi_1(X, 0)$ for all n and m (I suggest starting with n, m small).

Exercise 9.8: Let $X \subset \mathbb{R}^n$ be the union of a finite collection $\{X_i\}_{i=1}^m$ of open convex sets. Assume that $X_i \cap X_j \cap X_k \cong \emptyset$ for all i, j, k. Prove that X is simply-connected.

Hint: Use van Kampen and induction on m. If you get stuck, get some intuition by looking at the cases $m = 1, 2, 3, 4$ in $\mathbb{R}^2$ first.

The theorem of van Kampen for Π_1

Lecture 10

The theorem of van Kampen, in full generality, says that that the fundamental groupoid of a (nice) pushout is also a pushout. In this lecture we:

- Discuss pushouts in Grpoid (Section 10.1).
- State the fundamental groupoid version of the theorem of van Kampen (Theorem 10.8).
- State a variation in which we replace the fundamental groupoids by smaller equivalent groupoids (Theorem 10.11), making things more computable.
- Compute the fundamental group of the circle (Section 10.3) and see a couple of very fun (and deep!) applications (Subsection 10.3.1).
- Introduce cell attachments. This is the process of glueing an n-dimensional disc to a space (Definition 10.20).
- Particularise van Kampen to the setting of cell attachments, where it boils down to a simpler computation (Theorems 10.24, 10.23, 10.25, and 10.26).
- Introduce cell complexes, which are spaces built inductively by attaching discs (Definition 10.27).

The motto is that Algebraic Topology is not about so much about strange pathological spaces, but about cell complexes, which are spaces that are well-behaved locally, but have an interesting global theory of "holes".

10.1 Groupoids

We now address pushouts in Grpoid.

10.1.1 The coproduct

First we look at coproducts:

Lemma 10.1. *The coproduct in* Grpoid *is the disjoint union.*

Proof. Concretely: let $\mathcal{G} \rightrightarrows B$ and $\mathcal{G}' \rightrightarrows B'$ be groupoids. Then we must show that

$$\mathcal{G} \coprod \mathcal{G}' \rightrightarrows B \coprod B'$$

is their coproduct.

First observe that $\mathcal{G} \coprod \mathcal{G}' \rightrightarrows B \coprod B'$ is not connected. By construction,

$$\pi_0(\mathcal{G} \coprod \mathcal{G}') \simeq \pi_0(\mathcal{G}) \coprod \pi_0(\mathcal{G}').$$

That is to say, morphisms in $\mathcal{G}$ do not interact with those in $\mathcal{G}'$. This implies that any pair of functors from $\mathcal{G}$ and $\mathcal{G}'$ into a third groupoid $\mathcal{H} \rightrightarrows A$ factors uniquely through $\mathcal{G} \coprod \mathcal{G}'$. Details are left to the reader. □

Remark 10.2: Since groupoids are allowed to have multiple identities, the coproduct in Grpoid is much easier than the coproduct in Grp. In particular, observe that the inclusion functor Grp → Grpoid does not preserve the coproduct. △

10.1.2 The pushout

Despite Remark 10.2, pushouts in Grpoid are similar to those in Grp:

Definition 10.3. *Let $\mathcal{G} \rightrightarrows G$, $\mathcal{H} \rightrightarrows H$, and $\mathcal{I} \rightrightarrows I$ be groupoids. We will henceforth omit G, H and I from the notation for readability. Consider the following diagram in* Grpoid*:*

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{f_{IG}} & \mathcal{G} \\ {\scriptstyle f_{IH}}\downarrow & & \\ \mathcal{H} & & \end{array}$$

*We will now define the **amalgamated product** $\mathcal{G} *_{\mathcal{I}} \mathcal{H} \rightrightarrows U$ of $\mathcal{G}$ and $\mathcal{H}$ along $\mathcal{I}$, with respect to the functors f_{IG} and f_{IH}.*

We set U to be the pushout in Set *of $G \leftarrow I \rightarrow H$.*

The morphisms in $\mathcal{G} *_{\mathcal{I}} \mathcal{H}$ are words quotiented by moves. The words under consideration are:

- *We consider the alphabet whose letters are morphisms in $\mathcal{G}$ or $\mathcal{H}$.*
- *We say that a letter $\gamma \in \mathcal{G} \coprod \mathcal{H}$ is **composable** with another letter $\nu \in \mathcal{G} \coprod \mathcal{H}$ if $[\mathfrak{t}(\gamma)] = [\mathfrak{s}(\nu)] \in U$.*
- *We let W be the set of **admissible** words. A word is admissible if adjacent letters are composable.*
- *Given a word in W of the form $\gamma \cdots \nu$, its **source** is $[\mathfrak{s}(\nu)] \in U$ and its **target** is $[\mathfrak{t}(\gamma)] \in U$.*

We can then define concatenation of words: $\omega_1 \in W$ *is* ***concatenable*** *with* $\omega_2 \in W$ *if* $\mathfrak{t}(\omega_1) = \mathfrak{s}(\omega_2)$.

To yield the morphisms, we quotient W *by the relation generated by the moves:*

I. *Let* $u \in U$ *and let* x, y *be elements in* G *or* H *with* $[x] = [y] = u$*. Then* $\mathrm{id}_u := \mathrm{id}_x \simeq \mathrm{id}_y \in \mathcal{G} *_{\mathcal{I}} \mathcal{H}$.

II. *Given* $\gamma, \nu \in \mathcal{G}$ *composable in* $\mathcal{G}$*, the word* $\nu\gamma$ *with two letters is equivalent to the word* $\nu \bullet \gamma$ *with a single letter.*

II'. *Similarly, given* $\gamma, \nu \in \mathcal{H}$ *composable in* $\mathcal{H}$*, the word* $\nu\gamma$ *is equivalent to* $\nu \bullet \gamma$.

III. *The single word letters* $f_{IG}(\alpha)$ *and* $f_{IH}(\alpha)$ *are equivalent, for every* $\alpha \in \mathcal{I}$.

Morphisms are composed by concatenating representative words.

As in Grp, observe that $\mathcal{G} *_{\mathcal{I}} \mathcal{H}$ depends on f_{IG} and f_{IH}, even if they are missing from the notation.

Lemma 10.4. *Definition 10.3 is well-defined.*

Proof. This follows from the following observations:

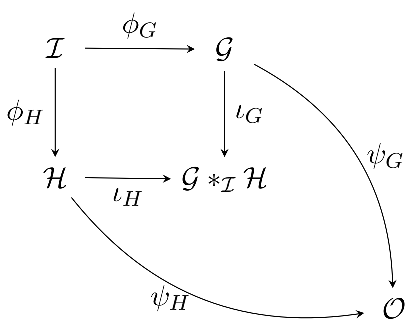
- Moves preserve admissibility.
- Given two words in W, their concatenation is also admissible (since its adjacent letters are still composable).
- Suppose that ω and ω' relate by moves, and α and α' relate by moves. Then $\omega\alpha$ relates to $\omega'\alpha'$ (just apply the relevant moves to each word).

□

Remark 10.5: Two morphisms $\gamma, \nu \in \mathcal{G}$ can be composable as letters in W even if they were originally not composable as morphisms in $\mathcal{G}$. The reason is that once we identify the sets of objects, it may hold that $[\mathfrak{t}(\gamma)] = [\mathfrak{s}(\nu)] \in U$ even if $\mathfrak{t}(\gamma)] \neq \mathfrak{s}(\nu) \in G$. △

Lemma 10.6. *Pushouts in* Grpoid *are amalgamated products.*

Proof. Consider groupoids $\mathcal{G} \rightrightarrows G$, $\mathcal{H} \rightrightarrows H$, and $\mathcal{I} \rightrightarrows I$. Furthermore fix a fourth auxiliary groupoid $\mathcal{O} \rightrightarrows O$. We are then given a commutative diagram



and we must show that there is a unique $\alpha : \mathcal{G} *_{\mathcal{I}} \mathcal{H} \to \mathcal{O}$ making the diagram commute. At the level of objects, we let α be the canonical map $G \coprod_I H \to O$ given by the pushout in Set. We now address morphisms.

We are forced to let α be the unique functor that evaluates on letters from $\mathcal{G}$ as ψ_G and on letters from $\mathcal{H}$ as ψ_H. We extend uniquely to words by applying α letter by letter and then composing in $\mathcal{O}$. Do note that this is well-defined: Our definition of α at the level of objects implies that admissible words are mapped by α to composable sequences of morphisms in $\mathcal{O}$.

It remains to show that α is well-defined once we quotient the set of words W; this we check one move at a time. This is done as in the case of groups, so we leave it to the reader. Checking move I implies that α preserves identities. α preserves composition because it already sent concatenation in W to composition in $\mathcal{O}$. It follows that α is a functor. □

Example 10.7: As we observed in Remark 10.2, the coproduct is not preserved by the inclusion Grp $\to$ Grpoid. This is related to the following phenomenon: $\emptyset$ is a groupoid, but not a group (as every group must have an identity). Given groups G and H, it follows that the diagram $G \leftarrow \emptyset \to H$ does not make sense in Grp, but it does in Grpoid (and there its pushout is the disjoint union).

However, if you compare Definitions 9.8 and 10.3 you can see that the pushout of $G \leftarrow \{e\} \to H$ in Grp and Grpoid is the same. △

For a more interesting example of pushout of groupoids you should read the proof of Theorem 10.13.

10.2 The theorem of van Kampen

We now state the fundamental groupoid version of van Kampen. It will be used to establish all other versions:

Theorem 10.8. *Let X be a space presented as an union $X = A \cup B$ of open subsets. Write $I = A \cap B$. Let $j_A : I \to A$, $j_B : I \to B$, $i_A : A \to X$, and $i_B : B \to X$ for the inclusions. Then:*

$$\begin{array}{ccc} \Pi_1(I) & \xrightarrow{(j_A)_*} & \Pi_1(A) \\ {\scriptstyle (j_B)_*}\downarrow & & \downarrow{\scriptstyle (i_A)_*} \\ \Pi_1(B) & \xrightarrow[(i_B)_*]{} & \Pi_1(X) \end{array}$$

is a pushout. In particular,

$$\Pi_1(X) \simeq \Pi_1(A) *_{\Pi_1(I)} \Pi_1(B).$$

We postpone the proof of Theorem 10.8 to Section 12.1.

10.2.1 Several basepoint version

As we observed in Corollary 6.2, Π_1 contains a lot of redundant information. Namely, within a path-component, all points have isomorphic fundamental groups. It is then sensible to try to throw away some of this information strategically:

Definition 10.9. *Let X be a space and $P \subset X$ a subspace. We write $\pi_1(X, P) \rightrightarrows P$ for the full subgroupoid of $\Pi_1(X)$ with object set P.*

That is: we keep all the arrows between the elements of P, but we discard all other points in X.

Remark 10.10: $\pi_1(X, P)$ is equivalent (Definition 7.1), as a category, to $\Pi_1(X)$. When $P \subset X$ contains exactly one point per path-component, $\pi_1(X, P) \rightrightarrows P$ is a skeleton (Definition 7.2) of $\Pi_1(X)$. △

Theorem 10.11. *We use the same notation as in Theorem 10.8 and we furthermore fix a subspace $P \subset X$. We work under the assumption that:*

- *A and B are open.*
- *P intersects each path-component of I, A and B non-trivially.*

Then:

$$\begin{array}{ccc} \pi_1(I, I \cap P) & \longrightarrow & \pi_1(A, A \cap P) \\ \downarrow & & \downarrow \\ \pi_1(B, B \cap P) & \longrightarrow & \pi_1(X, P) \end{array}$$

is a pushout. In particular:

$$\pi_1(X, P) \simeq \pi_1(A, A \cap P) *_{\pi_1(I, I \cap P)} \pi_1(B, B \cap P).$$

This result is established in Appendix 12.2. For now, let us explain why the theorem is plausible. Indeed, the second assumption says that each fundamental groupoid (say $\Pi_1(I)$) is being replaced by an equivalent one (say $\pi_1(I, I \cap P)$). Moreover, this is done in a manner that is coherent with the functors between them. Intuitively, we are not losing information, so the resulting diagram is still a pushout.

Remark 10.12: Observe that Theorem 9.11 (van Kampen for π_1) is the concrete case of Theorem 10.11 in which $P = \{p\}$. The assumptions on path-connectedness guarantee that P intersects every (i.e. the only) path-component of A, B, and I. △

10.3 The fundamental group of the circle

After much ado, we can finally state and prove:

Theorem 10.13. $\pi_1(\mathbb{S}^1, 1) \simeq_{\text{Grp}} \mathbb{Z}$.

Proof. This application is more difficult than the ones we have seen so far, since we cannot apply the group version of van Kampen. Indeed, cover $\mathbb{S}^1$ by two open intervals A and B by letting A be a neighbourhood of the upper hemisphere and B a neighbourhood of the lower one. Then $I = A \cap B$ has two components: one of them I_+ is an open interval around $1 \in \mathbb{S}^1$ and the other I_- is an open interval around -1. Since I is not path-connected, Theorem 9.11 does not apply[1].

Instead we consider the subset $P = \{1, -1\} \subset \mathbb{S}^1$. Our goal is to apply Theorem 10.11. Observe that $\pi_1(A, P)$ and $\pi_1(B, P)$ are both isomorphic to the pair groupoid of P (Lemma 6.15). I.e. $\pi_1(A, P)$ contains the morphisms $\{c_1, c_{-1}, a, a^{-1}\}$ representing, respectively, the constant paths in A with value 1 or -1, the unique class of path from 1 to -1 within A, and its inverse. Similarly, $\pi_1(B, P) = \{c_1, c_{-1}, b, b^{-1}\}$; now b is the unique path from -1 to 1 within B. Lastly, $\pi_1(I, P) = \{c_1, c_{-1}\}$, since the two points cannot be connected within I.

Using the fact that amalgamated products are pushouts in Grpoid (Lemma 10.6) and Theorem 10.11 we deduce that:

$$\pi_1(\mathbb{S}^1, P) \simeq \{c_1, c_{-1}, a, a^{-1}\} *_{\{c_1, c_{-1}\}} \{c_1, c_{-1}, b, b^{-1}\}.$$

Now it remains to spell out what this means using the definition of amalgamated product of groupoids (Definition 10.3).

The objects underlying $\pi_1(A, P)$, $\pi_1(B, P)$ and $\pi_1(I, P)$ are all P, so the set of objects of $\pi_1(\mathbb{S}^1, P)$ is also P. Let us now focus on the morphisms that go from 1 to 1 (i.e. $\pi_1(\mathbb{S}^1, 1)$!) These are words written using the letters a and b. Subsequent letters must be concatenable. This means that only b or a^{-1} can follow a. Similarly, only a or b^{-1} can follow b. Furthermore, since we are assuming that we begin and end at 1, the words we look at must begin in a or b^{-1} and finish in b or a^{-1}.

Putting these constraints together it follows that, once we simplify any word by cancelling inverses, it must be an alternation of a and b or an alternation of a^{-1} and b^{-1}. Let us list these words concretely:

$$\pi_1(\mathbb{S}^1, 1) \simeq \{\cdots, (ba)^{-3}(ba)^{-2}, (ba)^{-1}, 1, ba, (ba)^2, (ba)^3, \cdots\}$$

the composition being concatenation. I.e. we have produced the group $\langle (ba) \ |, \rangle$, with (ba) being a single symbol. This is just an involved way of writing $(\mathbb{Z}, +)$. ☐

Looking at the proof we can readily find a representative of the generator ba:

Corollary 10.14. *The class* $[\text{id}_{\mathbb{S}^1}]$ *is a generator in* $\pi_1(\mathbb{S}^1, 1)$.

[1] If it did, we would obtain a trivial fundamental group (since A and B are contractible), contradicting the statement.

More generally:

Corollary 10.15. *Let* $\gamma_k : \mathbb{S}^1 \to \mathbb{S}^1$ *be the map* $z \mapsto z^k$*. Then we can write out explicitly the isomorphism* $\pi_1(\mathbb{S}^1, 1) \simeq \mathbb{Z}$ *as*

$$[\gamma_k] \mapsto k.$$

You can also check manually that $[\gamma_k \bullet \gamma_l] = [\gamma_{k+l}]$.

Corollary 10.16. *The following statements hold:*

- $[\mathbb{S}^1, \mathbb{S}^1] \simeq_{\mathrm{Set}} \mathbb{Z}$.
- *In particular, every map* $\mathbb{S}^1 \to \mathbb{S}^1$ *is homotopic to one of the maps* γ_k *from Corollary 10.15.*
- $H_1(\mathbb{S}^1, \mathbb{Z}) \simeq \mathbb{Z}$.
- $H_1(\mathbb{S}^1, \mathbb{R}) \simeq \mathbb{R}$.

Proof. The last two statements follow from the definition of the first homologies. To establish the first and second we will prove that the forgetful function $\psi : \pi_1(\mathbb{S}^1, 1) \to [\mathbb{S}^1, \mathbb{S}^1]$ is bijective. Indeed, it is surjective because $\mathbb{S}^1$ is path-connected. Furthermore, two classes have the same image under ψ if and only if they are conjugate. However, conjugation in an abelian group is abelian. This implies that two classes have the same image if and only if they are the same, proving injectivity of ψ. □

Corollary 10.17. *The circle* $\mathbb{S}^1$ *is not a retract of* $\mathbb{R}^2$*.*

Proof. If there was a retraction $r : \mathbb{R}^2 \to \mathbb{S}^1$, it would hold that $r_* : \pi_1(\mathbb{R}^2, 1) \to \pi_1(\mathbb{S}^1, 1)$ is surjective (Corollary 5.22), but this cannot be true since the former is the trivial group and the latter is isomorphic to $\mathbb{Z}$. □

10.3.1 Applications

The following is the famous *Brouwer's fixed point theorem*:

Theorem 10.18. *Let* $f : \mathbb{D}^2 \to \mathbb{D}^2$ *be map of the closed disc. Then* f *has a fixed point.*

Proof. Suppose otherwise. Then we can define a map $\mathbb{D}^2 \to \mathbb{S}^1$:

$$g := \frac{f(z) - z}{|f(z) - z|}.$$

We restrict it to $\mathbb{S}^1$ to yield $h : \mathbb{S}^1 \to \mathbb{S}^1$.

Since $h = g|_{\mathbb{S}^1}$, we deduce that h is nullhomotopic (Exercise 2.2). We will now reach a contradiction by showing that h is homotopic to $\mathrm{id}_{\mathbb{S}^1}$, which we have just shown is not nullhomotopic (Corollaries 10.14 and 10.16).

The crucial observation is the following: Given $z \in \mathbb{S}^1$ and $w \in \mathbb{D}^2$, it holds that $\langle w, z \rangle \leq 1$ and the equality holds if and only if $w = z$. This is a consequence of Cauchy-Schwartz and it geometrically means that z is the only vector that is "furthest away from 0 in the direction of z". Since $f(z) \neq z$, it follows that

$$\langle f(z) - z, z\rangle = \langle f(z), z\rangle - 1 < 0,$$

i.e. $f(z) - z$ points (roughly) opposite to z.

$-z$ points exactly opposite to z. It follows that:

$$\langle (1-t)[f(z) - z] - tz, z\rangle = (1-t)\langle f(z), z\rangle - 1 < 0.$$

It implies that the vector $(1-t)[f(z) - z] - tz$ is never zero, so we can define:

$$h_t := \frac{(1-t)[f(z) - z] - tz}{|(1-t)[f(z) - z] - tz|} : \mathbb{S}^1 \to \mathbb{S}^1.$$

It satisfies $h_0 = h$ and $h_1(z) = -z$.

It remains to show that $[h_1] = [\mathrm{id}_{\mathbb{S}^1}]$. This can be shown using the further homotopy $(z \mapsto e^{t\pi i}z)_{t\in[0,1]}$. □

The other very cool application is the *fundamental theorem of algebra*:

Theorem 10.19. *Let $P : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of order exactly k. Then P has exactly k roots (counting with multiplicity). Identically, P can be factorised as a product of k first order polynomials.*

Proof. First observe that it suffices to show that a single root exists. Once that is established we can write $P = Q(z - z_0)$, with z_0 the root we found and Q of order exactly $k - 1$, so induction can be applied. The statement is trivially true for P of order one, which is the inductive hypothesis.

Suppose for contradiction that P has no roots. Then we can fix any radius r and define:

$$Q(z) := \frac{P(rz)}{|P(rz)|} : \mathbb{S}^1 \to \mathbb{S}^1.$$

It is nullhomotopic, since it extends to a map $\mathbb{D}^2 \to \mathbb{S}^1$, using the same formula. We will reach contradiction by showing that, for r sufficiently large, it holds that $[Q] = [z- > z^k]$, which is non-trivial (Corollary 10.15).

Observe that we can assume that the leading coefficient of P is 1, since that does not change the roots. We can thus write $P(z) = z^k + \sum_i a_i z^i$, for some coefficients a_i. The key remark is that, if $|z| = r$ is sufficiently large, we have that $|\sum_i a_i z^i| < |z|^k/2$ (left to the reader). This implies that $P_t(z) := z^k + (1-t)\sum_i a_i z^i$ is a homotopy of order k polynomials with $|P_t(z)| \neq 0$ over the sphere of radius r. This allows us to define

$$Q_t(z) := \frac{P_t(rz)}{|P_t(rz)|} : \mathbb{S}^1 \to \mathbb{S}^1$$

which is a homotopy exhibiting $[Q] = [Q_0] = [Q_1] = [z \mapsto z^k]$. □

I.e. the idea behind the proof is that, close to infinity, complex polynomials of order k loop around the origin k times.

10.4 Attaching cells

The following is our main tool to construct topological spaces:

Definition 10.20. *Consider a space X and a map $\psi : \mathbb{S}^{n-1} \to X$. Write $\iota : \mathbb{S}^{n-1} \to \mathbb{D}^n$ for the inclusion. Then we say that:*

- *The space*
$$Y := \text{pushout}(\mathbb{D}^n \xleftarrow{\iota} \mathbb{S}^{n-1} \xrightarrow{\psi} X)$$
*is the result of **attaching an n-cell** to X.*
- *$\mathbb{D}^n \to Y$ is the **characteristic map** of the n-cell that we have attached.*
- *ψ is the **attaching map** of the cell.*

0-cells, 1-cells, and 2-cells are often referred to as **vertices**, **edges**, and **faces**, respectively.

The stereotypical example of attaching a cell is:

Lemma 10.21. *Attaching an n-cell to $\{p\}$ produces a space homeomorphic to $\mathbb{S}^n$.*

Proof. The attaching map can only be the constant map $\psi = c_p$. It follows that

$$\text{pushout}(\mathbb{D}^n \overset{\iota}{\leftarrow} \mathbb{S}^{n-1} \overset{c_p}{\to} \{p\})$$

is the space one obtains from $\mathbb{D}^n$ by identifying all the points in $\mathbb{S}^{n-1}$, which is indeed the n-sphere. □

In particular, observe that the characteristic map is an inclusion in the interior of the cell, but not necessarily along the boundary.

10.4.1 Cell attachment and fundamental group

The following results explain how the fundamental group changes under cell attachment. This is highly dependent on the dimension of the cell to be attached. We will provide all the statements and postpone the proofs to Section 11.1.

Let X be a topological space, $\psi : \mathbb{S}^{n-1} \to X$ the attaching map of an n-cell, and $Y = \text{pushout}(X \leftarrow \mathbb{S}^{n-1} \to \mathbb{D}^n)$ the result of attaching along ψ. We write $\iota : X \to Y$ for the inclusion.

We begin with the attachment of a 0-cell. We note:

Lemma 10.22. *Let Y be the result of attaching a zero-cell to X. Then:*

- *$\pi_0(Y) \simeq \pi_0(X) \coprod \{.\}$.*
- *$\iota_* : \pi_0(X) \to \pi_0(Y)$ is the inclusion.*

There are two possible situations for 1-cells. We state these separately. The first case is that attaching connects two path-components that were previously separate, putting their fundamental groups together:

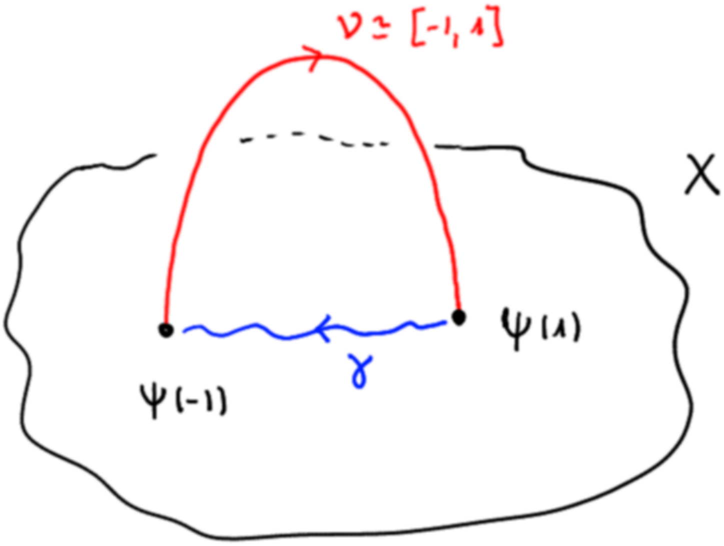


Figure 10.1: A 1-cell being attached to a single path-component. The new generator of π_1 is the concatenation of the cell itself (ν) and a path within X connecting the two points of the attaching (γ).

Theorem 10.23. *Let Y be the result of attaching a 1-cell to X using ψ. Suppose that $[\psi(1)] \neq [\psi(-1)] \in \pi_0(X)$. Then:*

- *$\iota_* : \pi_0(X) \to \pi_0(Y)$ is the quotient map of the identification $[\psi(1)] \cong [\psi(-1)] \in \pi_0(X)$.*
- *$\pi_1(Y, \psi(1)) \simeq \pi_1(X, \psi(1)) * \pi_1(X, \psi(-1))$.*
- *The pushforwards of the inclusions $\pi_1(X, \psi(\pm 1)) \to \pi_1(Y, \psi(1))$ are the inclusion monomorphisms.*

In this concrete case you should think of Y as a better behaved version of the wedge product. We are joining two components X_- and X_+ of X not by glueing them at a point (as we would do in the wedge) but by putting an interval between the two. The nice local structure of the interval allows us not to have any assumptions on the local structure of X_- and X_+ (unlike for the wedge).

The second case is that attaching creates a new loop, introducing a new generator into the fundamental group:

Theorem 10.24. *Let Y be the result of attaching a 1-cell to X using ψ. Suppose that $[\psi(1)] = [\psi(-1)] \in \pi_0(X)$. Then:*

- *$\iota_* : \pi_0(X) \to \pi_0(Y)$ is an isomorphism.*
- *$\pi_1(Y, \psi(1)) \simeq \pi_1(X, \psi(1)) * \mathbb{Z}$.*
- *$\iota_* : \pi_1(X, \psi(1)) \to \pi_1(Y, \psi(1))$ is the inclusion homomorphism.*
- *Fix $[\gamma] \in \pi_1(X, \psi(1), \psi(-1))$ and $[\nu] \in \pi_1(\mathbb{D}^1, -1, 1)$. Then, we can choose the generator of the term $\mathbb{Z}$ to be $[\nu][\gamma]$.*

See Figure 10.1.

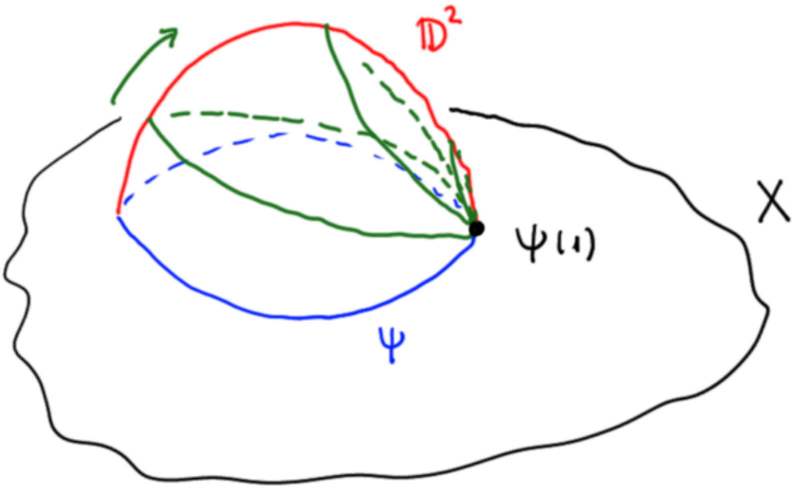


Figure 10.2: A 2-cell being attached. The new disc bounds the attaching loop ψ, which becomes nullhomotopic. A explicit (pointed) nullhomotopy across the disc is shown in green.

We now look into the attachment of a disc, a 2-dimensional cell. The boundary of $\mathbb{D}^2$ is nullhomotopic in $\mathbb{D}^2$, so attaching a 2-cell will make the class in X represented by its boundary trivial:

Theorem 10.25. *Let Y be the result of attaching a 2-cell to X using ψ. Then:*

- $\iota_* : \pi_0(X) \to \pi_0(Y)$ *is an isomorphism.*
- $\pi_1(Y, \psi(1)) \simeq \dfrac{\pi_1(X, \psi(1))}{[\psi]}$.
- $\iota_* : \pi_1(X, \psi(1)) \to \pi_1(Y, \psi(1))$ *is the quotient homomorphism.*

This means that $\pi_1(Y, \psi(1))$ has the same generators as $\pi_1(X, \psi(1))$, but one more relation (namely, $[\psi]$). See Figure 10.1.

Lastly, we study the attachment of higher-dimensional cells. These do not have any effect in the fundamental group. The idea is like in the previous result: attaching an n-cell, $n > 2$, will make its boundary in X be nullhomotopic, but its boundary is not a loop anymore, but an $(n-1)$-cell[2].

Theorem 10.26. *Let Y be the result of attaching an n-cell to X using ψ, with $n \geq 3$. Then;*

- $\iota_* : \pi_0(X) \to \pi_0(Y)$ *is an isomorphism.*
- $\iota_* : \pi_1(X, \psi(1)) \to \pi_1(Y, \psi(1))$ *is an isomorphism.*
- $\iota_* : [\mathbb{S}^{n-1}, X] \to [\mathbb{S}^{n-1}, Y]$ *sends* $[\psi]$ *to* $[c_{\psi(1)}]$*, the class of the constant map.*

[2] This goes beyond the scope of this course, but this suggests that you can remove classes in $\pi_{n-1}(X, x)$ ("higher holes") by attaching n-cells to X. This is not very surprising: attaching an n-cell basically means "filling in the hole using $\mathbb{D}^n$".

10.5 Cell complexes

We will henceforth be interested in spaces obtained by successively attaching cells. The recipe is the following:

Definition 10.27. *An N-dimensional* ***cell complex*** *is a space X that can be presented as:*

$$X_{-1} := \emptyset \subset X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_N = X$$

with each X_n obtained from X_{n-1} by attaching n-cells.

Concretely, this means that we have characteristic maps $\phi_j^n : \mathbb{D}^n \to X_n$ such that:

- *The attaching map $\psi_j^n = \phi_j^n|_{\partial\mathbb{D}^n}$ takes values in X_{n-1}.*
- *X_n is the pushout of $\coprod_j \mathbb{D}^n \longleftarrow \coprod_j \mathbb{S}^{n-1} \overset{\coprod \psi_j^n}{\longrightarrow} X$.*

The subspace X_n is often referred to as the ***nth-skeleton*** *of X.*

It is also common to refer to cell complexes as *cellular complexes* or *CW-complexes.*

In a cell complex, X_n is obtained from X_{n-1} by attaching all the n-cells at once. In particular: we cannot attach n-cells on top of one another! We always attach along smaller cells.

Lemma 10.28. *A 0-dimensional complex is a space X with the discrete topology.*

Proof. By assumption $X = X_0$ is obtained from $X_{-1} = \emptyset$ by attaching zero cells, which are points. It follows that X is the coproduct of a collection of points, so it has the discrete topology. □

We are not making any assumptions on the cardinality of X.

Lemma 10.29. *A 1-dimensional complex*

$$X := \text{pushout}(\coprod\{.\} \longleftarrow \coprod\{0,1\} \longrightarrow \coprod[0,1])$$

is a graph.

Proof. This is more of a definition than a lemma. It simply says that we start with a discrete space $X_0 := \coprod\{.\}$ of vertices. We then take a collection of intervals ("edges") and the attaching maps tell us how to identify their endpoints with vertices in X_0. □

10.6 Worked out example: the torus

The following explains how one presents a space as the result of attaching cells iteratively. This is then used to compute the fundamental group. This is one of the standard strategies to compute the π_1 of a space. The other strategy, which can be hassle often, is to apply the Theorem of van Kampen directly by finding a suitable cover by opens.

We study the space $\mathbb{T}^2$, using its usual depiction as the quotient $\mathbb{R}^2/\mathbb{Z}^2 = (\mathbb{R}/\mathbb{Z})^2$. This allows us to use the coordinates (x, y) in $\mathbb{R}^2$ to parametrise $\mathbb{T}^2$. Let us denote p for the point $(0, 0) \in \mathbb{T}^2$ (or rather, its equivalence class). Similarly, we write $a := \{y = 0\} \subset \mathbb{T}^2$ and $b := \{x = 0\} \subset \mathbb{T}^2$. These are subsets homeomorphic to $\mathbb{S}^1$, passing through p.

10.6.1 Step 1: Constructing the desired cell complex

The idea is that we will now construct a space S by iterated cell attachment. Later we will exhibit a homeomorphism $S \simeq \mathbb{T}^2$.

We begin with a single vertex q. We let $S_0 := \{q\}$.

We then consider two copies $A, B \simeq [0, 1] \simeq \mathbb{D}^1$ and the attaching maps $\phi_A, \phi_B : \{0, 1\} \to S_0$, which are constant. We write S_1 for the result of attaching A and B to S_0 using these two maps.

Lastly, we consider a square $F = [0, 1]^2$. By Exercise 10.4 it is homeomorphic to $\mathbb{D}^2$. In particular (Lemma 13.5), its boundary ∂F is homeomorphic to $\mathbb{S}^1$. We define an attaching map $\phi_F : \partial F \to S_1$ explicitly: $\phi_F(t, 0) := t \in A$, $\phi_F(1, t) := t \in B$, $\phi_F(1 - t, 1) := 1 - t \in A$, and $\phi_F(0, 1 - t) := 1 - t \in B$. We can then consider the result $S = S_2$ of attaching F to S_1.

10.6.2 Step 2: Proving it is homeomorphic to the torus

By construction, S is the quotient of $\tilde{S} := \{q\} \coprod A \coprod B \coprod F$ by the identifications given by the attaching maps ϕ_A, ϕ_B and ϕ_F. It follows that we can construct a map $f : S \to \mathbb{T}^2$ by constructing first a map $\tilde{f} : \tilde{S} \to \mathbb{T}^2$ and showing it descends to the quotient. Indeed, we define $\tilde{f}(q) := p$, $\tilde{f}|_A(t) := (t, 0) \in a$, $\tilde{f}|_B(t) = (0, t) \in b$, and $\tilde{f}|_F(x, y) = (x, y) \in \mathbb{T}^2$

By construction (exercise for you!) these maps agree on overlaps, and the pieces of $\tilde{S}$ are closed subsets covering S. By the pasting lemma we deduce that $\tilde{f}$ induces the desired continuous map f. Moreover, also by construction (exercise for you!) we get that f is a bijection. The following lemma implies that f is a homeomorphism:

Lemma 10.30. *Let X be compact, let Y be Hausdorff, and let $g : X \to Y$ be continuous. Then:*

- *If g is surjective, it is a quotient map.*
- *If g is bijective, it is a homeomorphism.*

Proof. We address the first item. Fix a subset $U \subset Y$. We must show that U is open if $g^{-1}(U)$ is open. Assuming this, we see that the complement of $g^{-1}(U)$, which is $g^{-1}(C)$, is closed. Since X is compact, this implies that $g^{-1}(C)$ is compact. Its image under g is also compact (by continuity) and, since g is surjective, this image is C. We deduce that C is compact, which allows us to conclude it is closed, now by Hausdorffness of Y. It follows that U is open.

For the second item, we use the first item to deduce that g is a quotient map. Since g is a bijection, it is then an open map. This precisely means that g^{-1} is continuous. □

10.6.3 Step 3: Applying van Kampen

We can now apply Theorems 10.23, 10.24, 10.25, and 10.26 systematically. We have that $\pi_0(S_0) \simeq \{.\}$ and it has vanishing π_1.

Since S_1 is obtained from S_0 by attaching two edges and there is a single path-component, we apply Theorem 10.24 twice. To be explicit about generators, we write $\iota_A : A \to S_1$ and $\iota_B : B \to S_1$ for the inclusions. These are loops based at q, so they represent classes $[\iota_A], [\iota_B] \in \pi_1(S_1, q)$. The theorem says that each attachment introduces a new generator in the fundamental group. Moreover, its last item describes such generator explicitly, namely:

$$\pi_1(S_1, q) \simeq \langle [\iota_A], [\iota_B] \mid \rangle.$$

Note that after each attachment we remain path-connected.

We then apply Theorem 10.25 to the attachment of F. We see ∂F as a loop with basepoint $(0, 0)$, given as the concatenation of four intervals (one per side of F). The attaching map $\phi_F : \partial F \to S_1$ was described explicitly earlier, and it restricts to the first interval as ι_A, to the second interval as ι_B, to the third interval as $\bar{\iota_A}$ and to the last interval as $\bar{\iota_B}$. This implies that $[\phi_F]$ is a class in $\pi_1(S_1, q)$ that can be explicitly written as the commutator:

$$[\phi_F] = [\iota_B]^{-1}[\iota_A]^{-1}[\iota_B][\iota_A] = [[\iota_A], [\iota_B]].$$

Theorem 10.25 then implies that:

$$\pi_1(S, q) \simeq \langle [\iota_A], [\iota_B] \mid [[\iota_A], [\iota_B]] \rangle \simeq \mathbb{Z}^2.$$

This concludes the exercise. You may nonetheless want to observe that:

Corollary 10.31. *The pushforward of the inclusion $S_1 \to S$ is precisely*

$$\pi_1(S_1, q) \simeq \mathbb{Z} * \mathbb{Z} \to \pi_1(S, q) \simeq \mathbb{Z}^2$$

the abelianisation map.

Proof. The pushforward of the inclusion is the obvious quotient:

$$\langle [\iota_A], [\iota_B] \mid \rangle \to \langle [\iota_A], [\iota_B] \mid [[\iota_A], [\iota_B]] \rangle.$$

□

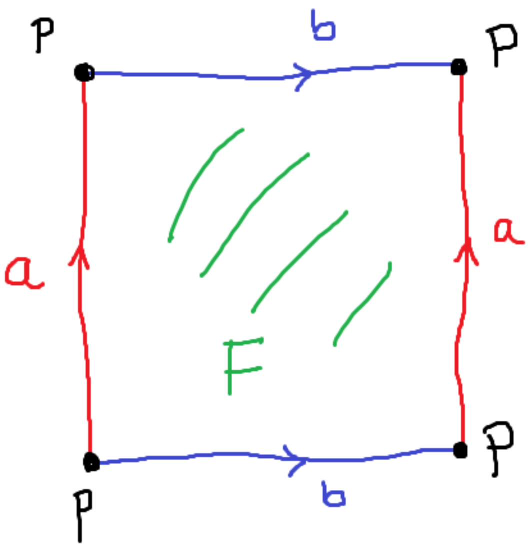


Figure 10.3: The standard cell structure on the torus $\mathbb{T}^2$. It has one vertex p, two edges a and b, and a face F.

10.7 Exercises

10.7.1 The torus

Exercise 10.1: Let $S := \mathbb{T}^2$ be the 2-torus, presented as in Section 10.6. Let $\iota : S_1 \to S$ be the inclusion of its 1-skeleton. Prove that there is no retraction $r : S \to S_1$.

Exercise 10.2: Identify $\mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z}$ and $\mathbb{T}^2$ with $\mathbb{R}^2/\mathbb{Z}^2$. Let a and b be two integers. Consider the curve $\gamma : (\mathbb{S}^1, 0) \to (\mathbb{T}^2, p)$ given by $\gamma(t) := (at, bt)$. Express $[\gamma] \in \pi_1(T^2, p)$ as a word in terms of generators.

Exercise 10.3: Fix complex coordinates in $\mathbb{D}^2$ and $\mathbb{S}^1$. Consider the torus $T^2 := \mathbb{S}^1 \times \mathbb{S}^1$ and the solid torus $S := \mathbb{S}^1 \times \mathbb{D}^2$. The former is a subspace of the latter and we write ι for the inclusion.

- Prove that S can be obtained from T^2 by first attaching a 2-cell and then a 3-cell.
- Use this to compute $\iota_* : \pi_1(T^2, x) \to \pi_1(S, x)$, with $x = (1, 1)$.

Hint: For the 3-cell you should invoke Exercise 10.4 below.

10.7.2 Spaces homeomorphic to the ball

Exercise 10.4: Suppose $X \subset \mathbb{R}^n$ is a compact subset with non-empty interior that additionally satisfies:

- There is $x \in X$ such that, for every other $y \in X$, the segment $[x, y)$ is contained in the interior of X.

Prove that X is homeomorphic to the closed ball $\mathbb{D}^n$.

Exercise 10.5: Use the previous exercise to show that cubes, convex polygons, and tetrahedra are homeomorphic to the closed ball.

10.7.3 Cell complexes

Endow each of the upcoming spaces with a cell structure. State how many cells of each dimension you use. Describe the attaching maps as explicitly as possible. Compute their fundamental groups. Compute their first homologies.

Exercise 10.6: Let X_k be as in Exercise 9.4. Consider the space

$$(Y_k, p) := (\mathbb{S}^1, 1) \vee (X_k, q).$$

Exercise 10.7: Let $f : [0, \pi] \to \mathbb{S}^2 \subset \mathbb{R}^3$ be the map $f(t) \to (\cos(t), \sin(t), 0)$. Consider the space:

$$W := \text{pushout}(\mathbb{S}^2 \xleftarrow{f} [0, \pi] \xrightarrow{f} \mathbb{S}^2).$$

Exercise 10.8: Let $\psi : \mathbb{S}^0 \to \mathbb{S}^2$ be the usual inclusion as the equator of the equator. Consider the space

$$X := \text{pushout}(\mathbb{S}^2 \xleftarrow{\psi} \mathbb{S}^0 \xrightarrow{\iota} \mathbb{D}^1)$$

with $\iota : \mathbb{S}^0 \to \mathbb{D}^1$ the inclusion.

Exercise 10.9: Let $Z := (T^2, p) \vee (\mathbb{D}^2, q)$. Here q is a point in the interior of $\mathbb{D}^2$.

Exercise 10.10: Let $Y := (\mathbb{S}^1, 1) \vee (\mathbb{RP}^2, q)$.

Exercise 10.11: Consider $\mathbb{RP}^n$. **Hint:** Show that $\mathbb{RP}^n$ can be obtained from $\mathbb{RP}^{n-1}$ by adding an n-cell. Inductively, this means that $\mathbb{RP}^n$ has one cell of each dimension up to n.

10.7.4 A more complicated 3-dimensional cell complex

Definition 10.32. *Fix a non-negative integer p and consider the following data:*

- *We let X and Y be two copies of the solid torus $\mathbb{D}^2 \times \mathbb{S}^1$.*
- *We will glue the two along $Z = \mathbb{S}^1 \times \mathbb{S}^1$.*

- *It is still convenient to use complex coordinates in* $\mathbb{D}^2$ *and* $\mathbb{S}^1$*.*
- *To glue we specify the maps*

$$\begin{array}{rcl} \iota_X : Z & \to & X \\ (z,w) & \mapsto & (z,w) \end{array}$$

and

$$\begin{array}{rcl} \iota_Y : Z & \to & Y \\ (z,w) & \mapsto & (z, z^p w^{-1}). \end{array}$$

Which are identifications of Z *with the boundaries of* X *and* Y*. We then define the* $(p,1)$*-lens space to be:*

$$L_{p,1} := pushout(X \leftarrow_{\iota_X} Z \rightarrow_{\iota_Y} Y).$$

Exercise 10.12: Compute the fundamental group and the first homologies of $L_{p,1}$. **Hint:** Use Exercises 10.2 and 10.3.

10.7.5 The line with two origins

In the upcoming exercises L denotes the line with two origins. Recall its construction. We set $A, B \cong \mathbb{R}$. Then we let

$$L := (A \coprod B)/\{A \ni x \cong x \neq 0 \in B\},$$

endowed the quotient topology induced from the disjoint union of A with B.

Exercise 10.13: Prove that L is the pushout of the diagram

$$\mathbb{R} \leftarrow \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R},$$

both arrows being the usual inclusion.

Exercise 10.14: For every X path-connected and Hausdorff, $[L, X] \cong \{.\}$. Deduce that L is not homotopy equivalent to $\mathbb{S}^1$.

Exercise 10.15: Let $p \in L$. Show that $\pi_1(L,p) \cong \mathbb{Z}$. **Hint:** Argue as we did for $\mathbb{S}^1$.

Exercise 10.16: There is a connected space X such that $[L, X] \ncong \{.\}$. **Hint:** Use the previous exercise.

Van Kampen for handle attachments and other applications

Lecture 11

Our goal going forward is to get familiar with more examples of topological spaces and apply van Kampen systematically to them. In this lecture we will:

- Prove the corollaries of van Kampen to the setting of cell attachments (Section 11.1).
- Use these results to compute the fundamental group of cell complexes (Section 11.2) and work out some applications, including computations for cell complexes with infinitely many cells (Section 11.4).

The theory of cell complexes is central to Algebraic Topology, but we will only look at the basics. For more information, you may want to refer to Hatcher [Hat02, Appendix, p. 519].

11.1 Van Kampen for handle attachments

We now address the proofs of Theorems 10.24, 10.23, 10.25, and 10.26.

11.1.1 Mapping cylinders and attachments

The strategy to follow is to apply the theorem of van Kampen (Theorems 10.11 and 9.11). This result is easiest to apply when we are able to cover our space of interest X using open pieces A, B such that $A \cap B$ is as simple as possible. We will now make a small detour to show that this can be assumed to be the case when attaching cells.

Lemma 11.1. *Consider a space X and an attaching map $\psi : \mathbb{S}^{n-1} \to X$. Denote the resulting space by Y.*

Then, Y is also the result of attaching a cell to a space $X' \supset X$ with an attaching map $\psi' : \mathbb{S}^{n-1} \to X'$ such that:

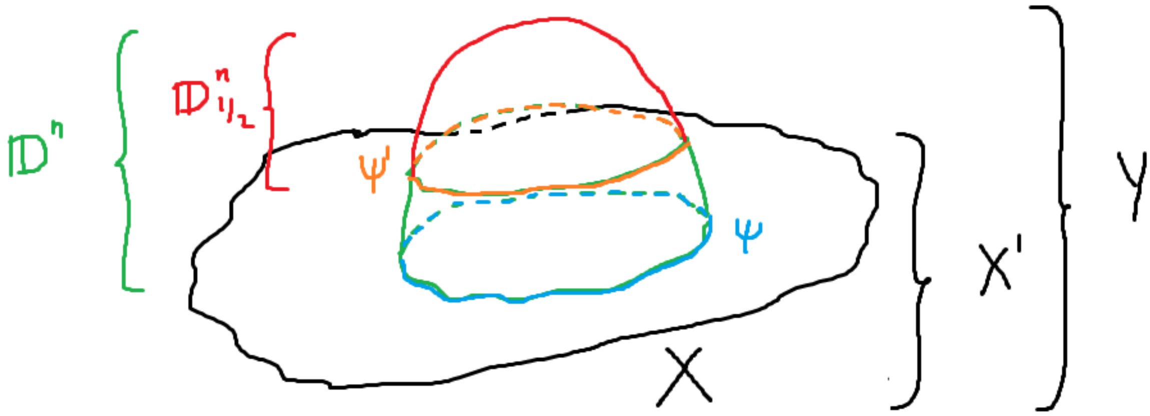


Figure 11.1: The attaching situation depicted in Lemma 11.1. The region $\mathbb{S}^{n-1} \times [1/2, 1)$ is shown as the complement of (the interior of) the disc $\mathbb{D}^n_{1/2}$ within the cell $\mathbb{D}^n$. Do note that the image of ψ is shown as a nice curve, but this map could potentially be very complicated. Similarly, X itself may have very complicated local topology.

- *ψ' is an inclusion.*
- *$\psi'(\mathbb{S}^{n-1}) \subset X$ has a neighbourhood homeomorphic to $\mathbb{S}^{n-1} \times [1/2, 1)$.*
- *X' deformation retracts to X.*

See Figure 11.1.

Proof. We can write the cell $\mathbb{D}^n$ as the union of the disc $\mathbb{D}^n_{1/2}$ and the collar $\mathbb{S}^{n-1} \times (1/2, 1]$ (here we are using spherical coordinates). This means that we can set $X' = \text{pushout}(\mathbb{S}^{n-1} \times (1/2, 1] \leftarrow \mathbb{S}^{n-1} \rightarrow X)$ and ψ' to be the inclusion of $\mathbb{S}^{n-1} \times \{1/2\}$ into X'. The first two properties then follow by construction. For the last property we observe that $\mathbb{S}^{n-1} \times (1/2, 1]$ deformation retracts to $\mathbb{S}^{n-1} \times \{1\}$ by pushing linearly in the second component. Using the Pasting Lemma (with the homotopy being the identity in X) provides the claimed deformation retraction. □

That is: X could potentially be a very complicated space locally, meaning that the image of ψ could have badly behaved neighbourhoods. However, the cell itself is great (it is a disc!). Attaching first a neighbourhood of its boundary we produce a space X', homotopy equivalent to X, such that the attaching region of the remainder of the cell is very nice.

One of the features that has appeared in the proof is:

Lemma 11.2. *Consider a map $f : S \rightarrow X$. Then, there is a space $C \supset X$, called the* ***mapping cylinder****, such that:*

- *C deformation retracts to X.*
- *There is a map $g : S \rightarrow C$ that is an inclusion.*
- *The deformation retraction takes g to f.*

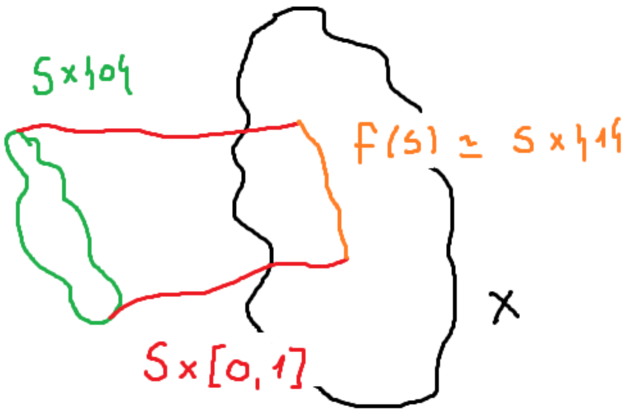


Figure 11.2: A mapping cylinder C, as in Lemma 11.2. The map f identifies $S \times \{1\}$ with $f(S) \subset X$. The cylinder deformation retracts to X. S includes into C as $S \times \{0\}$.

Proof. We set $C = \text{pushout}(X \leftarrow S \rightarrow S \times [0,1])$, the first arrow being f and the second arrow being the inclusion map taking S to $S \times \{1\}$. Then g is just the identification of S with $S \times \{0\}$. The claimed deformation retraction is the identity on X (as it should be!) and it pushes $S \times [0,1]$ to $S \times \{1\}$. □

11.1.2 Proofs

It is best to prove the case of higher cells first, then 2-cells, and lastly 1-cells. All the proofs follow the same overall strategy, but for lower-dimensions we will have to take care of some extra details. In all cases we assume that Lemma 11.1 has been applied. This produces a space X' homotopy equivalent to X, so the two have isomorphic fundamental groups. This allows us to just work with X'. The Lemma says that there is an open $\mathbb{S}^n \times [1/2, 1) \subset X'$ such that the attaching map ψ' of the smaller cell $\mathbb{D}^n_{1/2}$ is the inclusion of $\mathbb{S}^n \times \{1/2\}$.

Proof of Theorem 10.26. We cover Y using as opens $B = X'$ and the interior A of the (original) cell $\mathbb{D}^n$. The subspace A is contractible, B deformation retracts to X, and the intersection $I = A \cap B$ deformation retracts to the sphere $\mathbb{S}^{n-1}_{3/4}$. We fix a point $p \in \mathbb{S}^{n-1}_{3/4}$. It follows from Theorem 9.11 that:

$$\begin{aligned}\pi_1(Y, \psi(1)) &\simeq \pi_1(Y, p) \simeq \pi_1(A, p) *_{\pi_1(I,p)} \pi_1(B, p) \\ &\simeq 0 *_0 \pi_1(B, p) \simeq \pi_1(B, p) \simeq \pi_1(B, \psi(1)) \simeq \pi_1(X, \psi(1)).\end{aligned}$$

In the first step we used that the cell is path-connected, so $\psi(1)$ and p lie in the same component of Y. This was used again at the end of the proof for $B = X'$. In the middle we used that $\mathbb{S}^{n-1}$ is simply-connected if $n > 2$, so it adds no relations to the free product. □

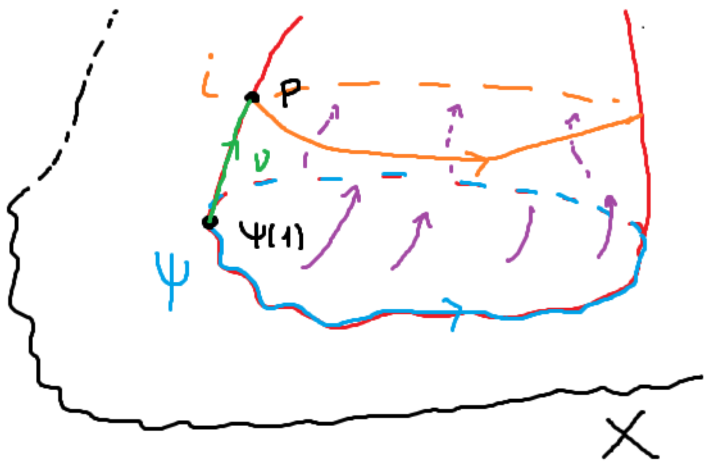


Figure 11.3: The situation explained in the proof of Theorem 10.25. We must relate the class $[\iota]$ of the sphere at radius 3/4 to the class of $[\psi]$. The two are conjugate via $[\nu]$. The homotopy exhibiting this is shown in purple.

Proof of Theorem 10.25. The same reasoning as before gives us

$$\pi_1(Y,\psi(1)) \simeq \pi_1(Y,p) \simeq 0 *_{\pi_1(I,p)} \pi_1(B,p)$$

but $\pi_1(I,p) \simeq \pi_1(\mathbb{S}^{n-1}_{3/4},p) \simeq \mathbb{Z}$ is now non-trivial. We denote by $\iota : \mathbb{S}^{n-1} \to B$ the inclusion of $\mathbb{S}^{n-1}_{3/4}$. We must understand then what the image of its pushforward

$$\iota_* : \pi_1(\mathbb{S}^{n-1},p) \to \pi_1(B,p)$$

is, because this is precisely what we are quotienting by. We verify that the image is generated by the class $\iota_*[\mathrm{id}_{\mathbb{S}^1}] = [\iota]$. It follows that:

$$\pi_1(Y,\psi(1)) \simeq \frac{\pi_1(B,p)}{[\iota]}.$$

To conclude the proof we take a path $\nu : [0,1] \to B$ connecting $\psi(1)$ to p (such a path can be found within $\mathbb{S}^n \times [1/2,1]$). Conjugating with $[\nu]$ yields an isomorphism $\beta_{[\nu]} : \pi_1(B,p) \to \pi_1(B,\psi(1))$ that satisfies $\beta_{[\nu]}([\iota]) = [\psi]$; see Figure 11.3. This yields:

$$\pi_1(Y,\psi(1)) \simeq \frac{\pi_1(B,\psi(1))}{[\psi]} \simeq \frac{\pi_1(X,\psi(1))}{[\psi]},$$

concluding the proof. □

Proof of Theorem 10.24. We use the same reasoning and notation as before, but $I \simeq \mathbb{S}^0_{3/4}$ has now two path-components. We write $p_\pm = \pm 3/4 \in P = \mathbb{S}^0_{3/4}$ for each of the two points. We then have that $\pi_1(A,P)$ is the pair groupoid of P; we denote $\alpha \in \pi_1(A,p_+,p_-)$ for the unique class of path connecting the two. We then apply Theorem 10.11 to deduce that $\pi_1(Y,p_+)$ consists of words such that:

- They begin with α or with a letter from $\pi_1(B, p_+, p_-)$.
- They finish with α^{-1} or with a letter from $\pi_1(B, p_-, p_+)$.
- They alternate between letters from p_+ to p_- with letters from p_- to p_+.

You may think of α as "half a letter", since it is not the class of a loop. To address this we fix an element $\beta \in \pi_1(B, p_-, p_+)$; it exists by path-connectedness of X (and thus B). See Figure 11.4. Given any word $\omega \in \pi_1(Y, p_+)$, we can produce a new word ω' by the following process:

- Whenever we encounter α or a letter in $\pi_1(B, p_+, p_-)$, we add the pair of letters $\beta^{-1}\beta$ immediately afterwards.
- We then group each occurrence of β with the letter immediately before. In the case of α this is done purely algebraically, regarding $(\beta\alpha)$ as a single letter. For letters in $\pi_1(B, p_+, p_-)$, concatenating with β amounts to concatenating in $\Pi_1(B)$, yielding an element in $\pi_1(B, p_+)$.

These two steps consist of moves, so ω' represents the same element as ω.

What we have just done is the following: We have considered the natural inclusion Ψ of the group

$$\pi_1(B, p_+) * \langle (\beta\alpha) \mid \rangle \simeq \pi_1(B, p_+) * \mathbb{Z}$$

into $\pi_1(Y, p_+)$ and we have shown that Ψ is surjective (and thus an isomorphism). This concludes the proof, since $\pi_1(Y, p_+) \simeq \pi_1(Y, \psi(1))$ by path-connectedness. □

Proof of Theorem 10.23. For this one write X_+ and X_- for the two components involved in the attaching. These are homotopy equivalent to $X_+ \cup [0, 1]$ and $X_+ \cup [-1, 0]$ where 1 is glued to $\psi(1)$ and -1 to $\psi(-1)$. The endpoint 0 in both intervals $[0, 1]$ and $[-1, 0]$ has an open neighbourhood $[0, 1/2)$ and $(-1/2, 0]$ that deformation retracts to it. We can therefore take 0 as the basepoint in both spaces and apply van Kampen for the wedge to yield the result. □

11.2 Fundamental group of finite cell complexes

Since cell complexes are built from the ground up via cell attachment, their fundamental groups can be computed completely by iteratively applying the results in Subsection 10.4.1.

Corollary 11.3. *Let X be a cell complex with finitely many cells. Write X_2 for its 2-skeleton. Fix a vertex $p \in X$. Then, the pushforward $\pi_1(X_2, p) \to \pi_1(X, p)$ is an isomorphism.*

Proof. X_n is obtained from X_{n-1} by successively attaching n-cells. Every time a cell is attached, assuming $n > 2$, we obtain an isomorphism between fundamental groups (Theorem 10.26). Inductively, it follows that $\pi_1(X_2, p) \to \pi_1(X_3, p) \to \cdots \to \pi_1(X_n, p)$ are all isomorpisms. □

The second statement reads:

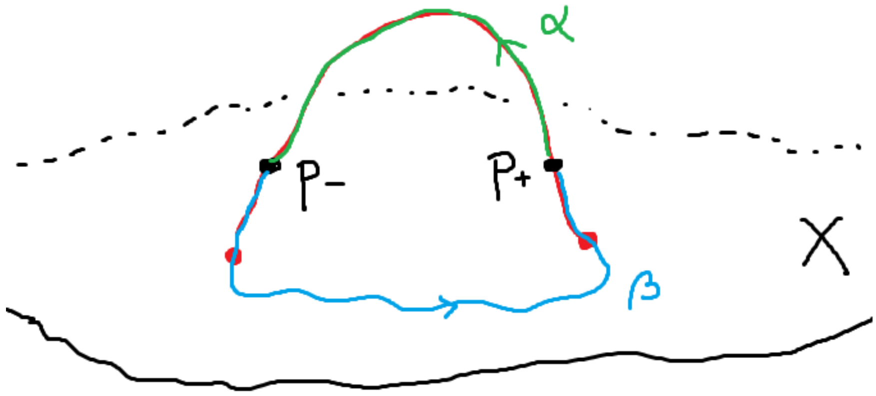


Figure 11.4: The situation explained in the proof of Theorem 10.24. We are attaching a 1-cell (red) to X. Inside of this cell we have two points p_+ and p_-, that together form $\mathbb{S}^0_{3/4}$. A path connecting them within the 1-cell is shown, labelled by its class α. The two points connect within X' as well, the class of the path is denoted by β. Together, $\beta\alpha$ form the class of a loop in $\pi_1(Y, p)$. This is the new generator introduced by the handle attachment.

Corollary 11.4. *Let X be a cell complex with finitely many cells. Write X_1 for its 1-skeleton. Fix a vertex $p \in X$. Then, the pushforward of the inclusion $\pi_1(X_1, p) \to \pi_1(X, p)$ is surjective.*

Proof. X_2 is obtained from X_1 by successively attaching 2-cells. Every time a cell is attached we are performing a quotient in the fundamental group (Theorem 10.25). Inductively, it follows that $\pi_1(X_1, p) \to \pi_1(X_2, p)$ is a surjective homomorphism. The result then follows from Corollary 11.3. □

These allow us to state the following beautiful result relating Group Theory to Topology:

Corollary 11.5. *Let G be a finitely presented group. Then there is a path-connected 2-dimensional cell complex X such that $\pi_1(X, p) \simeq G$.*

Proof. According to Lemma 7.16, G has a presentation $\langle A \mid R\rangle$. We can then define the 1-skeleton to be $X_1 = \vee_A(\mathbb{S}^1, 1)$. The only vertex is the wedge point p, and each copy of $\mathbb{S}^1$ contributes one 1-cell. According to Corollary 8.30, $\pi_1(X_1, p) \simeq \langle A \mid \rangle$.

Consider now each relation $r \in R$. This can be regarded as an element in $\pi_1(X_1, p)$ and thus be represented as $[\gamma]$ with $\gamma : \mathbb{S}^1 \to X_1$. This provides for us a collection $\{\psi_r\}_{r\in R}$ with each ψ_r and attaching map for a 2-cell in X_1. We let X_2 be the result of attaching said cells. According to Theorem 10.25, when we attach using ψ_r we must add $[\psi_r] = r$ as a relation in the fundamental group. Since R is finite we can therefore use induction to show $\pi_1(X_2, p) \simeq \langle A \mid R\rangle$. □

Remark 11.6: As you can see, all these statements included a finiteness assumption. This assumption can be removed, at the expense of slightly more work. In the upcoming sections we will see the ideas that go into this. △

11.3 Local properties of cell complexes

We will now go through some of key properties of cell complexes. The overall message is that cell complexes are very well-behaved locally.

You do not have to know the proofs in this section, but it will be useful for you to keep the statements in mind.

For the rest of the subsection we write X for a cell complex, leaving the rest of the data implicit.

Lemma 11.7. *The restriction $\phi^i_j|_{\mathring{\mathbb{D}}^i}$ of a characteristic map ϕ^i_j to the interior of a cell is a homeomorphism with its image.*

Proof. The quotient map that defines X is injective when restricted to the interior of each cell. The claim then follows from the fact that X has the quotient topology. □

The exact same argument, recalling that cell complexes are quotients of the disjoint union of their cells, shows that:

Lemma 11.8. *A subset $U \subset X$ of a cell complex is open if and only if $(\phi^i_j)^{-1}(U) \subset \mathbb{D}^i$ is open for each characteristic map.*

The local contractibility of discs can then be used to prove that:

Proposition 11.9. *X is locally contractible.*

Proof. Local contractibility means that every $p \in X$ has a system of neighbourhoods that are contractible. The idea is the following. First observe that the statement is true if $p \notin X_{N-1}$. If that is the case, p is contained in the interior of some N-cell σ, which is identified with $\mathbb{D}^n$ via its characteristic map $\psi : \mathbb{D}^n \to X$. Sufficiently small neighbourhoods of p in X correspond to neighbourhoods in the interior of σ and can therefore be assumed to be euclidean balls.

Suppose now that p is contained in the interior of some $(N-1)$-cell σ. Reasoning as above we can fix a contractible neighbourhood $U \subset \sigma$ of p, homeomorphic to an $(N-1)$-dimensional ball. Our goal now is to "thicken" this ball slightly to produce a neighbourhood of p within X. This will result in a neighbourhood V of p that intersects X_{N-1} in U, and intersects all N-cells incident to σ.

We use the following auxiliary claim. Let ϕ^N_j be the attaching maps of the N-cells and write $q_j = \phi^N_j(0) \in X$ for the centers of the cells. Recall that $\mathbb{D}^n \setminus \{0\}$ deformation retracts to $\partial\mathbb{D}^n$, simply by dilating towards the boundary. It follows that we can produce a deformation retract

$$r_s : X \setminus \{q_j\}_{j \in I_N} \to X,$$

using the Pasting Lemma, such that $r_s : V := r_1^{-1}(U) \to U$ is a deformation retraction. It follows that V is homotopy equivalent to U and thus contractible.

The general case follows similarly, inductively on the dimension of the cells. □

The deformation retract idea we have just encountered can also be used to prove:

Lemma 11.10. *X is Hausdorff.*

Proof. Given distinct points $p, p' \in X$ we must exhibit disjoint neighbourhoods. Suppose p is contained in the interior of a cell σ and p' is in the interior of cell σ'. In the previous proof we already observed that p has a neighbourhood $V \subset X$ that deformation retracts to a neighbourhood $U \subset \sigma$. In particular, V only intersects cells that are larger than σ and incident to it. Furthermore, U may be assumed to be as small as we want. V may be large, but we can shrink it towards U using the deformation retraction r_s. We reason similarly for p', yielding $U' \subset \sigma'$ and $V' \subset X$.

There are then three cases:

- $\sigma = \sigma'$. Then we can assume that U and U' are disjoint by shrinking them enough. The same is true for V and V' if we define both using the same deformation retraction.
- $\sigma \neq \sigma'$ but they have the same dimension. Then U' and U are disjoint automatically and we can once again assume V and V' are defined using the same deformation retraction, implying that they are also disjoint.
- σ and σ' have different dimensions. Then one of them is smaller in dimension, say σ. Given any compact subset K in the interior of the larger cells, we can apply a deformation retraction to produce a V is disjoint from K. If we take K sufficiently large, we can assume $p' \in K$ and $U' \subset K$, proving the claim.

□

11.4 Cell complexes with infinitely many cells

Let us fix a cell complex X. We allow X to have infinitely many cells.

11.4.1 Compactness and finiteness

Lemma 11.11. *Let A be a compact space. Let $f : A \to X$ be a map. Then (the image of) f intersects the interior of finitely many cells of X.*

Proof. For each cell whose interior is intersected by f, pick a point, yielding a subset $\Sigma \subset X$. According to Lemma 11.8, this subset is closed and so is its preimage $f^{-1}(\Sigma) \subset A$. By compactness of A, $f^{-1}(\Sigma)$ is compact. According to Lemma 11.10, any two points in Σ have disjoint neighbourhoods, so the same is true for $f^{-1}(\Sigma)$ by taking preimages. It follows that $f^{-1}(\Sigma)$ is compact and discrete, so it must be finite. □

Given a finite collection of cells I of X, we say that they *define* the subcomplex $Y \subset X$ given by the following inductive recipe. We let J be the smallest collection of cells in X such that the attaching map of every cell in Y intersects some cell in J. We can then write Z for the subcomplex defined by J (given to us by induction). Then Y is the union of Z and the cells in I.

Corollary 11.12. *Let I be a finite collection of cells. Then, the subcomplex they define is finite.*

Proof. J is finite thanks to Lemma 11.11. Furthermore, the largest cell in J has strictly smaller dimension than the largest cell in I (because cells attach along the lower skeleta). This means that the inductive process finishes in finitely many steps and involves only finitely many cells. □

Furthermore:

Corollary 11.13. *X is compact if and only if it has finitely many cells.*

Proof. Suppose X is compact. Then we can apply Lemma 11.11 to the identity map id_X and deduce that its image (which is X) intersects only finitely many cells (which are all of them). Conversely, X is constructed as a quotient of finitely many cells, all of which are compact, so it is itself compact due to the definition of the quotient topology. □

This corollary can then be used to deduce:

Example 11.14: We claim that $\mathbb{R}$ has a unique cellular structure up to isomorphism. Indeed, reasoning as above we see that any cell structure $X_0 \subset X_1$ must have infinitely many cells. Furthermore, we see that X_0 must have the discrete topology, so its vertices form a two-sided sequence

$$\cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots$$

that must go to $\pm\infty$ (otherwise there will be an accumulation point, leaving a half-line uncovered by edges). We can now map t_i to $i \in \mathbb{Z}$ and each interval $[t_i, t_{i+1}]$ to $[i, i+1]$. This is a cellular isomorphism with respect to the usual cellular structure. △

11.4.2 Fundamental group

A crucial consequence of Lemma 11.11 is that:

Corollary 11.15. *Fix a cell complex X and a class $\alpha \in \pi_1(X, x)$. Then, there is a compact subcomplex $Y \subset X$ such that α is in the image of*

$$\iota_* : \pi_1(Y, x) \to \pi_1(X, x).$$

Proof. α is represented by a loop $\gamma : (\mathbb{S}^1, 1) \to (X, x)$ which, by Lemma 11.11, intersects only finitely many cells. The complex they define, which we denote by Y, is finite (Corollary 11.12) and γ maps into it. This proves the claim. □

Which allows us to generalise Corollaries 11.3 and 11.4 to the case of arbitrarily many cells:

Corollary 11.16. *Let X be a cell complex. Write X_1 for its 1-skeleton. Fix a vertex $p \in X$. Then, the pushforward $\pi_1(X_1, p) \to \pi_1(X, p)$ is surjective.*

Proof. Every class $a \in \pi_1(X,p)$ is in the image of $\pi_1(Y,x) \to \pi_1(X,x)$, for some finite subcomplex Y. By van Kampen for cell attachments (Theorem 10.26), every element im $\pi_1(Y,x)$ is in the image of $\pi_1(Y_1,x)$, where Y_1 is the 1-skeleton of Y. It follows that a is in the image of $\pi_1(Y_1,x) \to \pi_1(X,x)$. □

Corollary 11.17. *Let X be a cell complex. Write X_2 for its 2-skeleton. Fix a vertex $p \in X$. Then, the pushforward $\pi_1(X_2,p) \to \pi_1(X,p)$ is an isomorphism.*

Proof. Corollary 11.16 shows that $\pi_1(X_2,p) \to \pi_1(X,p)$ is surjective, so we must prove injectivity. The reasoning is very similar. Any homotopy of loops taking place in X actually takes place in a finite subcomplex Y and, in turn, this homotopy can then be homotoped to lie in the 2-skeleton $Y_2 \subset X_2$. □

11.4.3 Sequences of subcomplexes

This line of reasoning allows us to compute the fundamental group of a cell complex as long as we can cover it by an increasing sequence of subcomplexes:

Corollary 11.18. *Suppose you can write X as the union of a sequence of cell complexes*

$$X_1 \subset X_2 \subset X_3 \subset \cdots$$

(here X_i does not mean the i-skeleton), all of which contain the basepoint $x \in X_1$. Suppose additionally that the pushforwards

$$(\iota_i)_* : \pi_1(X_i,x) \to \pi_1(X_{i+1},x)$$

of the inclusions $\iota_i : X_i \to X_{i+1}$ is injective. Then

$$\pi_1(X,x) \simeq \bigcup_{i=1}^{\infty} \pi_1(X_i,x).$$

Proof. Using the homomorphisms $\{(\iota_j)_*\}$ we see each $\pi_1(X_i,x)$ as a subgroup of the fundamental groups that come afterwards. Furthermore, each $\pi_1(X_i,x)$ also maps to $\pi_1(X,x)$ using the pushforward of the inclusion $X_i \to X$. Putting all of these together, we obtain a group homomorphism

$$\Psi : \bigcup_{i=1}^{\infty} \pi_1(X_i,x) \to \pi_1(X,x).$$

According to Corollary 11.15, Ψ is surjective. Injectivity follows from the fact that any nullhomotopy of a path into X must take values in some X_i (thanks to Lemma 11.11, since the domain of a homotopy is compact), so any class α in $\pi_1(X_i,x)$, that gets taken to zero in $\pi_1(X,x)$, must already be zero in some $\pi_1(X_j,x)$, and therefore $\alpha = 0$ by injectivity of the $\{(\iota_l)_*\}$. □

Which particularises to:

Corollary 11.19. *Suppose you can write X as the union of a sequence of simply-connected cell complexes*

$$X_1 \subset X_2 \subset X_3 \subset \cdots,$$

all of which contain the basepoint $x \in X_1$. Then X is also simply-connected.

11.4.4 Graphs and trees

These ideas allow us to improve on Lemma 8.30:

Corollary 11.20. *Let I be a set (of arbitrary cardinality). Then:*

$$\begin{aligned} \pi_1(\vee_I(\mathbb{S}^1, 1)) &\simeq *_I\mathbb{Z}, \\ H_1(\vee_I\mathbb{S}^1, \mathbb{Z}) &\simeq \mathbb{Z}^I, \\ H_1(\vee_I\mathbb{S}^1, \mathbb{R}) &\simeq \mathbb{R}^I. \end{aligned}$$

Proof. We argue similarly to Lemma 8.36 and Corollary 11.18. Given a subset $J \subset I$, we write $\iota^J : \vee_J(\mathbb{S}^1, 1) \to \vee_I(\mathbb{S}^1, 1)$ for the inclusion and r^J for the retraction inverting it from the left. It follows (Corollary 5.22) that r^J_* is surjective and ι^J_* is injective. Furthermore, for each J finite it holds that $\pi_1(\vee_J(\mathbb{S}^1, 1)) \simeq *_J\mathbb{Z}$, according to the Lemma 8.30.

All the maps ι_J are compatible with one another. Putting together all the pushforwards ι^J_* for J finite provides for us a group homomorphism[1]:

$$\Psi : \quad *_I\mathbb{Z} \to \pi_1(\vee_I(\mathbb{S}^1, 1)).$$

Which is injective because each ι^J_* was injective.

To conclude the proof we must prove surjectivity of Ψ. This follows from Lemma 11.11: any loop $\gamma : \mathbb{S}^1 \to \vee_I(\mathbb{S}^1, 1)$ intersects only finitely many 1-cells (say, the subset $J \subset I$). This means that the class $[\gamma]$ is contained in $*_J\mathbb{Z}$ and thus in $*_I\mathbb{Z}$. □

So it follows immediately:

Corollary 11.21. *Let I and J be sets with $|I| \neq |J|$. Then $\vee_I(\mathbb{S}^1, 1)$ and $\vee_J(\mathbb{S}^1, 1)$ are not homotopy equivalent.*

We are also interested in more general graphs:

Corollary 11.22. *Let X be a 1-dimensional path-connected cell complex. Then:*

- *There is a spanning tree $T \subset X$.*
- $\pi_1(X, p) \quad \simeq \quad *_{X\setminus T}\mathbb{Z}$.

[1] Here we are using that $*_I\mathbb{Z}$ is the *colimit* of all the $*_J\mathbb{Z}$ with J finite. A colimit is like the pushout of a more general diagram. The diagram we are dealing with is the diagram whose elements are the groups $\pi_1(\vee_J(\mathbb{S}^1, 1))$ and whose morphisms are the pushforwards of the inclusions. All of these map to $\pi_1(\vee_I(\mathbb{S}^1, 1))$. The claimed mapped Ψ is the group morphism given by the universal property of the colimit.

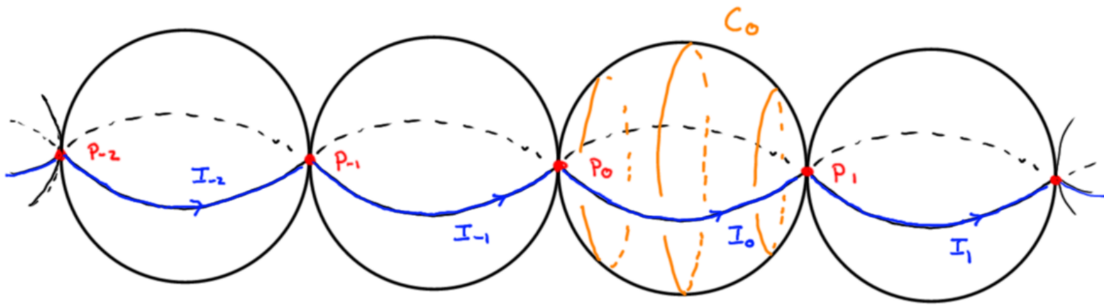


Figure 11.5: The space Y described in Section 11.5, together with its cell structure. The wedge points between different spheres are vertices. The front half of each equator is an edge. The remainder of each sphere is the interior of a face.

Proof. A tree is a graph that is simply-connected. A tree is spanning if it contains all vertices. To prove that a spanning tree T exists we apply Zorn's lemma to the set of all trees in X. This is a partially ordered set, ordered upwards by inclusion. A chain (i.e. a totally ordered subset) in this poset is thus an increasing sequence of trees. Given such chain, we can consider their union, which is a subcomplex of X. By Corollary 11.18 this union is again simply-connected and thus a tree. We have shown that each chain has an upper bound. Zorn's lemma then implies that there is a maximal element among all trees. This maximal tree must be spanning, since otherwise we would be able to build a bigger tree by adding one more edge.

For the second item we reason as in Corollary 11.20. We leave the details to the reader. □

11.5 Worked out example: an infinite string of spheres

The following example shows how one argues with cell complexes involving infinitely many cells and, particularly, how one computes their fundamental group.

Write $S_i \subset \mathbb{R}^3$ for the sphere of radius 1 and center $(2i, 0, 0)$. Consider the union

$$Y := \bigcup_{i \in \mathbb{Z}} S_i.$$

See Figure 11.5.

11.5.1 Step 1. Producing an abstract cell structure

We now endow Y with a cell structure. To do so we first produce a 2-dimensional cell complex S, abstractly. Later we provide a homeomorphism $f : S \to Y$.

We write $S_0 := \coprod_{i\in\mathbb{Z}}\{p_i\}$ for the 0-skeleton of S. For the 1-skeleton S_1 we consider the attachments $\phi_i : \{-1,1\} \to S_0$ given by $\phi_i(-1) = p_i$ and $\phi_i(1) = p_{i+1}$. This allows us to write:

$$S_1 := \text{pushout}(S_0 \overset{\coprod_i \phi_i}{\leftarrow} \coprod_{i\in\mathbb{Z}}\{-1,1\} \to \coprod_{i\in\mathbb{Z}}[-1,1]).$$

I.e. we have attached an edge between each two consecutive points. It is convenient to write $I_i := [-1,1]$ for the 1-cell between p_i and p_{i+1}, i.e. the domain of ϕ_i.

For the 2-skeleton S_2 we argue as follows. Recall that the square $C_i := [-1,1] \times [0,2\pi]$ is homeomorphic to the disc. Write (t,θ) for the coordinates in C (these will be spherical coordinates later). We define attachments $\psi_i : \partial C_i \to S_1$ of the form $\psi_i(-1,\theta) := p_i$, $\psi_i(1,\theta) := p_{i+1}$, $\psi_i(t,0) := t \in I_i$, and $\psi_i(t,2\pi) := -t \in I_i$. I.e. ∂C_i is sent to a loop that goes back and forth in I_i. We set:

$$S_2 := \text{pushout}(S_1 \overset{\coprod_i \psi_i}{\leftarrow} \coprod_{i\in\mathbb{Z}} \partial C_i \to \coprod_{i\in\mathbb{Z}} C_i).$$

11.5.2 Step 2. Identifying the cell structure with our space

We now define a homeomorphism $f : S \to Y$. To do so we first define a map g from the disjoint union of all cells D to Y. For the vertices, we set $g(p_i) := (2i-1,0,0)$, i.e. the points in which the spheres touch. For the edges we write

$$g(t) := (2i+t, \sqrt{1-t^2}, 0), \qquad \text{where } t \in I_i,$$

i.e. the image is half of the equator of the ith sphere. For the faces we set

$$g(t,\theta) := ((2i+t)\cos(\theta), \sqrt{1-t^2}\cos(\theta), \sin(\theta)), \qquad \text{where } (t,\theta) \in C_i$$

i.e. we parametrise the ith sphere using spherical coordinates.

The map g is surjective. We deduce that g is a quotient map using Lemma 10.30. Here there is a subtlety: D is not compact, so the Lemma does not apply directly. However, D can be exhausted by compacts (just take all the cells with $|i| \leq N$, for each natural number N), and the Lemma does apply to them. Being a quotient map is something you check locally in Y (by taking small opens), so the claim follows.

Now we observe that g restricted to the boundary of each cell is g itself. This implies that the identifications defining the quotient $D \to Y$ are the same as the identifications given by attaching (i.e. those that yield the quotient map $D \to S$). This implies that the induced map $f : S \to Y$ is a homeomorphism.

11.5.3 Step 3. Computing the fundamental group

Observe first that S_1 is homeomorphic to $\mathbb{R}$ and therefore contractible. This implies $\pi_1(S_1,p_0) \simeq 0$. The result then follows as an application of Corollary 11.16: the pushforward of the inclusion $\pi_1(S_1,p_0) \to \pi_1(S_2,p_0)$ is surjective so the target must also be zero.

11.6 Exercises

11.6.1 Cell complexes with infinitely many cells

Exercise 11.1: Let X be a tree (i.e. a 1-dimensional cell complex with no cycles). Prove that X is simply-connected. **Hint:** You can find a exhaustion by compacts of X in which each compact is itself a tree.

Exercise 11.2: Write $S_{i,j} \subset \mathbb{R}^3$ for the sphere of radius 1 and center $(3i, 3j, 1)$. Consider the union

$$Z := \{z = 0\} \coprod \bigcup_{i,j \in \mathbb{Z}} S_i.$$

Endow Z with a cell structure and prove that it is simply-connected.

The proof of van Kampen

Lecture 12

The theorem of van Kampen for groupoids (Theorem 10.8) is proven in Section 12.1. The version in which we replace the fundamental groupoid by an equivalent smaller groupoid (Theorem 10.11) is proven in Appendix 12.2.

12.1 Proof of Theorem 10.8

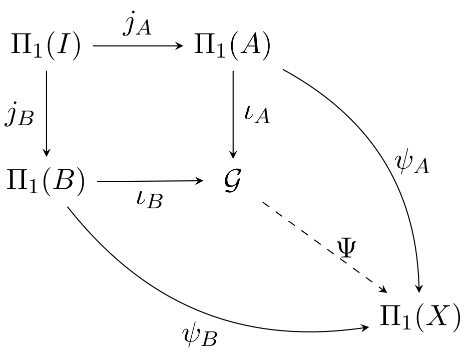
We dedicate this section to proving the van Kampen theorem for fundamental groupoids.

12.1.1 Setup

Recall that we have $X = A \cup B$ and we write $I = A \cap B$ for the intersection. All these pieces are open. Let us denote

$$\mathcal{G} := \text{pushout}(\Pi_1(A) \leftarrow \Pi_1(I) \rightarrow \Pi_1(B)) = \Pi_1(A) *_{\Pi_1(I)} \Pi_1(B)$$

for brevity. Due to the universal property of the pushout we have a unique functor $\Psi : \mathcal{G} \to \Pi_1(X)$ making the diagram commute:



Here j_A, j_B, ψ_A, ψ_B are pushforwards of the inclusions. ι_A and ι_B are the canonical maps into the amalgamated product.

We can then describe Ψ very explicitly. First observe that both $\mathcal{G}$ and $\Pi_1(X)$ have X as the set of objects. In the former case this follows from the explicit description we have of the amalgamated product (saying that the set of objects of the pushout is the pushout of the sets of objects). Ψ is simply the identity between objects then. At the level of morphisms, Ψ evaluates words in $\mathcal{G}$ by taking the individual letters (i.e. classes of paths $a \in \Pi_1(A)$ or $b \in \Pi_1(B)$) to their images $\psi_A(a), \psi_B(b) \in \Pi_1(X)$. The result is then a word of composable morphisms in $\Pi_1(X)$ that can then be concatenated.

The proof now amounts to showing that Ψ is an isomorphism of groupoids. We already know it is a morphism of groupoids, so it only remains to prove the two usual statements:

Proposition 12.1. *$\Psi : \mathcal{G} \to \Pi_1(X)$ is surjective.*

Proposition 12.2. *$\Psi : \mathcal{G} \to \Pi_1(X)$ is injective.*

To tackle surjectivity we must show that every class $[\gamma] \in \Pi_1(X)$ can be written as a word with letters in $\Pi_1(A)$ and $\Pi_1(B)$. To prove injectivity we must consider two elements ω_0, ω_1 in the amalgamated product $\mathcal{G}$ and assume that $[\gamma_0] = \Psi(\omega_0)$ and $[\gamma_1] = \Psi(\omega_1)$ are the same class in $\Pi_1(X)$. We should then show that ω_0 and ω_1 represent the same element in $\mathcal{G}$. That is, we write them as words using elements in $\Pi_1(A)$ and $\Pi_1(B)$ as letters and we have to check that they differ by a sequence of moves. This sequence will be provided to us by the homotopy $\Gamma : [0,1]^2 \to X$ connecting γ_0 and γ_1.

12.1.2 The Lebesgue covering lemma

We need some auxiliary statements allowing us to chop curves $\gamma : [0,1] \to X$ and homotopies $\Gamma : [0,1]^2 \to X$ into little pieces that map to either A or to B. Such statements follow from the so-called *Lebesgue covering lemma*:

Lemma 12.3. *Let A be a compact metric space and let $\{A_i\}$ be a cover by opens. Then there exists a number $\delta > 0$ such that every subset of diameter at most δ is contained in some A_i.*

Proof. If the cover has a single element we are done. Furthemore, we can assume that the cover is finite by compactness. We then consider the distance function $\chi_i : A \to [0, \infty)$ from a point $a \in A$ to the complement of A_i. It holds that $a \in A_i$ if and only if $\chi_i(a) > 0$. If χ_x is the distance function to $x \in A$, we have that $\chi_i = \inf_{x \notin A_i} \chi_x$. We see that χ_x is continuous thanks to the triangular inequality:

$$|\chi_x(a) - \chi_x(b)| < |\chi_a(b)| \overset{b \to a}{\longrightarrow} 0,$$

and since the bound is independent of x we deduce that χ_i is continuous as well.

We can now take the sum $\chi = \sum_i \chi_i$ and observe that χ is everywhere positive, since $\{A_i\}$ is a cover. By compactness, there exists $\delta > 0$ such that $\chi(a) > \delta$ for all $a \in A$, proving the claim. □

We will need the following consequence:

Corollary 12.4. *Let $X = \cup_j X_j$ be a union with open pieces. Let $\gamma : [0,1] \to X$ be a path. Fix any sufficiently large positive integer n. Then each $\gamma([i/n,(i+1)/n])$ is contained completely in some X_j.*

Proof. The subsets $A_j = \gamma^{-1}(X_j) \subset [0,1]$ are open and cover $[0,1]$. The result then follows from Lemma 12.3 by taking $n > 1/\delta$. □

And its generalisation to cubes, with the same proof:

Corollary 12.5. *Let $X = \cup_j X_j$ be a union with open pieces. Fix a map $\Gamma : [0,1]^m \to X$ and a sufficiently large positive integer n. Then each cube*

$$C_{i_1,\cdots,i_m} = [i_1/n,(i_1+1)/n] \times \cdots \times [i_m/n,(i_m+1)/n]$$

is mapped by Γ to some X_j.

12.1.3 Proof of surjectivity

Surjectivity is immediate thanks to Corollary 12.4:

Proof of the surjectivity Proposition 12.1. Consider an element $[\gamma] \in \Pi_1(X)$. According to Corollary 12.4 there are points

$$0 = s_0 < s_1 < \cdots < s_n = 1.$$

such that $\gamma([s_i, s_{i+1}])$ is contained fully in A or fully in B. The path $\nu_i(t) = \gamma(s_i + t(s_{i+1} - s_i))$ is a reparametrisation of $\gamma|_{[s_i,s_{i+1}]}$ and represents a class $[\nu]$ in either $\Pi_1(A)$ or $\Pi_1(B)$. It follows that γ is a reparametrisation of $\nu_{n-1} \bullet \cdots \bullet \nu_0$ so, at the level of classes:

$$[\gamma] = [\nu_{n-1}] \bullet \cdots \bullet [\nu_0].$$

We have exhibited $[\gamma]$ as a word written using classes coming from $\Pi_1(A)$ and $\Pi_1(B)$, so $[\gamma]$ is in the image of Ψ. This holds for all classes in $\Pi_1(X)$ and therefore Ψ is surjective. □

12.1.4 Proof of injectivity

We consider two elements ω, ω' of the amalgamated product $\mathcal{G}$. We assume that $\Psi(\omega) = \Psi(\omega')$ and therefore we must show that ω and ω' represent the same element in $\mathcal{G}$. This means that we have to exhibit a sequence of words, beginning at ω and ending at ω', such that subsequent words are related by the moves described in Definition 10.3, which we now particularise to the problem at hand:

(II) A class $[c_x] \in \Pi_1(A)$ represented by a constant path is equivalent to the empty word. Same for B.

(II) A word ab, with $a, b \in \Pi_1(A)$ composable, is equivalent to the one-letter word $a.b$ consisting of their composition. Same for B.

(III) A letter $\iota_A(x)$ in the image of $\iota_A : \Pi_1(A \cap B) \to \Pi_1(A)$ is equivalent to $\iota_B(x) \in \Pi_1(B)$.

Topological setup. We write out ω as a word $[\alpha_m] \cdots [\alpha_1]$ with each $[\alpha_i]$ a class in either $\Pi_1(A)$ or $\Pi_1(B)$. Similarly, we write out ω' as $[\beta_n] \cdots [\beta_1]$. In order to make the concrete groupoid explicit, we sometimes write a letter as $[-]_!$ with ! a label that can be either A or B. Assuming $\Psi(\omega) = \Psi(\omega')$ means that $\alpha_m \bullet \cdots \bullet \alpha_1$ and $\beta_n \bullet \cdots \bullet \beta_1$ are homotopic to each other relative endpoints. We write $\Gamma : [0,1]^2 \to X$ for this homotopy. Up to reparametrisation, we can assume that $\alpha_i(t) = \Gamma(i/m + t/m, 0)$ and $\beta_j(t) = \Gamma(j/n + t/n, 1)$.

We fix a sufficiently large multiple N of mn. By Corollary 12.5, Γ takes each square

$$C_{u,v} = I_u \times I_v = [u/N, (u+1)/N] \times [v/N, (v+1)/N]$$

to A or B (or possibly the intersection of the two). Up to choices, this determines a labelling that assigns A or B to each $C_{u,v}$. We denote the vertical segments by $V_{u,v} = I_u \times \{v/N\}$ and the horizontal segments by $H_{u,v} = \{u/N\} \times I_v$. These are the sides of the squares $C_{a,b}$. This is pictured in Figure 12.1. We can parametrise $V_{u,v}$ using $\nu^+_{u,v}(t) = (u/N + t/N, v)$ or its reverse $\nu^-_{u,v}$, each corresponding to one of the two possible orientations. Similarly, we can parametrise $H_{u,v}$ using $\chi^+_{u,v}(t) = (u, v/N + t/N)$ or the reverse $\chi^-_{u,v}$.

We can apply move (II) repeatedly to relate ω to the word $\omega_0 = [\Gamma \circ \nu^+_{N-1,0}] \cdots [\Gamma \circ \nu^+_{0,0}]$. Each $\Gamma \circ \nu^+_{j,0}$ is part of some α_k, so we label $[\Gamma \circ \nu^+_{j,0}]$ as A or B according to the label of $[\alpha_k]$. Similarly, ω' is related to the word $\omega'' = [\Gamma \circ \nu^+_{N-1,N}] \cdots [\Gamma \circ \nu^+_{0,N}]$. See Figure 12.1.

□

The idea of the proof is to match the topological side (the homotopy Γ) to the algebraic side (words using elements $\Pi_1(A)$ or $\Pi_1(B)$ as letters). The labelling of the squares and the segments is the first step in doing so.

Inductive argument. We order the pairs $\{(u,v)\}_{0 \le u,v \le N}$ increasingly using the lexicographic order. We are now going to find a sequence of words $\omega_{u,v}$ and paths $\gamma_{u,v} : [0,1] \to [0,1]^2$, indexed by the (u,v), such that:

- $\gamma_{u,v}$ is a concatenation of segments of the form $\nu^\pm_{a,b}$ and $\chi^\pm_{a,b}$.
- $\gamma_{u,v}$ begins at $(0,0)$ and finishes at $(1,0)$.
- Concatenation of segments in $\gamma_{u,v}$ corresponds to concatenation of letters in $\omega_{u,v}$.
- Each letter of $\omega_{u,v}$ is obtained from a segment of $\gamma_{u,v}$ as follows: Each appearance of $\nu^\pm_{a,b}$ in $\gamma_{u,v}$ corresponds to an appearance of the letter $[\Gamma \circ \nu^\pm_{u,v}]_!$ in $\omega_{u,v}$. The value of the label ! will be determined by the induction argument to be explained next. Things go analogously for each appearance of $\chi^\pm_{a,b}$.

In particular, $\Psi(\omega_{u,v}) = [\Gamma \circ \gamma_{u,v}]$. See Figure 12.2.

In the inductive step (u,v) we will be considering moves (in the algebraic side) and homotopies of paths (in the topological side) given by the square $C_{u,v}$. The one exception is the base case $(u,v) = (-1,0)$, where we simply set $\omega_{-1,0} = \omega_0$ and $\gamma_{-1,0} = (\Gamma \circ \nu^+_{N-1,0}) \bullet \cdots \bullet (\Gamma \circ \nu^+_{0,0})$.

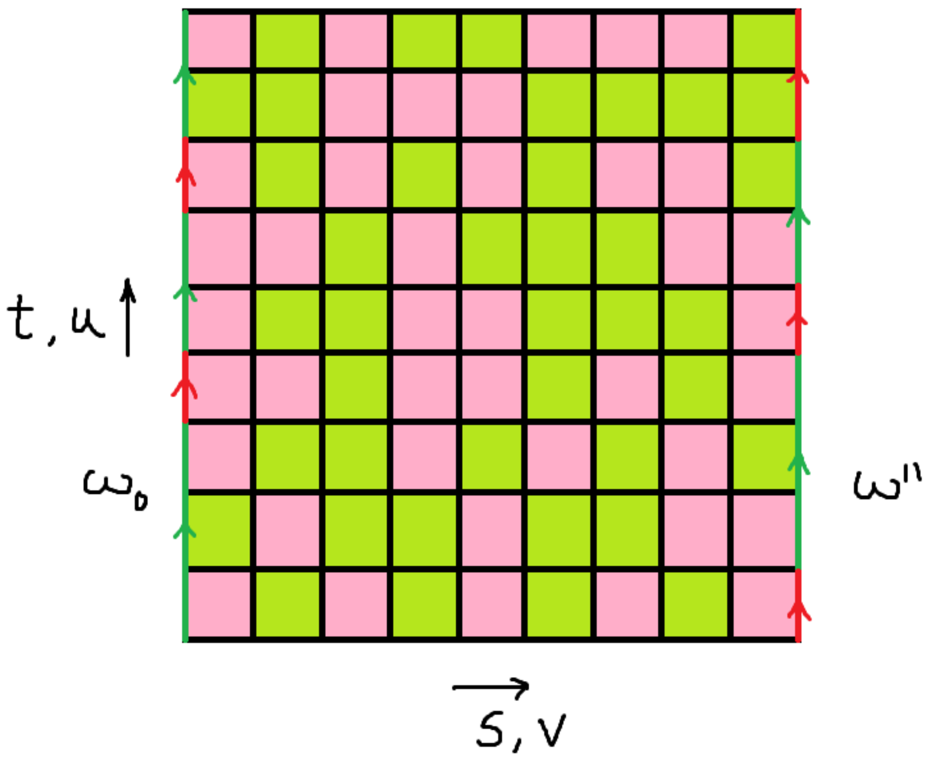


Figure 12.1: The domain $[0,1]^2$ of Γ. The homotopy variable s is shown from right to left. The squares $C_{u,v}$ appear coloured according to their label (light green for A, light red for B). At $s=0$ we see the paths $\{\nu^+_{u,0}\}$, whose classes $\{[\Gamma\circ\nu^+_{u,0}]\}$ concatenate to ω_0. Each path is coloured in green or red depending on whether the class $[\Gamma\circ\nu^+_{u,0}]$ belongs to $\Pi_1(A)$ or $\Pi_1(B)$. The same story applies at $s=1$, where we see the paths $\{\nu^+_{u,N}\}$ whose classes, appropriately labeled, concatenate to produce ω''.

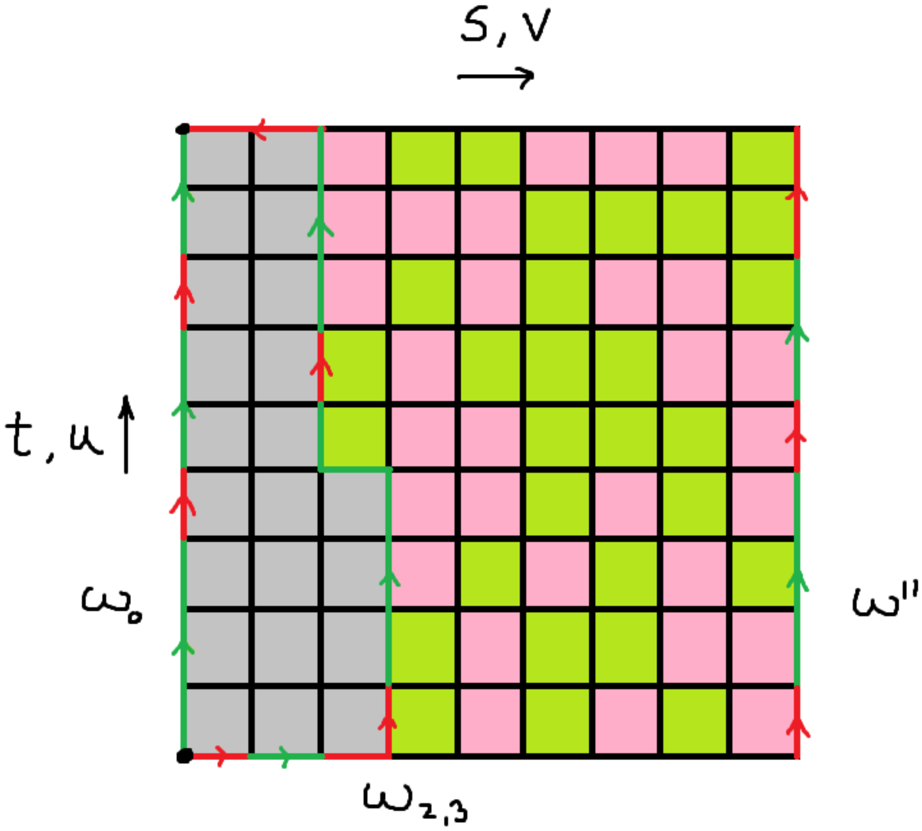


Figure 12.2: The inductive argument at $(u, v) = (2, 3)$. We have gone through the first two columns of squares plus a bit more (those are marked in gray). This has produced a curve $\gamma_{u,v}$, divided into segments, that goes from $(0, 0)$ to $(1, 0)$. The segments are labeled green or red depending on whether the corresponding letter in $\omega_{u,v}$ is a class in $\Pi_1(A)$ or $\Pi_1(B)$.

Consider the inductive step (u, v) and write (u', v') for the pair immediately before it. Let us suppose without loss of generality that $C_{u,v}$ is labeled as A. We focus on the square $C_{u,v}$ and we look at the segments of $\gamma_{u',v'}$ that are sides of $C_{u,v}$. It will follow from the upcoming argument that these sides appear consecutively in $\gamma_{u',v'}$, forming a curve S; see Figure 12.3. The segments of S correspond to letters in $\omega_{u',v'}$. Since they appear consecutively, they form a subword $\eta \subset \omega_{u',v'}$. Each letter q in η has a label, which may not be A. If that is the case, it means that the corresponding segment maps to B (as the label says) but also to A (because $C_{u,v}$ is fully contained in A). Since it maps to both, q is the image $\iota_B(p)$ of a class $p \in \Pi_1(A \cap B)$. We can then use move (III) to replace each such q by $\iota_A(p)$; see Figure 12.3. In this manner we obtain a word η', with letters coming only from $\Pi_1(A)$, that is equivalent to η. Since all the letters are in the same groupoid, we can compose them thanks to move (II), yielding another equivalent word η'', that has a single letter. By construction, $\eta'' = [\Gamma|_S]_A$.

The sides of $C_{u,v}$ not appearing in $\gamma_{u',v'}$ form a curve S' with the same endpoints as S. Observe that S and S' are homotopic to one another as curves in $[0, 1]^2$, relative endpoints. The homotopy is given by interpolating linearly within $C_{u,v}$. This implies that the letter $\eta'' \in \Pi_1(A)$ is exactly the same as $[\Gamma|_{S'}]_A$ (this is not a move, this is simply the definition of what a homotopy class relative endpoints is!). If S' consists of multiple segments, we can apply move (II) to η'' and produce a new word η''' whose letters correspond to said segments.

We conclude the induction by setting $\omega_{u,v}$ to be the word we obtain from $\omega_{u',v'}$ by replacing η by η'''. By construction, the two are related by moves. Analogously, we define $\gamma_{u,v}$ to be curve one obtains from $\gamma_{u',v'}$ by replacing the segment S for the segment S'. The two are homotopic relative endpoints. □

End of the proof of the injectivity Proposition 12.2. At the end of the argument we have produced a word $\omega_{N,N}$ that is related by moves to $\omega_{-1,0} = \omega_0$ and thus to ω. It remains to observe that $\omega_{N,N}$ is related to ω'' and thus to ω'. From the argument we see that

$$\omega_{N,N} = [\Gamma \circ \chi^-_{N,0}] \cdots [\Gamma \circ \chi^-_{N,N-1}][\Gamma \circ \nu^+_{N-1,N}] \cdots [\Gamma \circ \nu^+_{0,N}][\Gamma \circ \chi^+_{0,N-1}] \cdots [\Gamma \circ \chi^+_{0,0}]$$

but all the paths $\Gamma \circ \chi^{\pm}_{a,b}$ appearing in the formula are constant, since Γ was a homotopy relative endpoints and a equals 0 or N. This means that all these classes are the identity by move (I). We therefore have

$$\omega_{N,N} = [\Gamma \circ \nu^+_{N-1,N}] \cdots [\Gamma \circ \nu^+_{0,N}]$$

which looks the same as ω'', but we have to be careful with labels. Indeed, the letters of $\omega_{N,N}$ are labeled in some arbitrary way that may not match the labels in ω''. Nonetheless, we can reason as above and observe that if the labels do not match it is because the letter is in the image of $\Pi_1(A \cap B)$. We can thus apply move (III) and deduce that ω'' and $\omega_{N,N}$ are related to each other. This concludes the proof. □

12.2 Proof of Theorem 10.11

We will now establish Theorem 10.11 assuming Theorem 10.8. The proof is categorical in nature and:

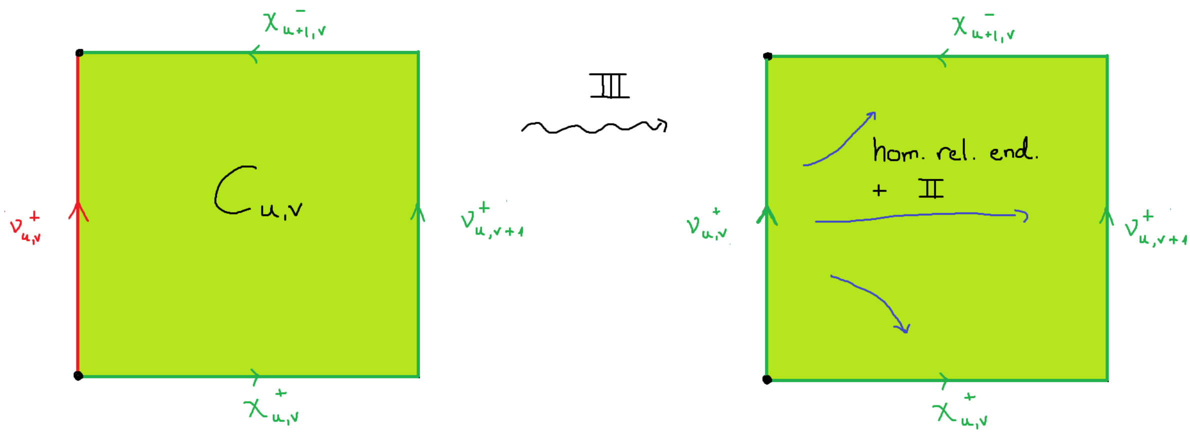


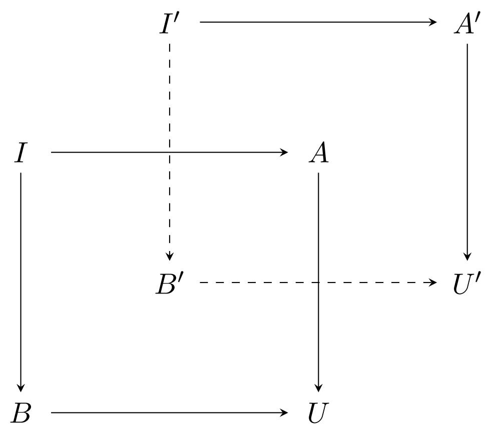
Figure 12.3: A possible configuration in a inductive step of the form $(0, v)$. In this case η consists of a single letter, namely $[\Gamma \circ \nu^+_{u,v}]_B$. Identically, S consists only of the segment $\nu^+_{u,v}$. The other three segments form S'. For $u \neq 0$, one gets a slightly different picture, with S consisting of two segments.
In this case the letter $[\Gamma \circ \nu^+_{u,v}]_B$ is labeled as B, but $C_{u,v}$ is labeled as A. We then have to apply move (III). This is allowed because $\Gamma \circ \nu^+_{u,v}$ takes values in $A \cap B$. This produces a new word η', which consists only of the letter $[\Gamma \circ \nu^+_{u,v}]_A$. Now that everything is labeled as A we can see that the square provides a homotopy from the right-hand-side interval to the other three. By move (II) this is equivalent to the three letter word $[\Gamma \circ \chi^-_{u+1,v}]_A[\Gamma \circ \nu^+_{u,v+1}]_A[\Gamma \circ \chi^+_{u,v}]_A$. This completes the inductive step.

The upcoming ideas will not be used elsewhere in the course.

12.2.1 Squares and their morphisms

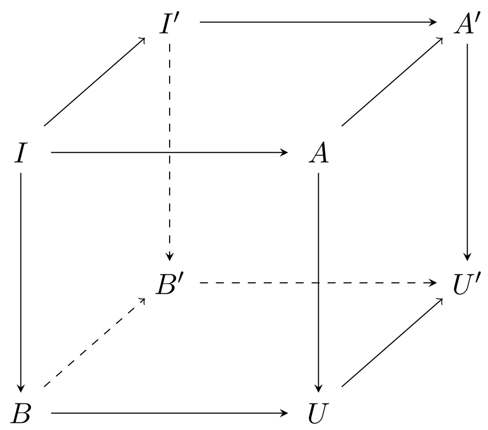
We first introduce the main categorical ingredient required for the proof:

Definition 12.6. *Let $\mathcal{C}$ be a category. Suppose we are given two square diagrams in $\mathcal{C}$ of the form*



Let us call "front square" S_f to the diagram formed by (I, A, B, U) and the morphisms between them. We will use "back square" S_b for the rest.

Then, a ***morphism*** *$S_f \to S_b$ is a diagram of the form*

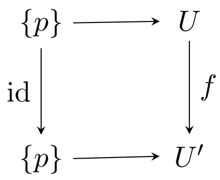


such that all arrows commute.

That is, a morphism of squares is a collection of four morphisms in $\mathcal{C}$, one per each corner, that is compatible in the sense that all possible compositions with the same source and target must be the same. You can imagine that one can provide a similar definition for diagrams of other shapes (not just squares), but this will be enough for our purposes.

Example 12.7: Let us work in Top. Suppose that all the maps appearing in the front and back squares are inclusions. Then, a morphism from the front to the back is equivalent to a map $f : U \to U'$ that respects this decomposition. I.e. $f(I) \subset I'$, $f(A) \subset A'$ and $f(B) \subset B'$. □

Example 12.8: As a concrete example, still in Top, we can take $I, A, I', A' = \emptyset$ and $B = B' = \{p\}$. Then, all the non-trivial information is condensed in the following diagram:



i.e. a morphism of squares in this case is a map from U to U' that commutes with the inclusion of p into both. This is simply a pointed map! □

Example 12.9: Let $\mathcal{C}$ be arbitrary. Suppose that the back square is constant. I.e. it satisfies $A' = B' = I' = U' = O$ and the corresponding morphisms are all the identity $\mathrm{id}_O : O \to O$. Then, a morphism of squares is very closely related to the pushout property: we have maps from A, B, and I into O that are compatible with each other and factor via a map $h : U \to O$. However, do note that h may not be the unique such morphism. □

Example 12.10: We can similarly assume the front square is constant and given by some object $O \in \mathcal{C}$. Then we have maps of O into A', B', and U', all of which factor through I'. □

You can imagine that we can set up a category whose objects are squares in $\mathcal{C}$ and whose morphisms are as we just defined. You can furthermore imagine that a similar construction will apply to any other "shape" of diagram in $\mathcal{C}$. We will not pursue this further. However, it is then sensible to wonder whether certain morphisms invert some others. We encounter then a familiar (and crucial) notion:

Definition 12.11. *Let $\mathcal{C}$ be a category and let S_f and S_b be squares in such a category. S_b is said to be a* ***retract*** *of S_f if there are morphisms of squares*

- $\iota : S_b \to S_f$ *"inclusion",*
- $r : S_f \to S_b$ *"retraction",*

such that $r \circ \iota : S_b \to S_b$ is the identity.

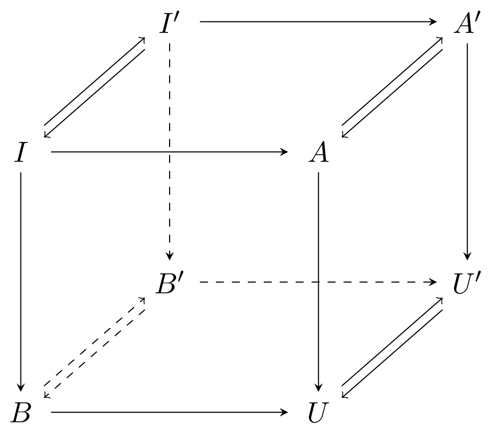
You can observe that a retract of squares is in particular a retract entry-by-entry, but it is a stronger notion (because the four retracts are coherent).

Example 12.12: In the situation described in Example 12.8, a retract is a pointed retract. □

12.2.2 Retracts of pushout diagrams

Here is the main technical result that we need:

Proposition 12.13. *Consider the diagram*



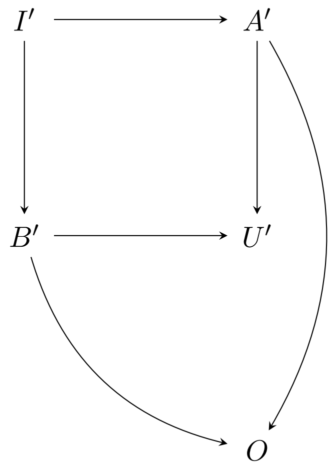
and assume that:

- *The front square is a pushout.*
- *The back square is a retract of the front square.*

Then, the back square is also a pushout.

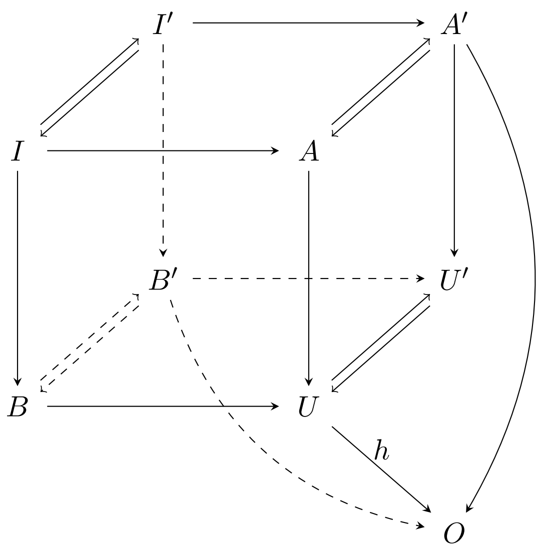
Proof. Let us name the morphisms appearing in the diagram. We write $r_I : I \to I'$ for the retract left-inverting the inclusion $\iota_I : I' \to I$. Similarly we write r_A, r_B, and r_U and analogously ι_A, ι_B, and ι_U. We use the same conventions from earlier; for instance f_{IA} is the map $I \to A$. For the back square we write $f'_{IA} : I' \to A'$ and so on. Before we begin the proof, do note that the diagram as whole is not commutative. For instance, the composition $\iota_I \circ r_I$ need not be the identity (but $\iota_I \circ r_I = \mathrm{id}_I$ by definition). The same remark applies to the other inclusion/retraction pairs. However, the front and back squares are commutative by themselves.

Suppose now that we have another object O, and corresponding morphisms $g'_{AO} : A' \to O$, g'_{BO}, and $g'_{IO} = g'_{AO} \circ f'_{IA} = g'_{BO} \circ f'_{IB}$. These, together with the back square, form a commutative diagram that we will call the “extended back diagram”:



We must construct a morphism $h' : U' \to O$ commuting with all others and then prove that h' is unique. This being true for all such objects O (and morphisms) will imply that the back square is a pushout diagram.

The first thing to do is observe that the front square, together with O and the morphisms $g_{AO} = g'_{AO} \circ r_A : A \to O$, $g_{BO} = g'_{BO} \circ r_B$, and $g_{IO} = g'_{IO} \circ r_I$ is also a commutative diagram, which we will call the "extended front diagram". This is the usual setup to apply the universal property of the pushout, which we know by assumption is satisfied by the front diagram. We deduce that there is a unique morphism $h : U \to O$ commuting with all others. The complete diagram at this point looks like:



The only candidate we have for $h' : U' \to O$ now is the composition $h' = h \circ \iota_U$. We must show that h' makes the extended back diagram commute and that h' is the unique morphism achieving this.

First we check commutativity. We will verify it only in the case $h' \circ \circ f'_{AU} = g'_{AU}$, leaving the others to the reader (which proceed analogously). Thus:

$$h' \circ f'_{AU} = h \circ \iota_U \circ f'_{AU} = h \circ f_{AU} \circ \iota_A = g_{AO} \circ \iota_A = g'_{AO},$$

where we first used the definition of h', then the fact that inclusions form a morphism of squares, then that the front square is a pushout, and lastly that r_A is a left-inverse to ι_A[1]. Overall, the key idea is that we can trade maps in the back square for maps in the front square, where commutativity holds. I recommend that you draw each of these steps in the diagram.

Lastly we address uniqueness. We first claim that h equals $h' \circ r_U$. This will follow if we prove that $h' \circ r_U$ makes the extended front diagram commute (by invoking the uniqueness of the pushout for the front square). We will check this commutativity in the concrete case $g_{AO} = (h' \circ r_U) \circ f_{AU}$, leaving the other checks to the reader once more. Indeed:

$$g_{AO} = g'_{AO} \circ r_A = h' \circ f'_{AU} \circ r_A = h' \circ r_U \circ f_{AU},$$

using the definition of g_{AO}, the fact that h' makes the extended back diagram commute, and lastly the fact that the retractions form a morphism of squares.

Suppose that there is some other morphism $\tilde{h}' : U' \to O$ that also makes the extended back diagram commute. The exact same reasoning shows that $h = \tilde{h}' \circ r_U$. We can then cancel using the inclusions from the right: $h' = h \circ \iota_U = \tilde{h}' \circ r_U \circ \iota_U = \tilde{h}'$, as desired. □

Remark 12.14: You may wonder whether the converse is true: does the back being a pushout imply that the front is a pushout as well? The answer is *no*. In the proof we used crucially that $r_A \circ \iota_A = \mathrm{id}_{A'}$ (and the analogous statements for B', U', I'). Trying to prove the converse would involve a similar computation with the other composition $\iota_A \circ r_A$, but this one need not be id_A. A more conceptual way to say this is that maps from the back square to O factor through the front square, but the converse need not be true.

As a concrete example: In Set you can take the back square to consist of singleton sets, so it is a pushout square, and take the front square to be an arbitrary (non-pushout) square. The retractions are the unique constant maps and the inclusions you can choose by picking an element of I. △

12.2.3 Proof of Theorem 10.11

First observe:

Example 12.15: By construction there is an inclusion functor $\iota : \pi_1(X, P) \to \Pi_1(X)$. It turns out that we can find a retraction (in fact many!) $r : \Pi_1(X) \to \pi_1(X, P)$ left-inverting

[1] This is the point where we use that the back is a retract of the front.

ι. The idea is rather geometric: for each point $x \in X$ we pick an element $p \in P$ and a class of path $[\gamma_{p,x}] \in \pi_1(X, p, x)$. That such a choice exists follows by the Axiom of Choice. We furthermore require that $[\gamma_{p,p}] = [c_p]$ for every $p \in P$.

Observe that this is the same argument that we used to construct an skeleton in an arbitrary category.

At the level of objects $X \to P$ we then set $r(x) = p$ to be the aforementioned choice. At the level of morphisms, for each $\nu \in \pi_1(X, x, y)$ we set

$$r([\nu]) = [\overline{\gamma_{r(y),y}}] \bullet [\nu] \bullet [\gamma_{r(x),x}] \in \pi_1(X, r(x), r(y)).$$

By construction, we have that $r \circ \iota$ is the identity functor $\pi_1(X, P) \to \pi_1(X, P)$. $\triangle$

Proof of Theorem 10.11. The conclusion follows as an application of Proposition 12.13 once we show that the diagram S_b:

$$\begin{array}{ccc} \pi_1(I, I \cap P) & \longrightarrow & \pi_1(A, A \cap P) \\ \downarrow & & \downarrow \\ \pi_1(B, B \cap P) & \longrightarrow & \pi_1(X, P) \end{array}$$

is a retract of the diagram S_f:

$$\begin{array}{ccc} \Pi_1(I) & \longrightarrow & \Pi_1(A) \\ \downarrow & & \downarrow \\ \Pi_1(B) & \longrightarrow & \Pi_1(X) \end{array}$$

There is a canonical inclusion $\iota : S_b \to S_f$, so we must find a retraction $S_f \to S_b$ inverting it from the left. We do so as in Example 12.15. We set $r(p) = p$ for each $p \in P$ and we assign to p the class of the constant path $[c_p]$. For each $i \in I$ we use the axiom of choice to pick some $r(i) \in P \cap I$; by assumption we can then choose a path $[\gamma_{p,i}] \in \pi_1(I, p, i)$ connecting the two. For each $a \in A$ we proceed similarly: if $a \in I$ then we just copy what we did in I. Otherwise we pick some $r(a) \in P \cap A$ and some $[\gamma_{p,a}] \in \pi_1(A, p, a)$. For B we proceed similarly. Since X is the union of A and B, there we do not have any further choice.

By construction, these choices are all coherent and therefore produce a morphism of squares at the level of objects. At the level of morphisms we reason as in Example 12.15, concatenating each class of path, on the left and the right, by the chosen $[\gamma_{p,-}]$. This yields thus a retract of squares and Proposition 12.13 concludes the proof. $\square$

Surfaces via planar representations

Lecture 13

Among all cell complexes, it is sensible to focus on subclasses with special properties, for instance, the class of manifolds. Manifold Topology is an extremely deep subfield Topology, and now we are just going to take the first steps into it. Namely, we will:

- Recall some basic notions about manifolds and manifolds with boundary (Section 13.1).
- Focus on surfaces (Section 13.2) and study them using *planar presentations* (a nicely behaved class of cell structures).

Our final goal, to be tackled in the next lecture, is to classify all compact surfaces up to homeomorphism.

13.1 Manifolds

Recall:

Definition 13.1. *A topological space M is a* ***manifold*** *of dimension n if the following properties hold:*

- *It is Hausdorff.*
- *It is second countable.*
- *Every point $p \in M$ has a neighbourhood homeomorphic to $\mathbb{R}^n$.*

The following confusing nomenclature is used often: a manifold is said to be **closed** if it is compact. Otherwise, we say that it is **open**.

For this definition to make sense we need to verify:

Theorem 13.2. *$\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $n = m$.*

Proof. We argue by contradiction, assuming there is homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^m$. Write g for the induced homeomorphism $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\}$. Observe that the source is homotopy equivalent to $\mathbb{S}^{n-1}$, thanks to the usual radial deformation retraction. Similarly,

$\mathbb{R}^m \setminus \{f(0)\}$ is homotopy equivalent to $\mathbb{S}^{m-1}$. We can then produce a homotopy equivalence $h : \mathbb{S}^{n-1} \to \mathbb{S}^{m-1}$ using g. The proof will be complete once we prove that this is only possible if $n = m$.

At this point we observe:

- $\mathbb{S}^{n-1}$ is path-connected if and only if $n > 1$.
- $\mathbb{S}^{n-1}$ is simply-connected if and only if $n > 2$.

This proves the claim for $n = 1, 2$. For the general case one has to study $\pi_a(\mathbb{S}^{n-1}, p) := [(\mathbb{S}^a, x), (\mathbb{S}^{n-1}, p)]$, which you will see in subsequent courses (so unfortunately we cannot prove this fully yet!). □

13.1.1 Manifolds with boundary

Lemma 13.5 below shows that the closed disc $\mathbb{D}^n$ is not an n-dimensional manifold. The reason is that the points in $\mathbb{S}^{n-1} \subset \mathbb{D}^n$ do not have a euclidean neighbourhood. This motivates us to introduce:

Definition 13.3. *The* ***upper half space*** *is*

$$\mathbb{H}^n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}.$$

The subspace $\mathbb{R}^{n-1} = \{x_1 = 0\}$ *is said to be the* ***boundary*** *of* $\mathbb{H}^n$.

Then:

Definition 13.4. *A topological space M is an n-dimensional* ***manifold with boundary*** *if the following properties hold:*

- *It is Hausdorff.*
- *It is second countable.*
- *Every point $p \in M$ has a neighbourhood homeomorphic to $\mathbb{R}^n$ or $\mathbb{H}^n$.*

A point $p \in M$ is a ***boundary point*** *if there is a neighbourhood (U, p) that is pointed homeomorphic to $(\mathbb{H}^n, 0)$. The points that have euclidean neighbourhoods are said to be* ***interior points****.*

For this to make sense we must verify that:

Lemma 13.5. *Let M be an n-dimensional manifold with boundary. M is partitioned into boundary and interior points.*

Proof. The idea now is to use the point-removal trick. Suppose that a point p is an interior point. Let U be any neighbourhood contained in a euclidean chart. We can then pick a small closed ball $B_\delta \subset U$, centered at p. It follows that $U \setminus \{p\}$ retracts to the sphere $\mathbb{S}^{n-1}_\delta$ radially.

Suppose that p is also a boundary point, for contradiction. We can then pick an open U homeomorphic to $\mathbb{H}^n$. We can now deformation retract $U \setminus \{p\}$ to the hemisphere $\mathbb{S}^{n-1} \cap \mathbb{H}^n$, also radially, which is contractible. It follows that $U \setminus \{p\}$ is contractible as well.

We have reached contradiction. First, observe that the $(n-1)$-sphere is not contractible. For $n = 1$ this follows because it is not path-connected. For $n = 2$, it is because it is not simply-connected. For higher n one would use the fact that $[\mathbb{S}^{n-1}, \mathbb{S}^{n-1}] \neq \{.\}$ (which you will learn in your master studies). Now, since the $(n-1)$-sphere is not contractible, $U \setminus \{p\}$ cannot retract to it. □

It then follows that:

Corollary 13.6. *Let M be an n-dimensional manifold with boundary. Its interior is an n-dimensional manifold.*

Proof. It is Hausdorff and second countable since it is a subspace of M. Moreover, all its points have euclidean neighbourhoods by assumption. □

Corollary 13.7. *Let M be an n-dimensional manifold with boundary. Its boundary is an $(n-1)$-dimensional manifold.*

Proof. It is Hausdorff and second countable since it is a subspace of M. All its points have neighbourhoods homeomorphic to $\mathbb{R}^{n-1} \subset \mathbb{H}^n$. □

13.2 Surfaces via planar presentations

We henceforth focus on:

Definition 13.8. *A 2-dimensional manifold is called a* ***surface****.*

One can speak of surfaces with boundary as well, but when we say *surface* we mean *without boundary.*

13.2.1 Planar presentations

It is very convenient to study surfaces (and manifolds in general) by putting cell structures on them. The simpler these are, the better. We thus introduce:

Definition 13.9. *A* ***planar presentation*** *is a space P obtained using the following recipe:*

- *We consider a regular n-gon F, which we draw in $\mathbb{R}^2$.*
- *Each side $\alpha \subset F$ is given an orientation (indicated by an arrow) and a label. The orientation gives us a unique parametrisation $f_\alpha : [-1, 1] \to \alpha \subset F$ of constant speed.*
- *P is the quotient of F obtained by identifying sides that have the same label. When identifying two sides α and β, we use $f_\alpha \circ f_\beta^{-1}$.*

Many examples are given in Section 13.3.

Recall that, according to Exercise 10.4, any convex polygon is homeomorphic to a disc. We can use this to deduce that planar presentations have canonical cell structures. This will be helpful in order to compute the fundamental group:

Lemma 13.10. *Let P be a planar presentation defined from the n-gon F. Let $\pi : F \to P$ be the quotient map. Then, the canonical cell structure in F descends to a cell structure on P.*

Proof. First observe that F is naturally a 2-dimensional cell complex, the corners being vertices, the sides being edges, and F itself being the unique face (here we use Exercise 10.4). This is just a cell structure on the 2-disc, adapted to the polygonal structure.

Then, we claim that the cell structure in F can be pushed, using π, to a quotient cell structure on P. Concretely, this means that:

- Given a vertex $p \in F$, we ask that its image $\pi(p) \in P$ is also a vertex.
- If $f_\alpha : [-1, 1] \to \alpha \subset F$ is the characteristic map of the edge α, then $\pi \circ f_\alpha$ is the characteristic map of an edge in P. Observe that two such characteristic maps agree if their corresponding sides in F have the same label.
- The quotient map π is the characteristic map of the unique 2-cell F.

I.e. the zeroth skeleton P_0 is $\pi(F_0)$ and similarly P_1 is $\pi(F_1)$. □

Since all vertices and edges of P appear at the boundary of the 2-cell F, and F has finitely many sides, we deduce that a planar presentation has finitely many cells, which implies (Corollary 11.13):

Corollary 13.11. *A planar presentation is compact and path-connected.*

13.2.2 Cyclic words

We now want to treat planar presentations in an algebraic manner. To this end, we introduce:

Definition 13.12. *Fix a finite alphabet I. A **cyclic word** (of length n) in the alphabet $I \coprod I^{-1}$ is a sequence of letters indexed by a cyclic group $\mathbb{Z}/n\mathbb{Z}$, without a preferred starting point. That is, the word $a_n \cdots a_1$ is identified with the word $a_{i-1} \cdots a_1 a_n \cdots a_i$, for each i.*

The concatenation of words (which involves choices, due to cyclicity) is the algebraic counterpart of the so-called connected sum of surfaces; it is discussed in Section 15.3. We will also discuss moves, which are the algebraic counterpart of the fact that a surface can be represented by multiple planar presentations; these appear in Section 14.1.

Lemma 13.13. *Fix a finite alphabet I. There is a 1-to-1 correspondence between:*

- *Planar presentations with labels I, up to rotation.*
- *Cyclic words in the alphabet $I \coprod I^{-1}$ in which each letter appears at least once.*

Proof. Suppose P is a planar presentation. We can produce a cyclic word by walking around the boundary of the polygonal face F in a counterclockwise manner. When we walk across a side α, we write down the label/letter $a_\alpha \in I$ if α is counterclockwise oriented. Otherwise we write a_α^{-1}. The word produced in this manner does not depend on the vertex we start from, due to cyclicity. Each label shows up at least once, by definition of planar presentation.

Conversely, given a word ω of length n and a regular polygon F with n sides, we can produce a planar presentation P with F as a face. We first label the sides of F counterclockwise, using

ω. A side is oriented counterclockwise if and only if the corresponding letter is in I. Once that is done, P is constructed, as a space, to be the quotient of F under the identifications given by the labels. □

Our results about van Kampen for cell attachments particularise to:

Corollary 13.14. *Let P be the planar presentation given by the cyclic word ω, written in the alphabet I and having a single vertex. Then:*

$$\pi_1(P, p) \simeq \langle I \mid \omega \rangle$$

Proof. If there is a single vertex p, each edge a, parametrised by $f_a : [0, 1] \to P_1$, is a loop. That is, the 1-skeleton P_1 is homeomorphic to $\vee_I(\mathbb{S}^1, 1)$. The theorem of van Kampen for 1-cell attachments says that the classes $[f_a]$ are the generators of $\pi_1(P_1, p) \simeq *_I\mathbb{Z}$.

$\pi_1(P, p)$ has the same generators, but additionally one relation per face attached. In this case, the unique face is F, and its attaching map is ω, which is a concatenation of edges. It follows that the unique relation is $[f_\omega]$, which is the word obtained from ω by replacing every appearance of the label a by $[f_a]$.

Now we compromise in order not to clutter notation: instead of writing $[f_a]$, we write a for the generator of π_1 associated to the edge a. Similarly, instead of writing $[f_\omega]$, we write ω. Note that this is quite natural, since a was a label anyway and ω was a word on labels (which we now see as not being cyclic, by starting at the vertex p). □

More generally:

Corollary 13.15. *Let P be the planar presentation given by the cyclic word ω. Suppose its 1-skeleton satisfies $\pi_1(P_1, p) \simeq \langle J \mid \rangle$, for some generating set J. Then, $\pi_1(P, p) \simeq \langle J \mid \omega \rangle$.*

13.2.3 When is a planar presentation a surface?

Planar presentations need not describe surfaces. However, there is an easy criterion that tells us whether they do:

Lemma 13.16. *Let P be the planar presentation given by the cyclic word ω. The space underlying P is a surface if and only if each letter in ω appears exactly twice.*

Proof. We consider first the only if direction. Consider a point p lying in the interior of an edge a, whose label appears n times. If $n = 1$, p resembles a boundary point in a surface (Lemma 13.5), so it cannot have a euclidean neighbourhood. If $n = 2$, great. If $n > 2$, we reason as follows.

Every sufficiently small neighbourhood U of p satisfies that $U \setminus \{p\}$ retracts to a graph X with fundamental group $*_{n-1}\mathbb{Z}$; see Figure 13.1. Suppose that p also has a neighbourhood V homeomorphic to $\mathbb{R}^2$. Then $V \setminus \{p\}$ deformation retracts to $\mathbb{S}^1$. Putting these facts together we obtain a retraction $r : V \setminus \{p\} \to X$, whose pushforward

$$r_* : \pi_1(V \setminus \{p\}, q) \to (X, q)$$

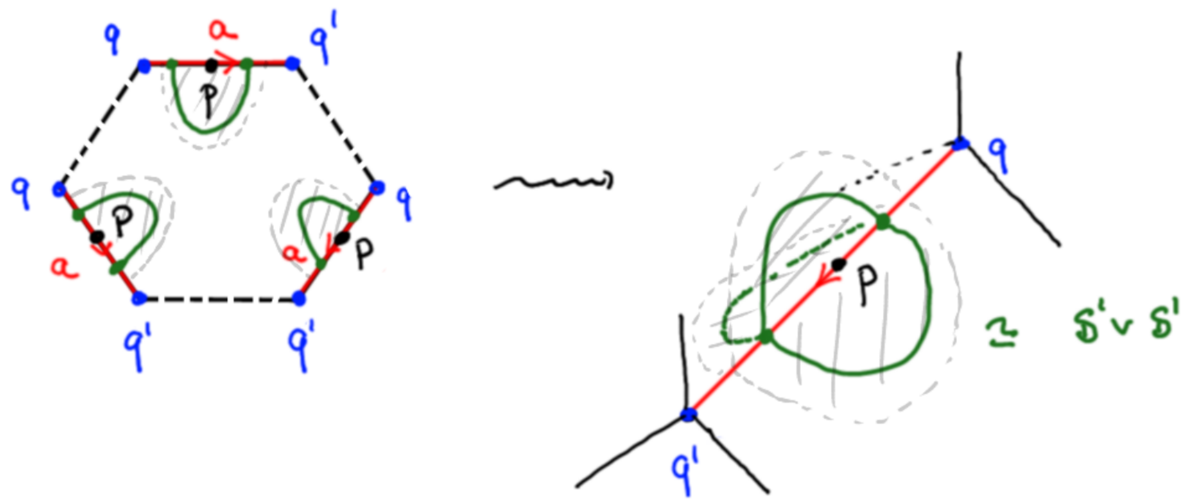


Figure 13.1: A planar presentation in which one of the labels, a, appears thrice. Given a point p in the interior of a, any neighbourhood (in gray) retracts to a graph consisting of two vertices and three edges. I.e. to a space homotopy equivalent to $\mathbb{S}^1 \vee \mathbb{S}^1$. This implies that p does not have a Euclidean neighbourhood.

must be surjective. However, the former group is isomorphic to $\mathbb{Z}$ and the latter to $*_{n-1}\mathbb{Z}$. This is not possible, since we can apply the abelianisation and tensor with $\mathbb{R}$ functors to r_* and obtain a surjective linear map $\mathbb{R} \to \mathbb{R}^{n-1}$, yielding a contradiction. Working it all out in detail is Exercise 13.2.

For the if direction, the proof amounts to exhibiting a euclidean neighbourhood for each point $p \in P$. Within the face F this is immediate. If p is in the interior of an edge a, this follows by glueing two half-discs incident to a. The remaining case is for p to be a vertex. By construction, p is an equivalence class of vertices of F. Namely, the equivalence class given by identifying different sides according to their labels. Given the collection $\{q_1, \cdots, q_l\}$ of vertices in F corresponding to p, we can consider the corners in P incident to each q_i and glue them along the sides according to the labels. Since each label appears twice, this will assemble a small disc neighbourhood of p. See Figure 13.2. □

13.2.4 Beyond regular n-gons

In upcoming arguments (Section 14.1) we will cut planar presentations (or rather, the polygons that define them) and reglue them using the identifications given by the labels. The result will in general not be a regular polygon. However, you may observe that:

Lemma 13.17. *Any two (strictly) star-shaped n-gons are homeomorphic.*

Proof. This was proven in Exercise 10.4, which states that both are discs. □

In fact, you can prove more: Any homeomorphism of the boundaries (which are piecewise

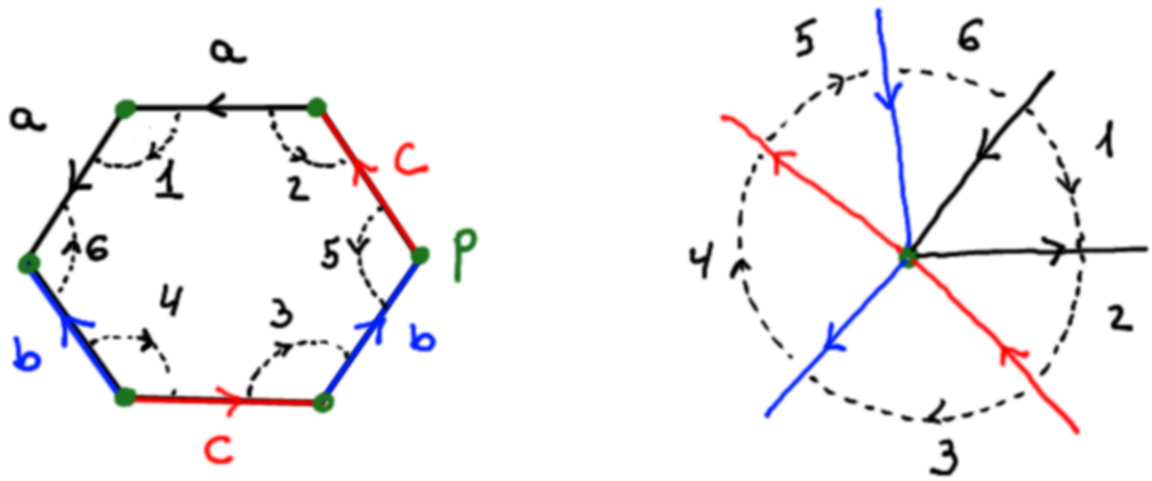


Figure 13.2: A planar presentation with six sides and three labels. The identifications between the sides imply that there is a single vertex, p. The corners incident to p can be glued to one another to assemble a disc neighbourhood.

circles) can be extended radially to a homeomorphism of the interiors. What this implies is that we do not have very careful about our polygons, since we can always reparametrise them to make them regular. This justifies the fact that we can do "arguments by picture".

13.3 Examples of planar presentations

13.3.1 The sphere

We begin with our favourite surface: the sphere. As we saw in Lemma 10.21, the 2-sphere is built by attaching a single face to a vertex. It follows that

Lemma 13.18. *The following planar presentations are homeomorphic to* $\mathbb{S}^2$*:*

- *The planar presentation with word* $\emptyset$*.*
- *The planar presentation with word* aa^{-1}*.*

Proof. The first item is precisely Lemma 10.21. The second item follows by consider the cell structure on $\mathbb{S}^2$ with two vertices (the poles), a single edge a (a meridian), and a single face. See Figures 13.3 and 13.4. □

Do note that these planar presentations have the "0-gon" and the "2-gon" as their face. You should not think too seriously about this and simply regard them as exceptional cases that fall outside the general definition we gave. Compare as well to the discussion in Subsection 13.2.4.

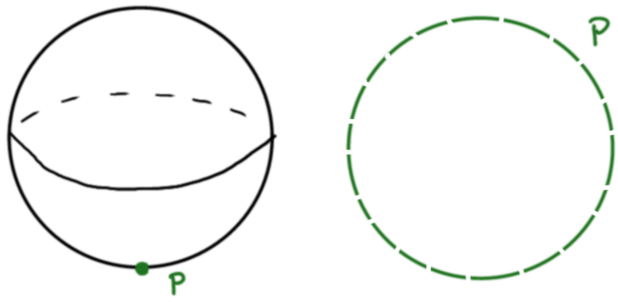


Figure 13.3: The planar representation, without edges, of the sphere. The “boundary of the 0-gon” is thus the vertex.

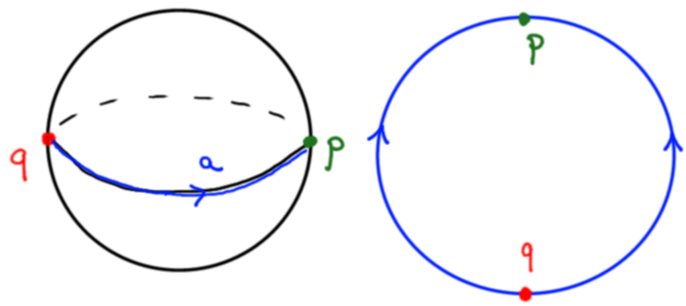


Figure 13.4: The planar representation of the sphere having two vertices (p and q) and one edge. The face is thus a 2-gon with both sides labelled as a.

13.3.2 The disc

There is also the exceptional case of the 1-gon:

Lemma 13.19. *The planar presentation with word a is homeomorphic to $\mathbb{D}^2$.*

13.3.3 The torus

In Section 10.6 we explained how the torus $\mathbb{T}^2$, using its usual presentation as a quotient of the square, can be endowed with the structure of a cell complex with one vertex, two edges, and one square face. It follows that:

Lemma 13.20. *The planar presentation with word $[a, b]$ is homeomorphic to $\mathbb{T}^2$.*

See Figure 10.3.

13.3.4 The projective plane

Another space you are familiar with is the projective plane $\mathbb{RP}^2$:

Lemma 13.21. *The planar presentation with word a^2 is homeomorphic to $\mathbb{RP}^2$.*

Proof. Recall that $\mathbb{RP}^2$ is the quotient of $\mathbb{S}^2$ under the antipodal identification $x \cong -x$, for all x. It follows that we can build a cell structure on $\mathbb{RP}^2$ from a cell structure on the sphere, as long as the latter is invariant under the antipodal map.

Consider then the following cell structure in $\mathbb{S}^2$: it has two vertices ($(1, 0, 0)$ and $(-1, 0, 0)$), two edges (the two sides of the equator, as divided by the vertices), and the two hemispheres (north and south). The antipodal map flips the vertices, edges, and hemispheres, keeping the cell structure invariant. It follows that $\mathbb{RP}^2$ has a cell structure with one vertex, one edge a, one face. Moreover, we see that the face F is attached along the word a^2, as claimed. See Figure 13.5. ☐

Once again, its face is a 2-gon.

13.3.5 The Klein bottle

Lastly, observe Figure 13.6 to deduce that:

Lemma 13.22. *The planar presentation with word $ba^{-1}ba$ is homeomorphic to the Klein bottle.*

13.4 Standard presentations of surfaces

Now that we have seen some examples, we are ready to introduce all the closed, path-connected surfaces. We do so by introducing their "standard planar presentations".

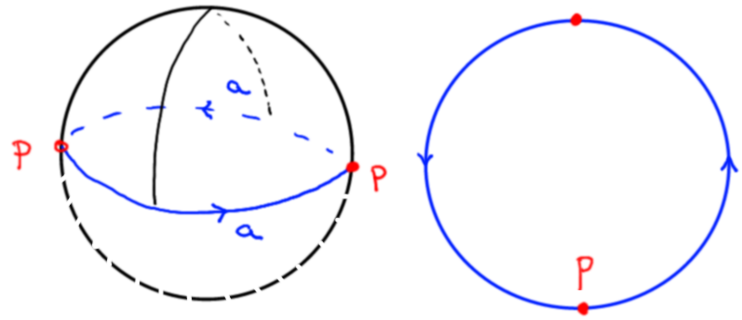


Figure 13.5: The standard planar representation of the projective plane. Both hemispheres of the sphere represent its face (under the antipodal identification).

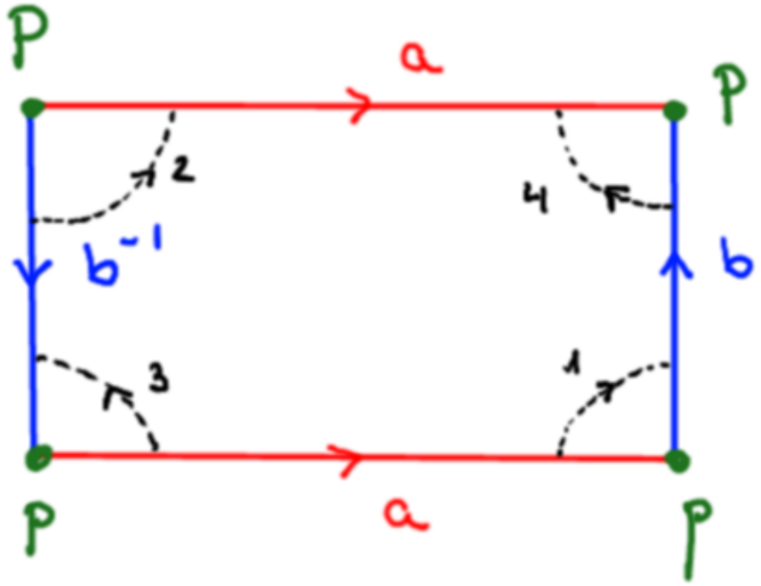


Figure 13.6: The standard planar representation of the Klein bottle.

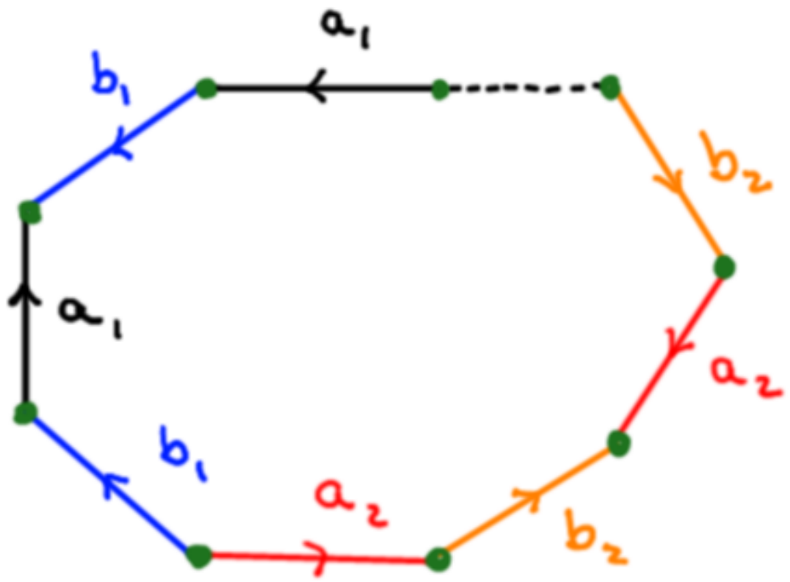


Figure 13.7: The standard planar representation of Σ_g, the closed orientable surface of genus g.

13.4.1 The compact orientable surfaces

Definition 13.23. *We say that $\mathbb{S}^2$ is Σ_0, the* ***closed orientable surface of genus*** *0.*

The name "orientable" will be explained in Subsection 15.1.2. Closed here means compact.

Definition 13.24. *For each $g \geq 1$ we define Σ_g, the* ***closed orientable surface of genus*** *g, as the surface given by the planar presentation with word $\prod_{i=1}^{g}[a_i, b_i]$. We call this presentation* ***standard****.*

It is not difficult to show, by following the identifications given by the labels, that the standard presentation of Σ_g has a single vertex. By construction, it has $2g$ edges, $a_1, b_1, \cdots, a_g, b_g$. See Figures 13.7 and 13.8. Note that:

Lemma 13.25. *Σ_1 is the 2-torus.*

And now we compute our favourite invariants:

Lemma 13.26. *The following results hold:*

- $\pi_1(\Sigma_g, p) \simeq \langle a_1, b_1, \cdots, a_g, b_g \mid \prod_{i=1}^{g}[a_i, b_i]\rangle$.
- $H_1(\Sigma_g, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$.
- $H_1(\Sigma_g, \mathbb{R}) \simeq \mathbb{R}^{2g}$.

Proof. The first result is immediate from Corollary 13.14. We can then abelianise, which amounts to introducing the commutators $[a_i, b_i]$ as relations, for all i. These new relations in particular imply the relation we already had. It follows that $H_1(\Sigma_g, \mathbb{Z})$ is the group with $2g$ generators, all of which commute, and no further relations. That is, $\mathbb{Z}^{2g}$. $\square$

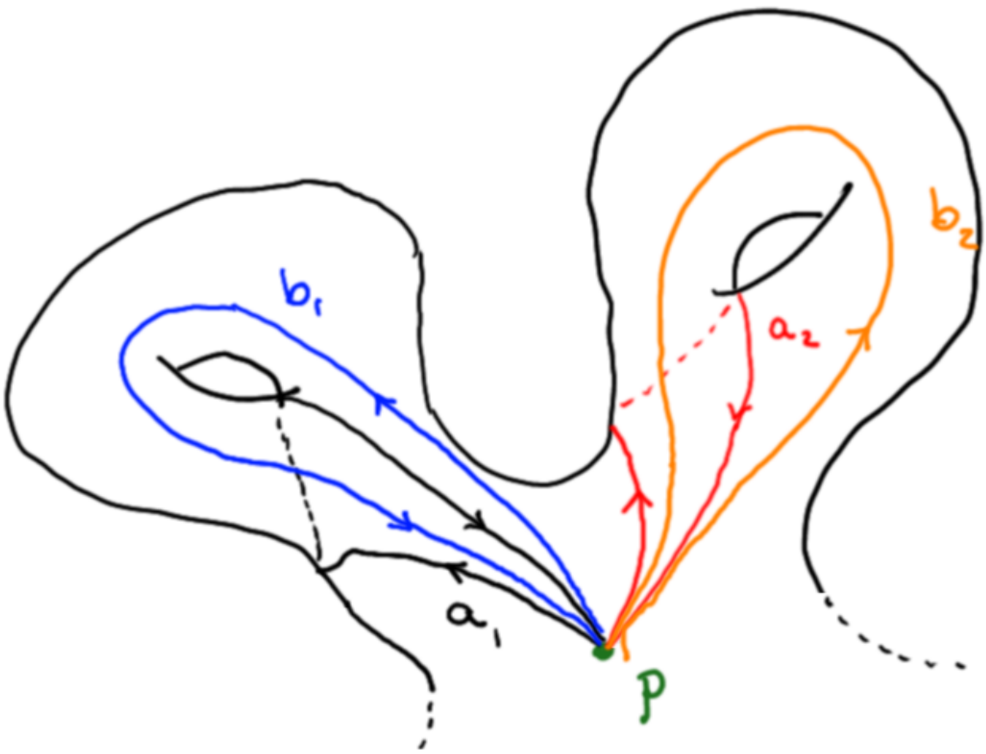


Figure 13.8: Σ_g, the closed orientable surface of genus g. Compare to Figure 13.7.

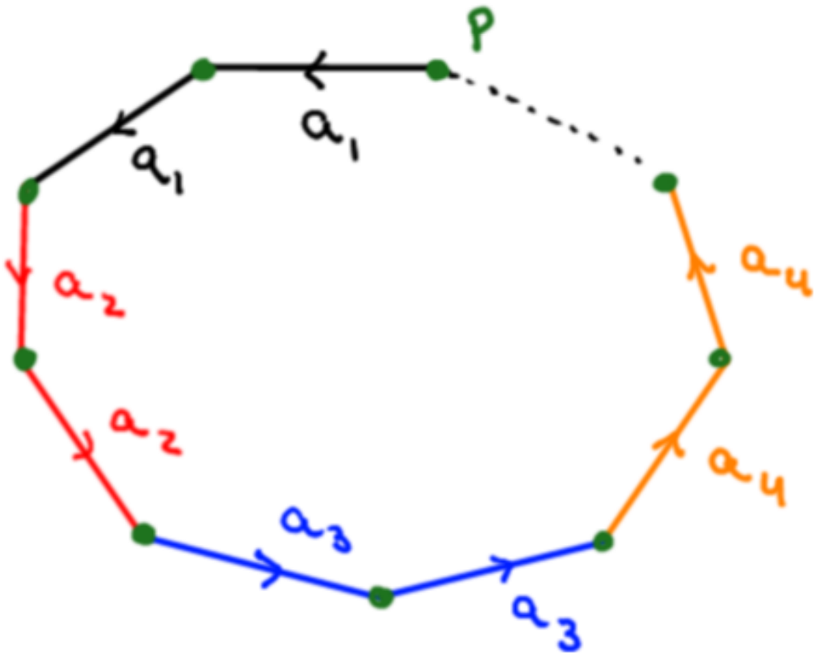


Figure 13.9: The standard planar representation of N_g, the closed non-orientable surface of genus g.

13.4.2 The compact non-orientable surfaces

Definition 13.27. *For each $g \geq 1$ we define N_g, the* ***closed non-orientable surface of genus*** *g, to be the surface given by the planar presentation with word $\prod_{i=1}^{g} a_i^2$. We call this presentation* ***standard****.*

By following the identifications we see that the standard presentation of N_g has a single vertex. By construction, it has g edges, $a_1, \cdots, a_g$. See Figure 13.9.

We readily see that:

Lemma 13.28. *N_1 is $\mathbb{RP}^2$.*

The following is left to you as Exercise 13.5. It follows very explicitly from the same arguments that prove the complete classification of surfaces (Theorem 15.8), but you can also deduce from the theorem itself.

Lemma 13.29. *N_2 is the Klein bottle.*

And here is the fundamental group:

Lemma 13.30. *The following results hold:*

- $\pi_1(N_g, p) \simeq \langle a_1, \cdots, a_g \mid \prod_{i=1}^{g} a_i^2 \rangle$.
- $H_1(N_g, \mathbb{Z}) \simeq \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$.
- $H_1(N_g, \mathbb{R}) \simeq \mathbb{R}^{g-1}$.

Proof. The first one is immediate from Corollary 13.14. For the second one, we abelianise, adding all the commutators $[a_i, a_j]$ as relations. Once we do that, we have the equality $(\prod_{i=1}^g a_i)^2 = \prod_{i=1}^g a_i^2$, so:

$$H_1(\Sigma_g, \mathbb{Z}) \simeq \langle a_1, \cdots, a_g \mid [a_i, a_j], (\prod_{i=g}^{1} a_i)^2 \rangle$$
$$\simeq \langle a_1, \cdots, a_{g-1}, b \mid [a_i, a_j], [a_i, b], b^2 \rangle \simeq \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z},$$

where in the second isomorphism we have changed the basis of the group by replacing a_g by $b = \prod_{i=1}^g a_i$. In the last isomorphism we observe that all generators commute, and the last one has order two, yielding the claim. □

13.4.3 π_1 is a complete invariant

Our previous computations show that:

Proposition 13.31. *All the surfaces*

$$\{\Sigma_g\}_{g=0}^{\infty} \cup \{N_g\}_{g=1}^{\infty}$$

appearing in Definitions 13.24 and 13.27 are not homotopy equivalent to each other. In particular, they are not homeomorphic.

Proof. According to Lemma 13.30, each $H_1(N_g, \mathbb{Z})$ has an element of order exactly 2. In contrast, the elements in $H_1(\Sigma_g, \mathbb{Z})$ have all infinite order. This means that N_g cannot be homotopy equivalent to $\Sigma_{g'}$, for any g and g'. Similarly, the Σ_g are all different to each other, since $H_1(\Sigma_g, \mathbb{R})$ has dimension $2g$. Lastly, $H_1(N_g, \mathbb{R})$ having dimension $g - 1$ shows that all N_g are distinct. □

So we have an infinite family of distinct surfaces. We will show in Section 15.2 that every closed surface is homeomorphic to one of these.

13.5 Worked out example: Retracts and planar presentations which are not surfaces

We now study planar presentations for spaces that need not be surfaces:

Lemma 13.32. *Let X_k be the planar presentation given by the cyclic word a^k. Then:*

- *X_k is a surface if and only if $k = 2$.*
- *X_1 is the disc $\mathbb{D}^2$.*
- *X_2 is $\mathbb{RP}^2$.*
- *$\pi_1(X_k, p) \simeq \mathbb{Z}/k\mathbb{Z}$.*
- *X_k does not retract to its 1-skeleton.*

Proof. According to 13.16, a planar presentation yields a surface exactly when all labels appear twice. This proves the first item.

Now we observe that, regardless of k, the 1-skeleton of X_k is $\mathbb{S}^1$, since it consists of exactly one vertex and one edge. In the concrete case of X_1, attaching a face along that one edge yields precisely the disc. The third item is also immediate.

For the fundamental group computation we apply Corollary 13.14, to see that a group presentation is $\langle a \mid a^k \rangle$. This also allows us to prove the last item: suppose that there is a retract $r : X_k \to S$, with S the 1-skeleton. Then $r_* : \pi_1(X_k, p) \to \pi_1(S, p)$ would be surjective, but this is not possible since the source is the cyclic group of order k (in particular, it is finite) and the target is $\mathbb{Z}$. □

The following computation is not about planar presentations, but it shows how one may reason about retracts in a slightly more complicated situation (compared to the examples we have seen before):

Lemma 13.33. *Consider $Y_k = (X_k, p) \vee (\mathbb{S}^1, 1)$ and let $S \subset Y_k$ be the 1-skeleton of X_k. Then, Y_k does not retract to S.*

Proof. One can argue in two different ways. Let us present the more general one first.

Using the previous lemma and van Kampen for the wedge, we compute

$$\pi_1(Y_k, p) \simeq \langle a, b \mid a^k \rangle$$

so a represents the generator coming from X_k and b the generator from $\mathbb{S}^1$. Since the 1-skeleton of X_k is a circle, we have $\pi_1(S, p) \simeq \mathbb{Z}$. Suppose for contradiction that a retract $r : Y_k \to S$ exists. Then it would follow that $r_* : \pi_1(Y_k, p) \to \pi_1(S, p)$ is surjective. Now we cannot derive a contradiction purely from size considerations, since $\pi_1(Y_k, p)$ is larger than $\pi_1(S, p)$. Instead, we recall that r_* has to be a left-inverse to i_*, with $i : S \to Y_k$ the inclusion. Let us thus compute i_* explicitly. Since S is the 1-skeleton of X_k, we have that the generator $a \in \pi_1(X_k, p)$ is precisely given by the generator in $\pi_1(S, p)$, which we therefore identify with $\langle a \mid \rangle$. It follows that r_* has to send $a \in \pi_1(Y_k, p)$ to $a \in \pi_1(S, p)$, but this cannot be the case since a has order k in the source, but infinite order in the target. This proves the claim.

Alternatively (but this is specific to this concrete example), one can note that a retraction $Y_k \to S$ in particular yields a retraction $X_k \to S$, which does not exist by the previous exercise. □

13.6 Exercises

13.6.1 Not surfaces

Exercise 13.1: Construct a 2-dimensional cell complex using a single vertex, a single edge, and two faces, but which is not a surface.

Exercise 13.2: Let e be an edge in a planar representation P. Suppose that the face of P is incident to e along n sides. Fix a point $p \in \overset{\circ}{e}$ and let $U \ni p$ be a small neighbourhood of p in P. Prove that $U \setminus \{p\}$ retracts to a 1-dimensional cell complex with two vertices and n edges.

13.6.2 Surfaces

Exercise 13.3: Show that glueing a disc and a (closed) Möbius band along their boundaries yields a projective plane. Use this to compute the fundamental group of the projective plane.

Exercise 13.4: Show that glueing two (closed) Möbius bands along their boundary yields a Klein bottle.

Exercise 13.5: Prove Lemma 13.29, saying that the surface N_2 is the Klein bottle.

Exercise 13.6: Let S be a compact surface and $p \in S$ a point.

- Prove that $S \setminus p$ is homotopy equivalent to a graph.
- Compute the fundamental group of $S \setminus p$.

Exercise 13.7: Let P be a planar presentation with word ω such that there is a label a that appears exactly once. Prove that P is homotopy equivalent to a graph.

Moves for planar representations

Lecture 14

We continue our study of surfaces, our goal being to prove the classification of path-connected, compact surfaces.

To this end, we introduce a series of combinatorial *moves*, which allow us to switch between different planar representations of the same surface (Section 14.1). We first introduce the two *basic* moves (Section 14.1). These are then combined to yield the three *compound* moves that the classification theorem relies upon (Section 14.2).

The upcoming moves change the planar presentation, but not the surface it represents (up to homeomorphism).

14.1 Basic moves for planar presentations

14.1.1 Basic move I: cutting and pasting

The first move reads:

Definition 14.1. *Let P be the planar presentation of a surface. Let $a \in I$ be one of the labels. Let d be a diagonal dividing the face F into two pieces, each containing one of the a labels. Consider the planar presentation P' produced by:*

- *Cutting P along d, yielding two polygonal pieces, each with one side labelled as d.*
- *Pasting the two pieces along the two a sides, yielding a polygon.*

Then P' is said to have been obtained from P by ***cutting and pasting****.*

First note that cutting and pasting is reversible: P can be obtained from P' by cutting along a (which is now a diagonal) and pasting d. Furthermore, observe that P' is not a convex polygon a priori, but we can replace it by one (Subsection 13.2.4). See Figure 14.1.

Lemma 14.2. *Let P and P' relate to each other via cutting and pasting. Then, the surfaces underlying the two are homeomorphic.*

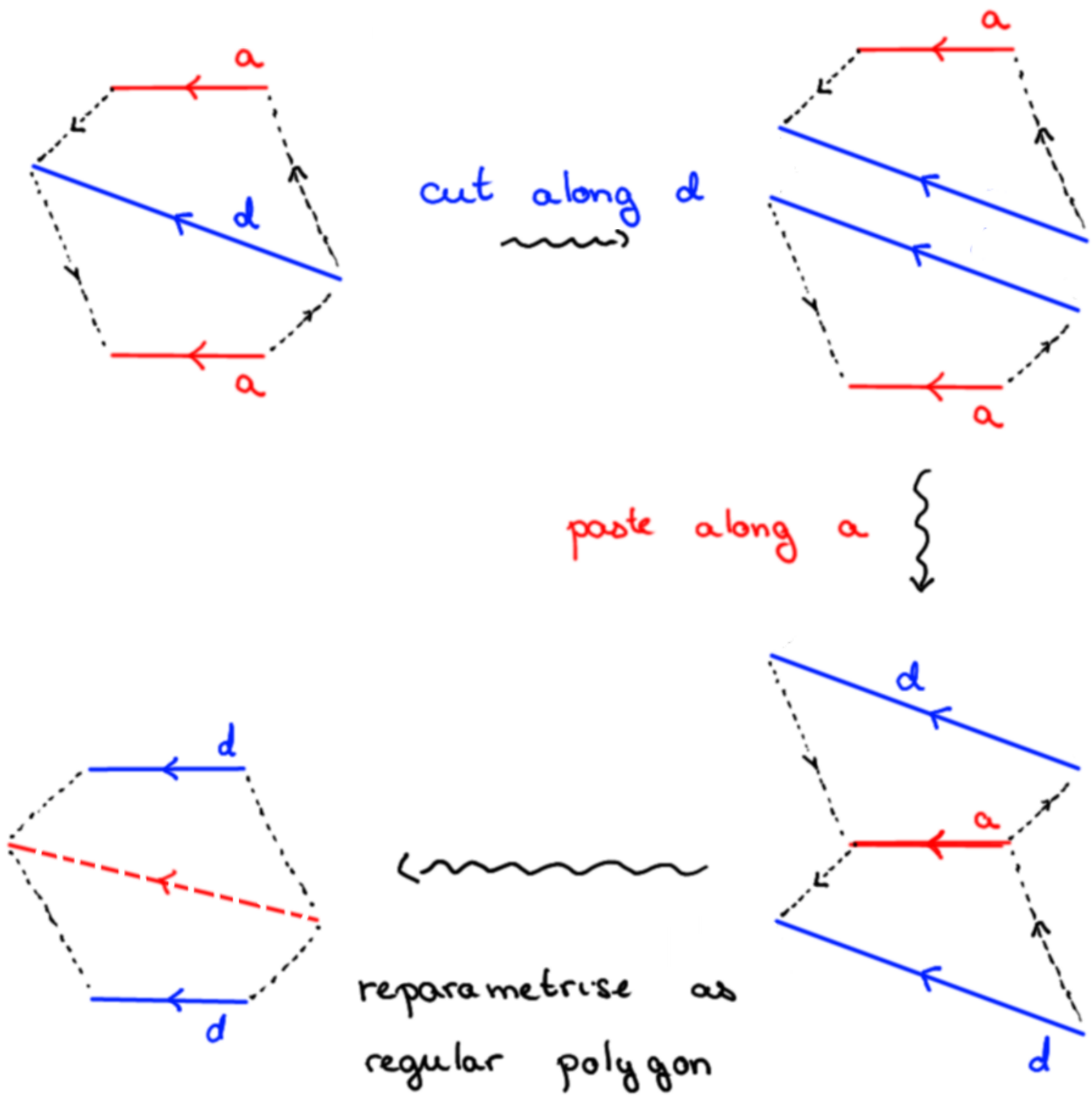


Figure 14.1: Two planar presentations of the same surface, related by cutting along d and pasting along a (or viceversa).

Proof. In some sense there is nothing to prove: cutting is something we do to the face F, but not to P. The new label d precisely says that the diagonal we cut has to still be glued, as it was in P. Similarly, the glueing we did along a was simply performing the identification forced upon by the label a.

Alternatively, you may argue as follows: cutting F produces two polygonal pieces A and B. Each piece maps to the face F' of P' canonically. These maps are compatible with the identifications given by the labels, meaning that there is a well-defined bijection between the quotients P and P'. It is continuous by the pasting lemma. Since both spaces are Hausdorff and compact, the map is a homeomorphism. □

14.1.2 Basic move II: contraction

The second move reads:

Definition 14.3. *Let ω_0 and ω_1 be words that share no letters, both representing surfaces. Let a be a letter not present in either word. Let P be the planar presentation associated to $\omega_1 a^{-1}\omega_0 a$. Let P' be the presentation associated to $\omega_1\omega_0$.*

*Then, P' is said to have been obtained from P by **contracting**. Conversely, P is obtained from P' by **expanding**.*

By construction, contracting and expanding are inverses to each other. See Figures 14.2 and 14.3. You may observe that expanding ω to $\omega a^{-1}a$ amounts to performing connect sum with $\mathbb{S}^2$ (concretely with its presentation with word $a^{-1}a$).

As before:

Lemma 14.4. *Let P' be obtained from P by contracting. Then, the underlying surfaces are homeomorphic.*

Proof. We can assemble a neighbourhood of the edge a as shown on the top right of Figure 14.2. This is done by taking neighbourhoods in F of the two a-sides, as well as neighbourhoods of the corresponding endpoint vertices p and q (by chasing around the corners of F), and glueing them along the identifications.

Within this model we delete a and the vertex q. We then use straight lines to connect to p the edges formerly incident to q. For this to work, we have to first create some room by deleting the ends of the edges. This is shown in the middle right image.

The changes made in the model can now be translated to the polygon F. Effectively, we are adding/subtracting polygonal pieces in each corner incident to q. This is shown in the bottom image. One can then reparametrise the polygon to make it regular once again. Note that all this cutting and glueing changes the face F, but not P itself. □

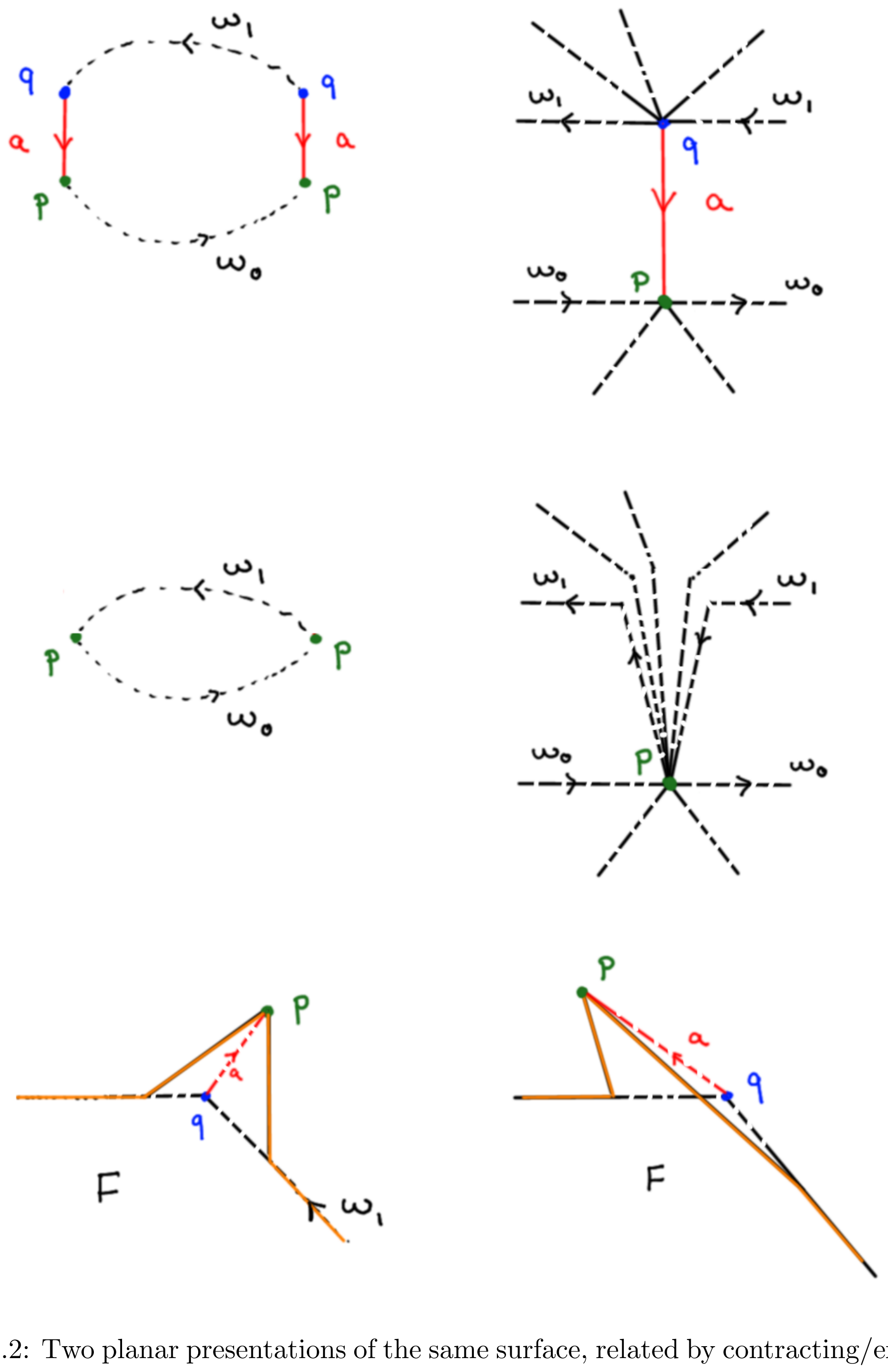


Figure 14.2: Two planar presentations of the same surface, related by contracting/expanding. The starting presentation P is shown in the top row, containing the edge a to be contracted. In the top right we see a model neighbourhood around a. In the second row we see the resulting presentation P'; the right image depicts how the vertex q has been deleted in the model. The changes in the model have to be translated into a change of the face; this is shown in the bottom row.

Figure 14.3: Two planar presentations of the same surface, related by contracting/expanding in the simple case in which the word is $\omega a^{-1}a$. In this case we do not have to modify the edges incident to q, it suffices that we delete p instead.

14.2 Compound moves for planar presentations

14.2.1 Compound move A: handle detection

We can now cut and paste repeatedly in order to simplify the word of a given planar presentation:

Definition 14.5. *Let P be the planar presentation of a surface associated to the cyclic word $b^{-1}\omega_3 a^{-1}\omega_2 b\omega_1 a\omega_0$. Here ω_0, ω_1, ω_2, and ω_3 are words, possibly sharing letters. a and b are additional letters.*

Let c and d be letters not appearing in the words ω_i. Then, the presentation P' with cyclic word $[c,d]\omega_1\omega_2\omega_3\omega_0$ is said to be obtained from P by ***handle detection****.*

In particular, observe that P' is the connected sum of the standard presentation of $\mathbb{T}^2$ and the planar presentation with word $\omega_3\omega_2\omega_1\omega_0$. This justifies the name: we have detected that our surface contains a $\mathbb{T}^2$ summand, and we call this a *handle*.

Lemma 14.6. *Let P' be obtained from P by handle detection. Then, the underlying surfaces are homeomorphic.*

Proof. Figure 14.4 shows that performing handle detection amounts to cutting and pasting twice. The result then follows from Lemma 14.2. □

14.2.2 Compound move B: crosscap detection

Instead of detecting a torus, in the non-orientable case we can try to detect whether our surface has an $\mathbb{RP}^2$ as a summand:

Definition 14.7. *Let P be the planar presentation of a surface associated to the cyclic word $a\omega_1 a\omega_0$. Here ω_0 and ω_1 are words, possibly sharing letters. a is a label.*

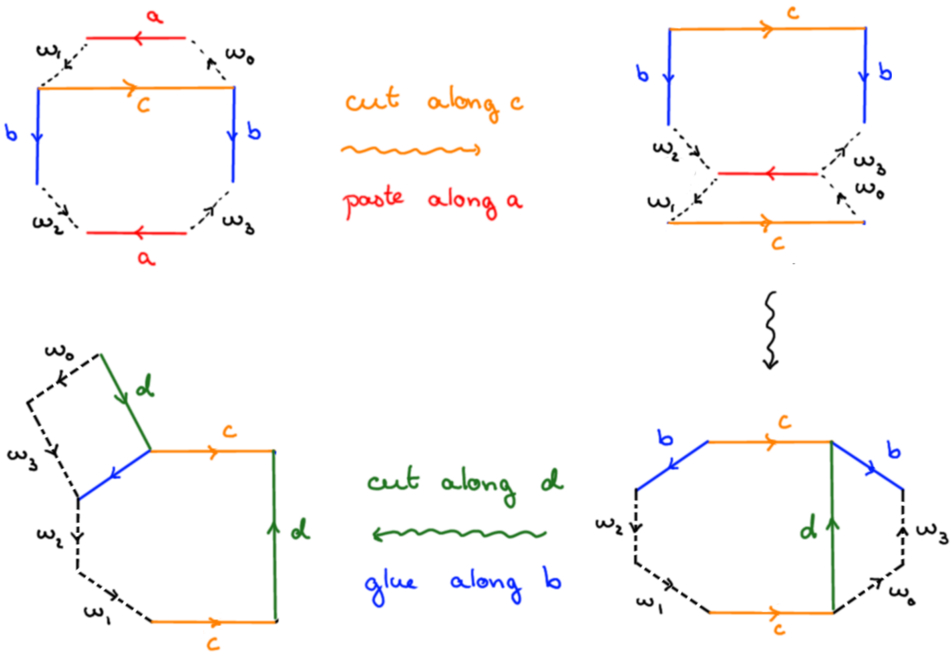


Figure 14.4: Two planar presentations of the same surface, related by handle detection. In the intermediate step we make the face into a regular polygon. The last step in which this is done again is not shown.

Let b be a letter not appearing in the words ω_i. Then, the presentation P' with cyclic word $b^2\omega_1\omega_0^{-1}$ is said to be obtained from P by ***crosscap detection****.*

The $\mathbb{RP}^2$ summand in our surface is called a *crosscap*.

Lemma 14.8. *Let P' be obtained from P by crosscap detection. Then, the underlying surfaces are homeomorphic.*

Proof. Figure 14.5 shows that performing crosscap detection amounts to cutting and pasting once. The result then follows from Lemma 14.2. ☐

14.2.3 Compound move C: handle trading

Lastly, we introduce a move that allows us to replace handles by crosscaps, as long as a crosscap is already present:

Definition 14.9. *Let P be the planar presentation of a surface associated to the cyclic word $[b, c]a^2\omega$. Here ω is some word.*

Let i, j, k be letters not appearing in ω. Then, the presentation P' with cyclic word $k^2j^2i^2\omega$ is said to be obtained from P by ***handle trading****.*

Lemma 14.10. *Let P' be obtained from P by handle trading. Then, the underlying surfaces are homeomorphic.*

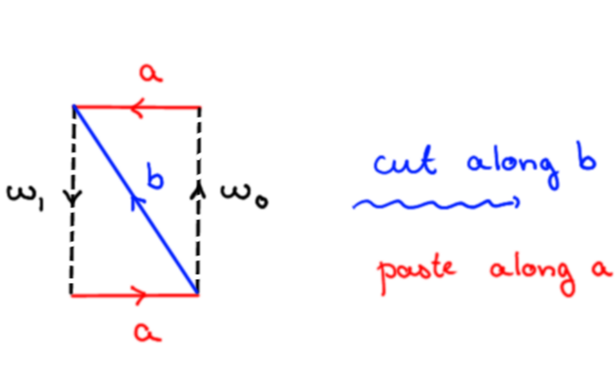


Figure 14.5: Two planar presentations of the same surface, related by crosscap detection.

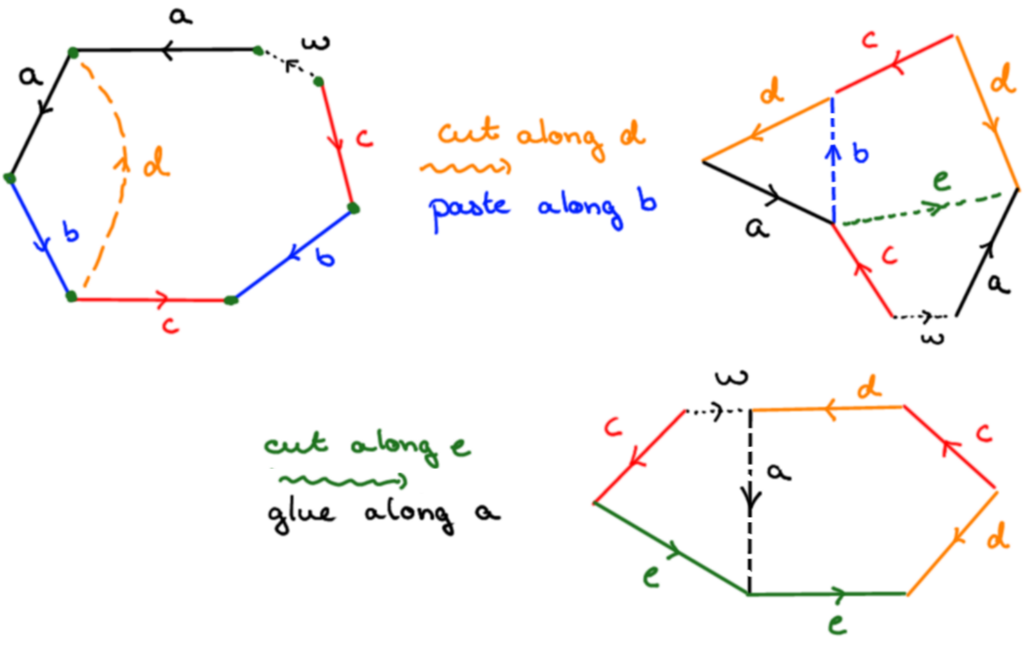


Figure 14.6: The first cut and paste move needed to perform handle trading, followed by the first crosscap detection.

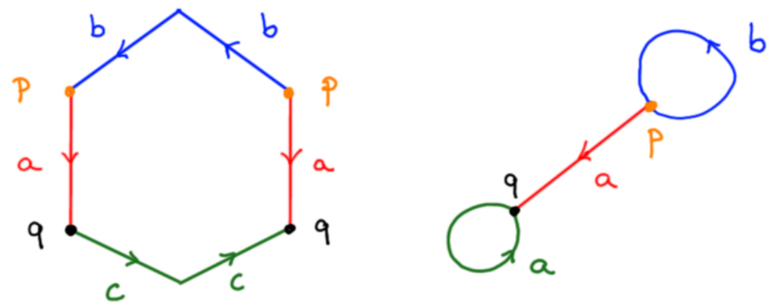


Figure 14.7: The planar presentation discussed in Section 14.3. It has two vertices and three edges. On the right, its 1-skeleton.

Proof. Handle trading begins with a cut and paste move, depicted in Figure 14.6. Once that is accomplished, one is now able to perform crosscap detection three times, with respect to a (also shown in the figure), then c, then d. The result then follows from Lemmas 14.2 and 14.8. □

In particular, we have established Corollary 15.20.

14.3 Worked-out example of planar presentations

Consider the planar presentation P shown in Figure 14.7, with cyclic word $\omega = a^{-1}c^2ab^2$. The label a separates ω into the subwords b^2 and c^2, which do not share labels. We are thus in a setting in which contraction can be applied (Definition 14.3), yielding the presentation P' with cyclic word c^2b^2. This is the standard presentation of N_2, up to relabelling.

Since P and P' are homeomorphic (Lemma 14.4), their fundamental groups agree. It follows that we know what the fundamental group of P, since we already computed it for N_2 (Lemma 13.30).

However, let us try to apply Corollary 13.15 directly, which computes the fundamental group of any planar presentation, and see that we get the same answer. Observe that P has two vertices p and q, so we cannot apply Corollary 13.14. Now we can draw the 1-skeleton P_1; this is depicted on the right hand side of Figure 14.7. We claim that:

Lemma 14.11.

$$\pi_1(P_1, p) \simeq \langle b, c' \mid \rangle.$$

Proof. Indeed, P_1 is a graph, so its fundamental group has no relations (Corollary 11.22). Moreover, its fundamental group is computed by taking a maximal tree $T \subset P_1$, which in this case is the edge a, and then observing that every further attachment of an edge yields a new generator. b is attached to p, forming a loop, so is represents a generator. c is attached to q, so it is a loop, but not based at p. The corresponding loop based at p is instead $c' := aca^{-1}$, which is obtained from c by performing change of basepoint using the edge a. □

P is obtained from P_1 by attaching the face F. Its attaching gives us a single relation, which is the word ω. Now we have to be careful: as written, ω is not a word on the symbols b and c'. However, we can expand:

$$\omega = a^{-1}c^2ab^2 = (a^{-1}ca)(a^{-1}ca)b^2 = (c')^2b^2.$$

Let us explain what this means. ω represents the homotopy class of the attaching of F to P_1 along ∂F. We think of it as a word on the labels a, b, and c, which are homotopy classes in $\Pi_1(P)$. The class b lives in the group $\pi_1(P, p)$, the class c lives in $\pi_1(P, q)$, and a lives in $\pi_1(P, q, p)$. The class c' is in $\pi_1(P, p)$ as well, and is a conjugate of c via a. It follows that we are allowed to rewrite $\omega \in \Pi_1(P)$ using moves, and it is best to express it in terms of generators of $\pi_1(P, p)$. This is what we just did. Then we deduce:

$$\pi_1(P, p) \simeq \langle b, c' \mid (c')^2b^2\rangle,$$

which we recognise as the fundamental group of N_2.

Classification of path-connected, compact surfaces

Lecture 15

In this lecture, we:

- Tackle the classification of closed path-connected surfaces up to homeomorphism (Theorem 15.8) using moves.
- Introduce two invariants of surfaces: Euler characteristic and orientability (Section 15.1).
- Define an operation on path-connected surfaces, called the connected sum (Section 15.3).

Even though Euler characteristic, orientability, and connected sum will be defined via planar presentations, we will show that they are preserved by moves. From this it will follow that they are actually intrinsic concepts for surfaces. This idea (defining an invariant via a concrete combinatorial model and then proving that the definition does not depend on the model chosen), appears in many other places in Topology.

15.1 Invariants of surfaces

In this course we have studied the fundamental group, which is now our favourite invariant of spaces. We have also defined the first homology, which is easier to handle but slightly less powerful. These invariants can be specialised to surfaces, as we did above. We now define two further invariants, the Euler characteristic and the orientation.

15.1.1 Euler characteristic

Consider:

Definition 15.1. *Let X be a cell complex with finitely many cells. Its* ***Euler characteristic*** *is defined as:*

$$\chi(X) := \sum_{i=0}^{\dim(X)} (-1)^i |\{\text{cells of dimension } i\}|,$$

where $|-|$ indicates taking the cardinality.

A priori, the Euler characteristic appears to be an invariant of the cell structure, and not of the space itself. One can show that it is an invariant up to homotopy equivalence. We will prove it only for surfaces (Theorem 15.13 below).

Proposition 15.2. *Let Σ_g and N_g be endowed with their standard planar presentation. Then $\chi(\Sigma_g) = 2 - 2g$ and $\chi(N_g) = 2 - g$.*

Proof. Σ_g has a vertex and a face, which count positively, and $2g$ edges, which count negatively. Similarly, N_g has one vertex, one face, and g edges. □

15.1.2 Orientability

Definition 15.3. *Let P be the planar presentation of a surface. We say that P is* ***orientable*** *if every label appears once with each orientation. Otherwise we say it is* ***non-orientable****.*

Once again, orientability seems to depend on the cell structure of P, and not just on the surface itself. However, this is not the case (Theorem 15.14).

By inspection we see that:

Proposition 15.4. *Let Σ_g and N_g be endowed with their standard planar presentation. Σ_g is orientable and N_g is not.*

Which justifies the name we gave to these surfaces.

The following lemma, and the subsequent discussion, provides some insight on the geometric meaning of orientability:

Lemma 15.5. *Let P be a non-orientable planar presentation. Then, there is an embedding $M \to P$, with M the Mobius band.*

Proof. Recall than an embedding is an injective map that is a homeomorphism with its image. If P is not orientable, there is a label a that appears twice with the same orientation. Then we can consider a band connecting the two sides labelled as a. Identifying the two sides also identifies the ends of the band, which closes up to yield a Mobius band. See Figure 15.1. □

Remark 15.6: In your course on differentiable manifolds you may have seen orientability defined as: *one can cover the manifold by charts such that the transition functions have positive determinant.* We now relate this differentiable notion to Definition 15.3, which is combinatorial in nature.

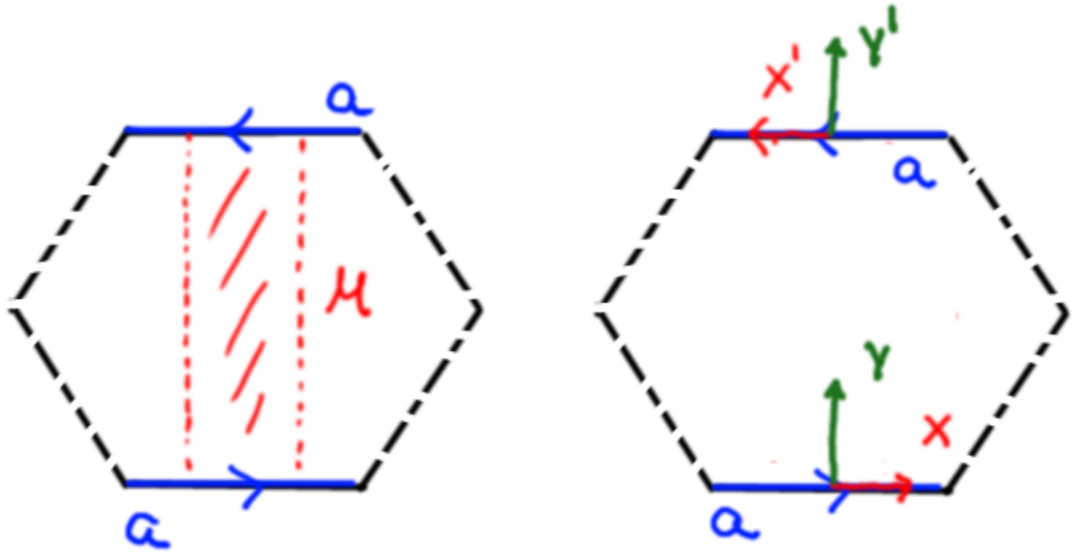


Figure 15.1: A planar representation with a label appearing twice with the same orientation. This allows us to find a Mobius band (left) as well as two see that the standard orientation $\{X, Y\}$ flips upon crossing the identification (right).

Let F be the polygonal face of P. You can imagine F being oriented by the standard basis of $\mathbb{R}^2$. We can now study whether this orientation remains consistent when we identify sides using their labels.

Given a side α, choose a unit vector X parallel to it, compatible with its orientation. There is then a unique vector Y that is orthogonal to X, of unit length, such that $\{X, Y\}$ is positively oriented. Find then the side β that is glued to α according to the labels. The identification between the two takes X to a vector parallel X' to β and oriented compatibly. The glueing moreover takes Y to a vector Y' orthogonal to β; Y' points into F if and only if Y points outwards. Now we conclude: $\{X', Y'\}$ is also positively oriented if and only if α and β have opposite orientations. △

Remark 15.7: In Figure 13.2 we explained how to assemble a neighbourhood of a vertex in a planar presentation. This was a little disc, whose boundary is a circle. In the figure we chose an orientation of this circle (clockwise, in the right hand side of the picture). The lack of orientability is now visible as follows: at each corner of the polygon we see a little piece of circle, with its orientation. The surface is non-orientable precisely when these orientations do not match (as is the case in the figure). This was also the case for the Klein bottle (Figure 13.6). △

15.2 Classification of surfaces

The classification theorem states:

Theorem 15.8. *Let S be a compact, path-connected surface. It is homemomorphic to one of the surfaces in the collection $\{\Sigma_g\}_{g=0}^{\infty} \cup \{N_g\}_{g=1}^{\infty}$.*

15.2.1 The proof

Theorem 15.9. *Given a compact, path-connected surface S, there is a planar presentation P whose underlying space is homeomorphic to S.*

This is a classic result of Radó. Its proof is beyond the scope of these notes. The rough idea is the following: one can cover S using finitely many charts $\{U_i\}$. Each chart we can subdivide (using its own coordinates) into little cubes using a scaling of the standard lattice. These subdivisions can now be intersected in S but, since the transition functions are just continuous, the cuts can be very complicated. Now we use the Jordan curve theorem (which says that every closed curve in the plane bounds a disc) to perturb these cuts and make them "intersect transversely". Effectively, this divides the surface S into discs, separated by segments. Each disc can then be identified with a polygon. One can then carefully remove segments in order to leave one big polygon with sides identified, i.e. a planar presentation.

According to Theorem 15.9, Theorem 15.8 will follow once we establish the following propositions:

Proposition 15.10. *Every orientable presentation of a surface relates to one of the standard presentations $\{\Sigma_g\}_{g=0}^{\infty}$.*

Proof. Consider the following inductive statement: Every orientable word $\omega'\omega$ such that:

- ω' and ω do not share any letters,
- ω has length at most n,

is equivalent, via moves, to $\omega'\widetilde{\omega}$, with $\widetilde{\omega}$ of length at most n and moreover of the form $\prod_i [a_i, b_i]$.

The inductive case is $n \leq 4$. If $n = 2$, orientability implies that $\omega = a^{-1}a$, so contraction can be applied to remove it. If $n = 4$, up to relabelling, it must be the case that $\omega = [a, b]$ or $b^{-1}ba^{-1}a$. In the former case we are done. In the latter we can apply contraction twice to remove ω.

For the general case, pick a label $a \in \omega$. Then we have two cases. The first is that there is some label b such that the two sides labelled a separate ω into two subwords, one containing b and the other b^{-1}. We then apply handle detection to a and b. This yields a new equivalent word $\omega'[c, d]\omega''$, with ω'' of smaller length. The second case is that a separates ω into two subwords that share no letters. This implies that contraction can be applied to remove the a label, yielding the shorter word $\omega'\widetilde{\omega}$. In both cases the inductive hypothesis applies.

The result follows by applying the inductive statement with ω' the empty word. □

And:

Proposition 15.11. *Every non-orientable presentation of a surface relates to one of the standard presentations* $\{N_g\}_{g=1}^{\infty}$.

Proof. Consider the following inductive statement: Every word $\omega'\omega$ such that:

- ω' and ω do not share any letters,
- ω has length at most n,

is equivalent, via moves, to $\omega'\widetilde{\omega}$, with $\widetilde{\omega}$ a concatenation of commutators and squares (and moreover of length at most n).

The inductive case is $n \leq 4$. We have already done the orientable cases. The non-orientable ones (up to relabelling) are $\omega = a^2$, $\omega = b^2a^2$, and $\omega = b^{-1}aba$. We only need to handle the last one, which is addressed by applying crosscap detection to a. For the general case, there are two situations: if ω is orientable, we apply the orientable statement. Otherwise, there is some label $a \in \omega$ such that the two a sides appear with the same orientation. It follows that we can apply crosscap detection and then the inductive hypothesis.

We now apply the inductive statement with $\omega' = \emptyset$. This says that every presentation is equivalent to a presentation that is a connected sum of tori and projective planes. Since our starting presentation was non-orientable, there is at least one projective plane. We can now apply handle trading until all handles have been replaced by crosscaps. □

This concludes the proof of Theorem 15.8. Moreover, we can combine these two propositions with Proposition 13.31 to deduce that:

Corollary 15.12. *Two presentations are related if and only if the corresponding surfaces are homeomorphic.*

Proof. If two presentations relate to a standard one, they relate to one another by concatenating the two sequences of moves. Moreover, all the standard presentations have different fundamental groups so they are not homeomorphic. □

15.2.2 Invariants under moves

We now prove that Euler characteristic and orientability do not depend on the planar presentation and are intrinsic to the surface.

Theorem 15.13. *Let P and P' be planar presentations of the same surface. Then $\chi(P) = \chi(P')$.*

Proof. According to Corollary 15.12, we must show that χ is invariant under moves. Suppose P and P' relate by a single basic move. If the two relate by cutting and pasting, we see that the two have the same amount of vertices, edges and a single face. The result follows. If they relate via contracting we see that P' has one vertex and one edge less, so the Euler characteristic is the same. □

We can thus define $\chi(S) := \chi(P)$ for any presentation P of the surface S.

Theorem 15.14. *Let P and P' be planar presentations of the same surface. Then P is orientable if and only if P' is orientable.*

Proof. Once again it is enough to show invariance of orientability under a single move. Suppose P is an orientable presentation. If P' relates to it by cutting and pasting we see that gluing along a will result into the label d appearing once with each orientation. All other labels remain as they were. It follows that P' is orientable as well. In the case of contraction the statement is immediate, since the contracted label disappears and all others stay with the same orientation. □

In particular we can define orientability of a surface S using the orientability of any of its presentations P.

This implies that the Euler characteristic and orientability are, together, a complete invariant of compact surfaces:

Corollary 15.15. *Two compact surfaces are homeomorphic if and only if they are both orientable (or both not) and they have the same Euler characteristic.*

Proof. According to Theorem 15.8, all surfaces are homeomorphic to one of $\{\Sigma_g\}_{g=0}^{\infty} \cup \{N_g\}_{g=1}^{\infty}$. The latter are indeed distinguished by the claimed invariants. □

A neat consequence of this is that we do not need fundamental group to classify the compact surfaces (but it clarifies things!) Indeed, Theorem 15.8 does not use π_1. Corollary 15.12 does, since it uses π_1 to distinguish the standard planar presentations. However, Corollary 15.15 states that this can also be done using Euler characteristic and orientability.

15.3 Connected sum

Consider the following concept: Given two manifolds M and M' of the same dimension we can: (1) remove small open balls $B \subset M$ and $B' \subset M'$ from each, (2) identify the spheres $\partial B \simeq \partial B'$. See Figure 15.2. This is called the *connected sum*.

15.3.1 Via planar presentations

We now study the connected sum in the setting of *compact and path-connected* surfaces, where it can be described algebraically using the formalism of planar presentations. The connected sum of surfaces that may be open and have boundary is discussed in Definition 15.27 and the exercises that follow it.

Definition 15.16. *Let P and P' be planar presentations of surfaces. Write ω and ω' for the corresponding cyclic words. The planar presentation $P\#P'$ described by the word $\omega'\omega$ is said to be the* ***connected sum*** *of P and P'.*

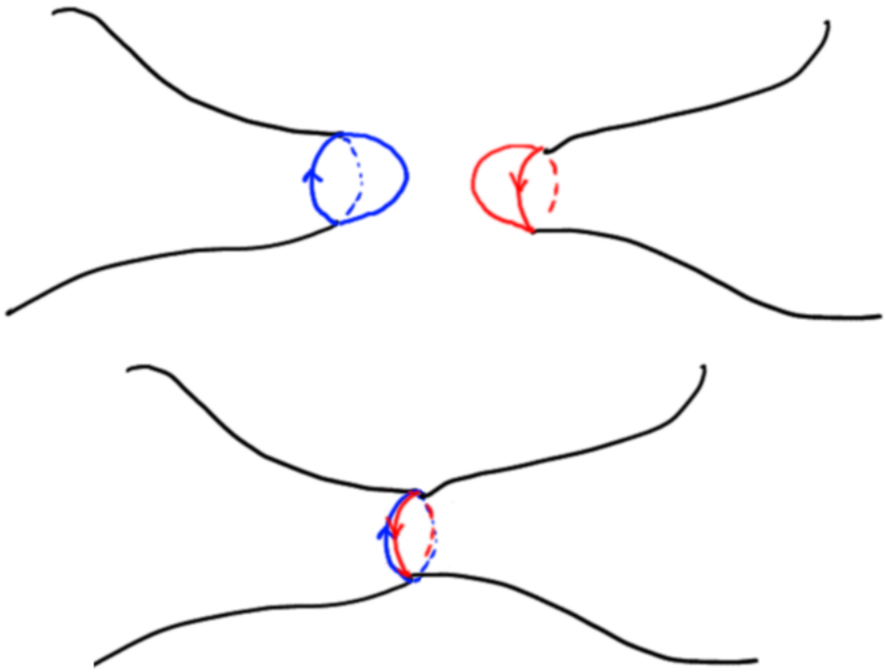


Figure 15.2: Two surfaces and two discs within. We remove the interior of the discs and glue the boundaries, yielding a new surface, the connected sum.

See Figure 15.3. Observe that the space underlying $P\#P'$ is a surface as well, since each label appears twice.

Even though the definition seems to depend on the choice of planar presentation, we have that:

Theorem 15.17. *Consider P, Q, P', and Q', planar presentations of surfaces. Suppose that P and P' relate via moves. Suppose similarly that Q and Q' relate via moves. Then $P\#Q$ relates to $P'\#Q'$.*

Proof. Given a connected sum $P\#Q$ and a move for P, we can apply it leaving Q as it was. The same applies to moves for Q. The claim follows. □

Which implies that we can define the connected sum of two surfaces S and T by taking planar presentations P and Q and defining $S\#T$ to be the space presented by $P\#Q$.

Using the standard planar presentations and the classification Theorem 15.8 we deduce:

Corollary 15.18. *$\Sigma_k\#\Sigma_{k'}$ is homeomorphic to $\Sigma_{k+k'}$.*

And similarly in the non-orientable case:

Corollary 15.19. *$N_k\#N_{k'}$ is homeomorphic to $N_{k+k'}$.*

The interesting case is when we sum an orientable surface and a non-orientable surface:

Corollary 15.20. *Let $k' > 0$. Then $\Sigma_k\#N_{k'}$ is homeomorphic to $N_{2k+k'}$.*

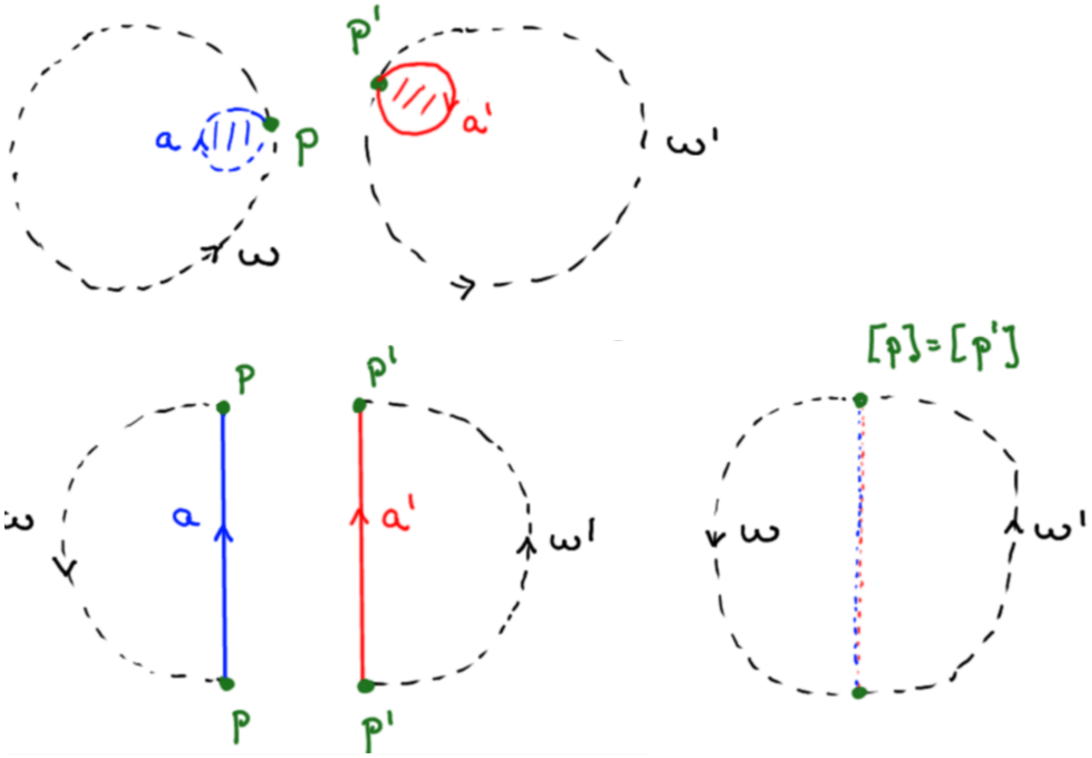


Figure 15.3: Two planar presentations (top left) and their connected sum (bottom right) at the vertices p and p'. Geometrically, we are removing two small discs, incident to the vertices, and identifying their boundaries a and a'.

Proof. $N_{2k+k'}$ is non-orientable and has Euler characteristic $2 - 2k - k'$, as seen using its standard presentation. The same is true about $\Sigma_k \# N_{k'}$. The result follows from Corollary 15.15. □

A particularly simple case reads:

Corollary 15.21. *Let Σ be a closed surface. Then $\Sigma \# \mathbb{S}^2$ is homeomorphic to Σ.*

This is the case since the sphere is represented by the planar presentation corresponding to the empty word.

15.3.2 The monoid of surfaces

We have thus deduced that the connected sum is an operation on surfaces. Recall that a **monoid** is a set endowed with an operation that is associative (but may have no identity nor inverses).

Corollary 15.22. *Consider the set $\mathcal{M}$ consisting of all compact, path-connected surfaces up to homeomorphism. Then, $(\mathcal{M}, \#)$ is a commutative monoid with identity.*

Proof. The sphere is the identity, according to Corollary 15.21. Commutativity follows from the fact that the presentations $P\#Q$ and $Q\#P$ are described by the same word, due to cyclicity. □

Much like for groups, we can discuss the generators of a monoid:

Corollary 15.23. *$(\mathcal{M}, \#)$ is generated by $\mathbb{RP}^2$ and $\mathbb{T}^2$. Moreover:*

- *N_g is the connected sum of g copies of $\mathbb{RP}^2$.*
- *Σ_g is the connected sum of g copies of $\mathbb{T}^2$.*

Proof. The two items follow from the description of the standard presentations and imply the claim about being generators. □

Having described the generators, we describe the relations:

Corollary 15.24. *$(\mathcal{M}, \#)$ is generated by $\mathbb{RP}^2$ and $\mathbb{T}^2$, with relation*

$$\mathbb{RP}^2 \# \mathbb{T}^2 \simeq \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2.$$

Proof. The claimed relation is the identity $\Sigma_1 \# N_1 \simeq N_3$, which was established in Corollary 15.20. That there are no further (independent) relations follows from the fact that we can now write every surface uniquely as $(\#_g \mathbb{T}^2) = \Sigma_g$, or $(\#_g \mathbb{T}^2) \# \mathbb{RP}^2 = N_{2g+1}$, or $(\#_g \mathbb{T}^2) \# \mathbb{RP}^2 \# \mathbb{RP}^2 = N_{2g+2}$, all of which are distinct. □

Corollary 15.25. *The closed orientable surfaces form a commutative monoid with identity isomorphic to $(\mathbb{Z}^{\geq}, +)$.*

Corollary 15.26. *The closed non-orientable surfaces form a commutative monoid without identity isomorphic to $(\mathbb{Z}^{>}, +)$.*

15.4 Exercises

15.4.1 Classification of surfaces

In the following exercises you should keep the following facts in mind:

- The classification of closed surfaces up to homeomorphism is exactly the same as up to homotopy equivalence.
- Planar presentations yield compact spaces. In particular, an open surface does not have a planar presentation.

Exercise 15.1: Are there non-homeomorphic closed surfaces A and B that become homotopy equivalent after removing a point?

Exercise 15.2: Are there non-homeomorphic closed surfaces A and B such that $A\#T^2$ and $B\#T^2$ are homeomorphic?

Exercise 15.3: Find two surfaces with boundary A and B that are not homeomorphic to each other but they are homotopy equivalent.

Exercise 15.4: Let S be the one of the planar representations shown in the Figure below.

- Is it a surface? Check that each point in S has a neighbourhood homeomorphic to a ball in the plane.
- Compute the Euler characteristic.
- Is it an orientable surface?
- Write down a group presentation for the fundamental group of S.
- Compute the first homologies.
- Determine all the g and g' such that S is homeomorphic to Σ_g or $N_{g'}$.

If you have the energy, relate it to the "standard" planar representation of S using moves.

15.4.2 Connected sum

The following was depicted in Figure 15.2.

Definition 15.27. *Let A and B be surfaces, not necessarily closed, possibly with boundary. We define $A\#B$ as follows. Find a closed disc $D_A \subset A$ and a closed disc $D_B \subset B$ both contained in the interior (i.e. disjoint from the boundary). Remove their interiors, and identify $\partial D_A \subset A$ with $\partial D_M \subset M$ via your favourite homeomorphism.*

Note: We have seen the connected sum $A\#B$ of compact surfaces, via planar representations, in Definition 15.16. You can use that the two notions are equivalent for compact surfaces; you do not need to prove it.

Exercise 15.5: Prove that $A\#B$ is a surface with boundary.

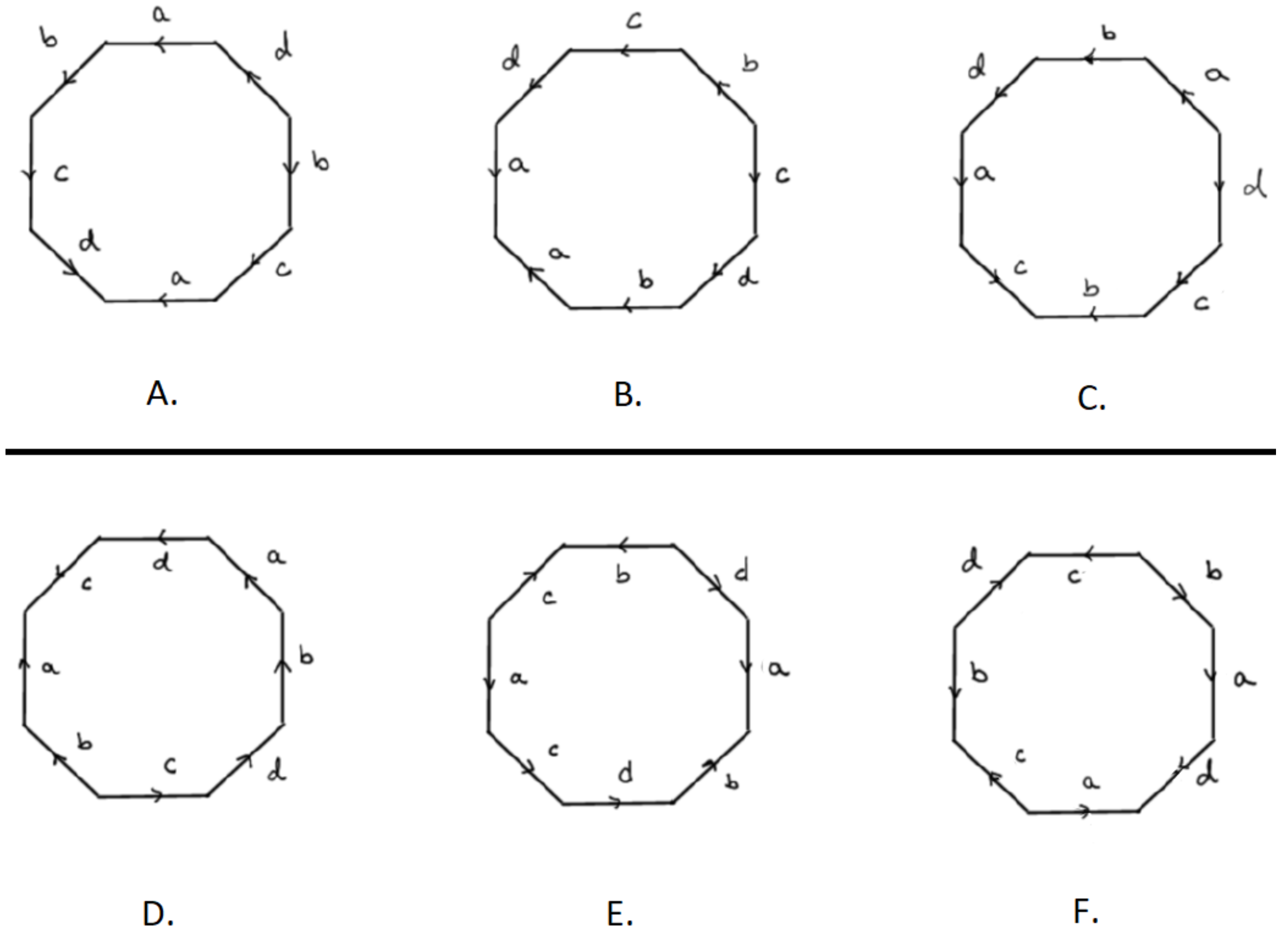


Figure 15.4: Planar representations for Exercise 15.4.

Exercise 15.6: Let M denote the closed Möbius band. Prove that there are non-homeomorphic closed surfaces A and B such that $A\#M$ and $B\#M$ are homeomorphic. **Hint:** Use Exercise 13.3.

Exercise 15.7: Let C denote the open cylinder. Prove that are there non-homeomorphic compact surfaces A and B such that $A\#C$ and $B\#C$ are homotopy equivalent.

Covering spaces

Lecture 16

Suppose we consider a pointed space (X, x) and we fix a subgroup $H \subset \pi_1(X, x)$. One can then pose the following question: Can we find a pointed space (Y, y), naturally associated to X, such that $\pi_1(Y, y) \simeq H$? In fact, can we do it so that there is a map $\pi : (Y, y) \to (X, x)$ such that

$$\pi_* : \pi_1(Y, y) \to \pi_1(X, x)$$

is an inclusion with image exactly H? A particularly important case is that of the trivial subgroup $\{e\}$. Then we are asking whether we can find a simply-connected space naturally associated to (X, x).

The final goal of the course is to prove that this is indeed the case, as long as X is nice enough. In this lecture we will:

- Define what covering spaces are (Section 16.1). Roughly speaking, they are spaces that locally resemble X and whose fundamental groups correspond naturally to subgroups of $\pi_1(X, x)$.
- Prove that covering spaces satisfy the so-called *unique homotopy lifting property* (Section 16.2). It says that homotopies in a covering space are in correspondence with homotopies in X. This has very strong consequences regarding the topology of covering spaces (Section 16.3).
- Associate a subgroup of $\pi_1(X, x)$ to each covering space (Proposition 16.36 and Corollary 16.39).

The correspondence between subgroups and covering spaces will be fully established in the next lecture.

16.1 Covering spaces

The idea behind the covering space $\pi : (Y, y) \to (X, x)$ associated to the subgroup H is that (Y, y) should resemble (X, x) at a local level, but Y has been "unwrapped" in order to replace

$\pi_1(X, x)$ by the subgroup H. When $H = \{e\}$, we are meant to unwrap completely, so Y is simply connected and every non-trivial loop in (X, x) corresponds to a path in (Y, y) that does not close up.

16.1.1 The definition

We need the following preliminary concept:

Definition 16.1. *Fix a map $\pi : Y \to X$. A subset $U \subset X$ is said to be* ***evenly covered*** *if there is a discrete space S and a homeomorphism ψ making the following diagram commute:*

$$\begin{array}{ccc} U \times S & \xrightarrow[\simeq]{\psi} & \pi^{-1}(U) \\ {\scriptstyle \pi_U}\downarrow & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow[\text{id}]{} & U \end{array}$$

Then:

Definition 16.2. *We say that $\pi : Y \to X$ is a* ***covering map*** *if every point in X has an evenly covered neighbourhood. We also say that Y is a* ***covering space*** *of the base X.*

That is, Y resembles X locally. Given an evenly covered subset $U \subset X$, its preimage looks like a bunch of copies $U \times S$ of U. Each copy $U \times \{s\}$, $s \in S$, is called a **sheet**. See Figure 16.1. Do note that we do not ask S to be non-empty, so Y could be the empty space.

A case that will be particularly important is:

Definition 16.3. *Fix a path-connected space X. A covering map $\pi : Y \to X$ is the* ***universal cover*** *of X if Y is simply-connected.*

We write "the" and not "a" universal cover, because we will see later (Proposition 17.12) that the universal cover (whenever it exists) is unique up to isomorphism. Moreover, we ask for path-connectedness because the universal cover will be path-connected.

16.1.2 Covering spaces of $\mathbb{S}^1$

As you read on, it is convenient that you keep the following example (the covering spaces of the circle) in mind at all times. See Figure 16.2.

Lemma 16.4. *The following is a covering map:*

$$\begin{array}{rcl} \pi : \mathbb{R} & \to & \mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z} \\ t & \mapsto & [t]. \end{array}$$

And it is in fact the universal cover of $\mathbb{S}^1$.

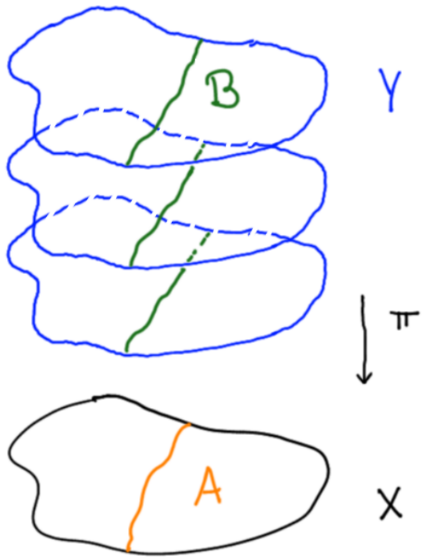


Figure 16.1: A covering map $\pi : Y \to X$. An evenly-covered open $U \subset X$ is shown, with three sheets from Y above. We also depict a subspace $A \subset X$ and its preimage $B \subset Y$, which is a covering space. The intersection $A \cap U$ is also evenly covered.

Proof. Any interval $(a, b) \subset \mathbb{R}$ of length less than 1 will project to an interval I in $\mathbb{S}^1$ whose preimage is the union

$$\coprod_{k \in \mathbb{Z}} (a + k, b + k),$$

meaning that I is evenly covered. The second claim follows from the fact that $\mathbb{R}$ is contractible and thus simply-connected. □

If we take complex coordinates in $\mathbb{S}^1$, we can write instead:

Lemma 16.5. *In complex coordinates, the universal cover of Lemma 16.4 reads:*

$$\begin{array}{rcl} \exp : \mathbb{R} & \to & \mathbb{S}^1 \subset \mathbb{C} \\ t & \mapsto & e^{2\pi i t}. \end{array}$$

We can now construct all other path-connected covering spaces of $\mathbb{S}^1$:

Lemma 16.6. *Fix a positive integer k. Then, the following map is a covering space with k sheets:*

$$\begin{array}{rcl} \pi_k : \mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z} & \to & \mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z} \\ [t] & \mapsto & [kt]. \end{array}$$

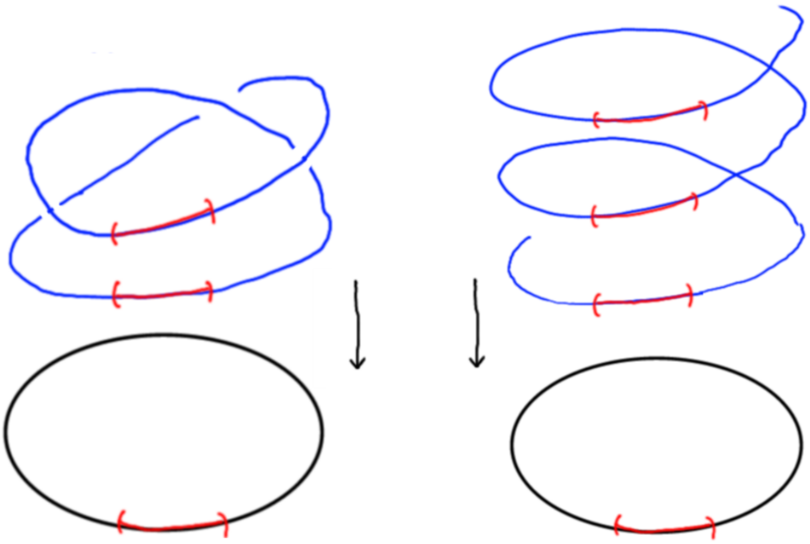


Figure 16.2: On the right $\mathbb{R}$ is shown as a spiral covering $\mathbb{S}^1$; it has infinitely many sheets. On the left, the circle covers itself with two sheets.

Alternatively, in complex coordinates, the map π_k can be expressed as:

$$\begin{aligned} f_k : \mathbb{S}^1 &\to \mathbb{S}^1 \subset \mathbb{C} \\ z &\mapsto z^k. \end{aligned}$$

Proof. Let $I \in \mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z}$ be an interval of length $h < 1/k$. Then its preimage under π_k consists of k disjoint intervals of length h/k, proving the evenly covered property. □

That is: it is better to speak of covering maps, and not covering spaces, since it is the map that matters (given that a given space can be covering for multiple maps).

Each of the covering maps f_k corresponds to a subgroup:

Corollary 16.7. *The pushforward*

$$(f_k)_* : \pi_1(\mathbb{S}^1, 1) \simeq \mathbb{Z} \to \pi_1(\mathbb{S}^1, 1)$$

sends the generator $a = [\mathrm{id}_{\mathbb{S}^1}]$ to a^k. In particular, the image of $(f_k)_*$ is the subgroup $k\mathbb{Z}$.

Observe that these are indeed all the subgroups of $\mathbb{Z}$. Moreover, note that $k\mathbb{Z}$ has index k (i.e. the cardinality of the quotient $\mathbb{Z}/k\mathbb{Z}$ is k), which is precisely the number of sheets of f_k. This is a general phenomenon (Proposition 16.34).

16.1.3 Basic properties of covering spaces

We now state various results, which basically amount to saying that Y and X look like one another when Y is a covering space. The following states that the covering property restricts nicely to subspaces:

Lemma 16.8. *Let $\pi : Y \to X$ be a covering map and $A \subset X$ a subspace. Then the restriction*

$$\pi : \pi^{-1}(A) \to A$$

is a covering map.

Proof. Given a point $p \in A$, take an evenly covered neighbourhood $U \subset X$. This means that $\pi^{-1}(U)$ is homeomorphic to $U \times S$, with S discrete. It follows that $\pi^{-1}(A \cap U)$ is homeomorphic to $(U \cap A) \times S$, so $U \cap A$ is an evenly covered neighbourhood of p in A. □

This implies that:

Corollary 16.9. *Let $\pi : Y \to X$ be a covering map. Then the evenly-covered opens in X form a basis of the topology.*

Proof. Given a point $x \in X$, take an evenly covered neighbourhood $U \subset X$. Every neighbourhood of x contained in U is evenly covered, so all of them form a basis at x. □

Similarly:

Corollary 16.10. *Let $\pi : Y \to X$ be a covering map. Then the sheets in Y over evenly-covered opens of X form a basis of the topology of Y.*

Proof. Given a point $y \in Y$, take an evenly covered neighbourhood $U \subset X$ of $\pi(y)$. Then we can take the sheet $U' \subset Y$ of U that contains y. Every neighbourhood V of y contained in U' is itself a sheet over its projection $\pi(V)$, so all of them together form a basis of the topology at y. □

Which means that:

Corollary 16.11. *Let $\pi : Y \to X$ be a covering map. Then, it is an open map (it takes opens to opens).*

Proof. It is enough to show that it maps basis elements of the topology in Y to opens in X. This is clear, since a sheet V over an open U is mapped to U. □

The covering property also interacts nicely with Hausdorffness:

Lemma 16.12. *Let $\pi : Y \to X$ be a covering space. Then Y is Hausdorff if and only if X is Hausdorff.*

Proof. Suppose X is Hausdorff. Let y and y' be two distinct points in Y. We must show that a pair of disjoint neighbourhoods exists. There are two cases. The first is that y and y' project to the same point $x = \pi(y) = \pi(y')$. Then we take U an evenly-covered neighbourhood of x and we let V and V' be the sheets over it passing via y and y', respectively. The evenly covered condition says that $\pi|_V : V \to U$ and $\pi|_{V'} : V' \to U$ are homeomorphisms, and from this it follows that V and V' must be disjoint, since y and y' are distinct. The second case is that $x = \pi(y)$ and $x' = \pi(x')$ are distinct. Then they have disjoint neighbourhoods U and U', so their preimages are disjoint.

Assume now that Y is Hausdorff. Then, given x and x' in X, we can pick arbitrary preimages y and y', find disjoint neighbourhoods V and V' that are also sheets, and deduce that $\pi(V)$ and $\pi(V')$ are disjoint neighbourhods of x and x' in X. □

Second countability is also compatible with the covering property:

Lemma 16.13. *Let $\pi : Y \to X$ be a covering space with countably many sheets. Then X is second countable if and only if Y is second countable.*

Proof. If X has a countable basis, it has a countable basis consisting of evenly-covered opens. Each such open has countably many preimages. The product of two countable sets is countable, so the basis of Y consisting of sheets is also countable. For the converse, just project the basis of sheets to the basis of evenly-covered opens. □

Observe that the exact same reasoning applies to first countability.

Putting the previous facts together:

Lemma 16.14. *Let $\pi : Y \to X$ be a covering space with countably many sheets. Then Y is a manifold if and only if X is a manifold.*

Proof. X and Y are locally homeomorphic, so one is locally euclidean if and only if the other one is. Hausdorffness and second countability we addressed above. □

In particular:

Corollary 16.15. *The covering spaces of a surface are also surfaces.*

The following result is the main tool to be used to construct covering spaces in practice. It says that we can construct covering spaces cell by cell, when the base is itself a cell complex.

Proposition 16.16. *Let X be a CW complex and let $\pi : Y \to X$ be a covering space. Then, there is a unique cell structure on Y compatible with π. Concretely, this means that for each characteristic map $\tilde{\psi} : \mathbb{D}^n \to Y$ of a cell, the projected map $\psi := \pi \circ \tilde{\psi} : \mathbb{D}^n \to X$ should also be a characteristic map.*

Proof. The desired conclusion forces us to define the n-th skeleton Y_n as the preimage of X_n. We will now explain how Y_n is obtained from Y_{n-1} by attaching n-cells, but we will not be able to complete the argument rigorously, since it needs the results of Section 17.1. The general idea is that, given a cell $\psi : \mathbb{D}^n \to X$ passing through $x \in X_{n-1}$ and a point $y \in \pi^{-1}(x)$, there is a unique cell $\tilde{\psi} : \mathbb{D}^n \to Y$ passing through y and satisfying $\pi \circ \tilde{\psi} = \psi$. This means that $\tilde{\psi}$ is a *lift* of ψ. This concept is introduced in Definition 16.17 below and most of covering space theory boils down to understanding it well. The reader should read ahead and come back once they have understood the prerequisites.

Here is thus the proof. The existence of a unique lift $\tilde{\psi}$ passing through y is a consequence of the lifting criterion (Theorem 17.1), using the fact that $\mathbb{D}^n$ is simply-connected. Since the restriction of ψ to the boundary is the attaching map, which takes values in X_{n-1}, it follows that $\tilde{\psi}$ restricted to the boundary takes values in Y_{n-1}. With this, we have exhibited Y_n as a union of cells.

It remains to show that Y indeed has the quotient topology inherited from the disjoint union of all cells, up to the identifications given by the attaching. This we can check locally: given a point $y \in Y$, we pick a small open U intersecting only cells incident to y. We can moreover assume that U is one of the sheets over the evenly covered $\pi(U)$. Since $\pi(U)$ is obtained from the (parts of) cells it intersects by quotienting, the same is true for U. □

16.2 The homotopy lifting property

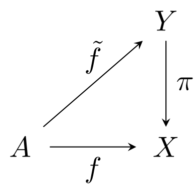
The theory of covering spaces is powered by the fact that covering spaces satisfy the so-called *unique homotopy lifting property* (UHLP). Recall that, given any map, we can always use it to pushforward paths, loops, and homotopies thereof. The UHLP will allow us to go the other way around and lift paths and homotopies from the base (X, x) to the covering space (Y, y). This has very strong consequences at the level of fundamental group (and in fact also for higher homotopy groups); see Section 17.4.

16.2.1 Lifts

First we introduce:

Definition 16.17. *Fix a map $\pi : Y \to X$. A map $\tilde{f} : A \to Y$ is a **lift** of $f : A \to X$ with respect to π if $f = \pi \circ \tilde{f}$.*

Identically, the three maps fit into the commutative diagram:



We are mostly interested in the pointed case:

Definition 16.18. *Fix a map $\pi : (Y, y) \to (X, x)$. A map $\tilde{f} : (A, a) \to (Y, y)$ is a **lift** of $f : (A, a) \to (X, x)$ with respect to π if $f = \pi \circ \tilde{f}$.*

The computation $[f] = [\pi \circ \tilde{f}] = [\pi] \circ [\tilde{f}] = \pi_*([\tilde{f}])$ proves that:

Lemma 16.19. *Suppose $f : A \to X$ admits a lift $\tilde{f}$ with respect to $\pi : Y \to X$. Then $[f] \in [A, X]$ is the image of $[\tilde{f}] \in [A, Y]$ via $\pi_* : [A, Y] \to [A, X]$.*

Which explains the relevance of lifts: they allow us to relate the "holes" in X to the "holes" in Y. In the concrete case of the fundamental group:

Corollary 16.20. *Suppose $\gamma : (\mathbb{S}^1, 1) \to (X, x)$ admits a lift $\tilde{\gamma} : (\mathbb{S}^1, 1) \to (Y, y)$ with respect to $\pi : (Y, y) \to (X, x)$. Then $[\gamma] \in \pi_1(X, x)$ is the image of $[\tilde{\gamma}] \in \pi_1(Y, y)$ via $\pi_* : \pi_1(Y, y) \to \pi_1(X, x)$.*

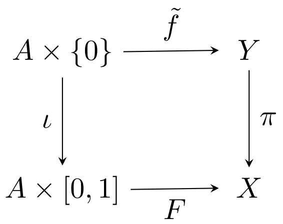
16.2.2 The homotopy lifting property

Suppose f and f' are homotopic as maps $A \to X$, and f has a lift $\tilde{f}$, as above. Then $[f'] = [f] = \pi_*([\tilde{f}])$. This does not necessarily mean that f' itself has a lift. For instance:

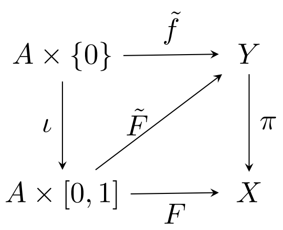
Example 16.21: Let A and Y be the point and X be $\mathbb{R}$. We let f and π be the map with image $0 \in \mathbb{R}$. Then f lifts to the unique map $A \to Y$. Any other map f', with image different from 0, is homotopic to f, but has no lift. △

However, some maps $\pi : Y \to X$ guarantee that f' has a lift if f does. This deserves a name:

Definition 16.22. *A map $\pi : Y \to X$ satisfies the **homotopy lifting property** if any commutative diagram*



extends to



That is: we are given a homotopy F of A into X and a lift $\tilde{f}$ of its starting map $f = F(-, 0)$. Then, the property says that we are able to find a homotopy $\tilde{F}$ of $\tilde{f}$, lifting F.

We will be interested in the stronger property:

Definition 16.23. *A map $\pi : Y \to X$ satisfies the **unique homotopy lifting property** if the homotopy lifting property holds and, moreover, the lift $\tilde{F}$ is unique.*

A concrete instance of the HLP that will be of interest to us is when A is simply a point.

Definition 16.24. *A map $\pi : Y \to X$ satisfies the **path lifting property** if, given a path $\gamma : I = [0, 1] \to X$ and a lift $\tilde{\gamma}(0)$ of its initial point $\gamma(0)$, we can always find a lift $\tilde{\gamma}$ completing the diagram:*

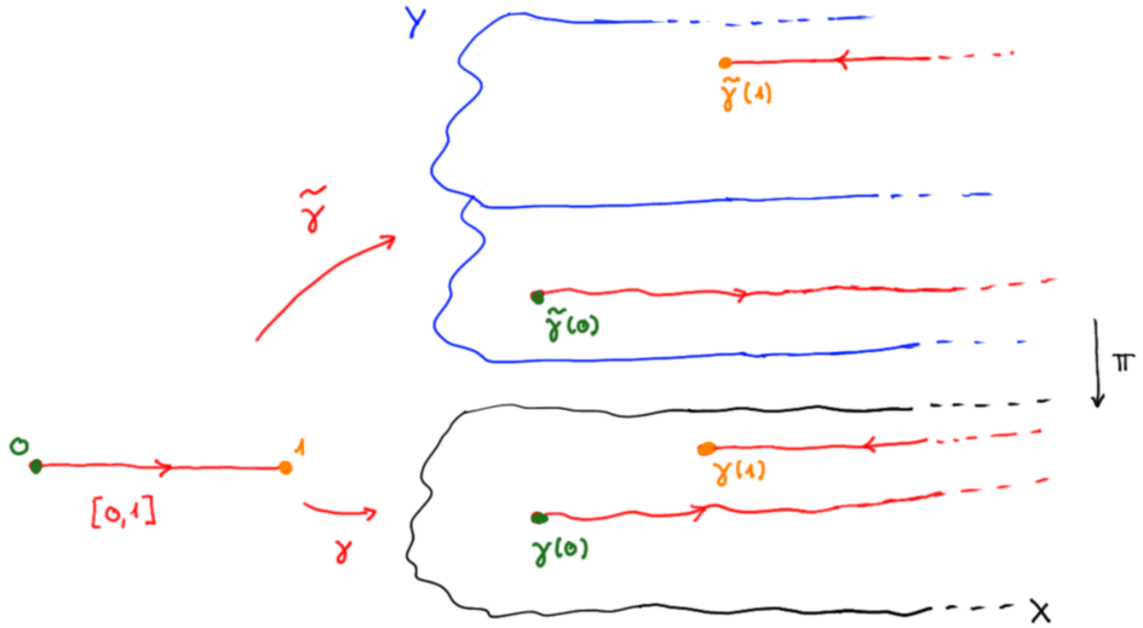
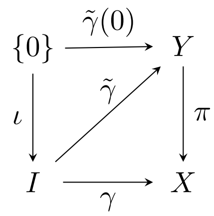


Figure 16.3: A covering space $\pi : Y \to X$, a path $\gamma : [0,1] \to X$, and a lift $\tilde{\gamma} : [0,1] \to Y$. An evenly-covered open of X is shown, with two sheets on top. The curve γ exits the evenly-covered open, which allows its lift to switch from one sheet to the other.



See Figure 16.3.

16.3 UHLP for covering spaces

The key result in the study of covering spaces reads:

Theorem 16.25. *Covering spaces satisfy the unique homotopy lifting property (UHLP).*

Idea of the proof. One first shows that a path γ can be lifted uniquely, once we choose a lift $\tilde{\gamma}(0)$ of its initial point $\gamma(0)$. The main insight is that this is true locally. Concretely: According to the Lebesgue covering lemma, the domain of γ can be divided into intervals I_i so that $\gamma|_{I_i}$ takes values in an evenly-covered open U_i. We can thus consider $U_0 \ni \gamma(0)$ and the sheet U_0' over it containing $\tilde{\gamma}(0)$. We then use the homeomorphism ϕ between U_0 and U_0' to define $\tilde{\gamma}|_{I_0} = \phi \circ \gamma|_{I_0}$. Observe that this is the unique way to define a continuous lift, since all other preimages of $\gamma|_{I_0}$ live in other sheets, which are disconnected from $\tilde{\gamma}(0)$. One then continues by induction on i.

For the general case, where we have a homotopy F and a lift $\tilde{f}$ of $f = F(-,0)$, one argues similarly. The idea is that one can use the compactness of $[0,1]$ to cover the domain A with little opens V so that $F|_{V\times[0,1]}$, which we can think of as a family of paths parametrised by V, can be lifted at once, using our earlier reasoning. Uniqueness implies that all these local lifts agree on overlaps. □

The interested reader may want to refer to [Hat02] for extra details.

16.3.1 Uniqueness of lifts

We now explore some consequences of Theorem 16.25, particularly at the level of uniqueness:

Lemma 16.26. *Suppose $\pi : (Y,y) \to (X,x)$ is a covering space. Let A be path-connected. Then, any two lifts $h, h' : (A,a) \to (Y,y)$ of $f : (A,a) \to (X,x)$ agree.*

Proof. Since A is path-connected, given any point b we can find a path $\gamma : [0,1] \to A$ that connects a to b. Then we observe that $h \circ \gamma$ and $h' \circ \gamma$ are both lifts of $f \circ \gamma$. Since π satisfies the unique path lifting property, we deduce that $h \circ \gamma = h' \circ \gamma$. In particular, $h(b) = h \circ \gamma(1) = h' \circ \gamma(1) = h'(b)$. Since b was arbitrary, the claim follows. □

A concrete case is:

Corollary 16.27. *Suppose $\pi : (Y,y) \to (X,x)$ is a covering space. Let A be path-connected and $f : (A,a) \to (X,x)$ be the constant map. Then the unique lift of f is the constant map with image y.*

16.3.2 Holonomy

The UHLP implies that there is a close relation between paths in X and points/sheets in the covering space Y. Namely, given a loop $\gamma : [0,1] \to X$ and a lift $\tilde{\gamma}$, it may be the case that $\tilde{\gamma}(1) \neq \tilde{\gamma}(0)$ That is: the lifted path changes sheet and does not close up anymore. This is precisely a measure of how the fundamental group of X is becoming unwrapped in the covering space Y. We study this using the following construction:

Definition 16.28. *Let $\pi : Y \to X$ be a covering map and $\gamma : [0,1] \to X$ a path from a to b. The **holonomy** of π along γ is the map:*

$$\mathrm{hol}_\gamma : \pi^{-1}(a) \to \pi^{-1}(b)$$

defined by $z \mapsto \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the unique lift of γ starting at $z \in \pi^{-1}(a)$.

We will now prove that the holonomy satisfies some pretty strong algebraic properties.

Using Corollary 16.27 (constant paths lift to constant paths) we first deduce:

Corollary 16.29. *The holonomy* hol_{c_a} *of a constant path is the identity.*

Moreover:

Lemma 16.30. *If* γ *and* ν *are homotopic relative endpoints,* $\mathrm{hol}_\gamma = \mathrm{hol}_\nu$*.*

Proof. Let a and b be the endpoints of γ and ν. Fix a point $z \in \pi^{-1}(a)$. By assumption we have a homotopy relative endpoints Γ between γ and ν. The unique homotopy lifting property yields unique lifts $\tilde{\gamma}, \tilde{\nu} : [0,1] \to Y$ starting at z. The UHLP also lifts Γ to a homotopy $\tilde{\Gamma} : [0,1] \times [0,1] \to Y$ starting at $\tilde{\gamma}$.

Since $\tilde{\Gamma}(-,1)$ starts at z and is a lift of ν, it must be $\tilde{\nu}$, by uniqueness of the lift. Moreover, since $\Gamma(1,-)$ and $\Gamma(0,-)$ are constant (because Γ is a homotopy relative endpoints), their lifts are also constant (Corollary 16.27). This implies that $\tilde{\Gamma}$ is also a homotopy relative endpoints. In particular, the endpoints of $\tilde{\gamma}$ and $\tilde{\nu}$ are the same. □

We have thus shown that hol_γ only depends on $[\gamma]$. This means that the holonomy is best understood through the fundamental groupoid $\Pi_1(X)$. Now we show the concatenation in $\Pi_1(X)$ is compatible with the holonomy:

Lemma 16.31. *Let* $[\gamma] \in \pi_1(X,a,b)$ *and* $[\nu] \in \pi_1(X,b,c)$*. Then*

$$\mathrm{hol}_{\nu\cdot\gamma} = \mathrm{hol}_\nu \circ \mathrm{hol}_\gamma : \pi^{-1}(a) \to \pi^{-1}(c).$$

Proof. Suppose $z \in \pi^{-1}(a)$. Let $\tilde{\gamma}$ be the lift of γ starting at a. By definition, it finishes at

$$w := \tilde{\gamma}(1) = \mathrm{hol}_\gamma(z) \in \pi^{-1}(b).$$

We can then take the lift $\tilde{\nu}$ of ν, starting at w. It ends up in

$$\tilde{\nu}(1) = \mathrm{hol}_\nu(w) = \mathrm{hol}_\nu \circ \mathrm{hol}_\gamma(z) \in \pi^{-1}(c).$$

The conclusion follows from the fact that $\tilde{\nu} \cdot \tilde{\gamma}$ is a lift (and thus the unique lift) of $\nu \cdot \gamma$ starting at z, so:

$$\mathrm{hol}_{\nu\cdot\gamma}(z) = (\tilde{\nu} \cdot \tilde{\gamma})(1) = \tilde{\nu}(1) = \mathrm{hol}_\nu \circ \mathrm{hol}_\gamma(z).$$

□

In particular:

Corollary 16.32. hol_γ *is the inverse of* $\mathrm{hol}_{\bar{\gamma}}$*. In particular, the two are bijections.*

Proof. Since $[\gamma]$ and $[\bar{\gamma}]$ are inverses we have that $\mathrm{hol}_{\bar{\gamma}} \circ \mathrm{hol}_\gamma = \mathrm{hol}_{c_a}$, which is the identity. The same holds for $\mathrm{hol}_\gamma \circ \mathrm{hol}_{\bar{\gamma}}$. □

All these properties can be summarised as:

Theorem 16.33. *Let* $\pi : Y \to X$ *be a covering space. There is a functor*

$$\mathrm{hol} : \Pi_1(X) \to \mathrm{Set}$$

that sends:

- *A point* $x \in X$ *to its fibre* $\pi^{-1}(x)$*.*
- *A class* $[\gamma] \in \pi_1(X,x,x')$ *to the bijection* hol_γ*.*

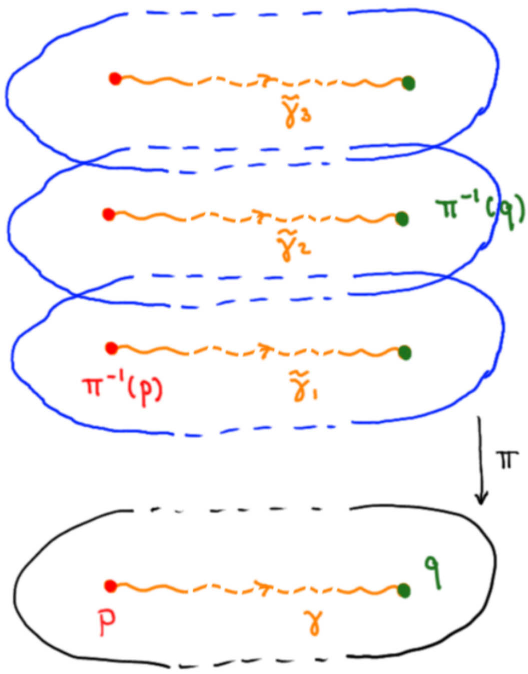


Figure 16.4: A covering space $\pi : Y \to X$. In order to relate the number of sheets over the points q and p we pick a path $\gamma : [0, 1] \to X$ from p to q and consider all its lifts. Each lift identifies a pair of points.

16.3.3 Sheets

The following result states that the "number of sheets" is independent of the evenly covered open we consider, as long as X is path-connected.

Proposition 16.34. *Let $\pi : Y \to X$ be a covering map, with X path-connected. Then, the cardinality of $\pi^{-1}(a)$ is independent of the point $a \in X$.*

Proof. Suppose a and b are points in X and consider a path γ from a to b. Then hol_γ is the required bijection between their fibres. The idea of the proof is shown in Figure 16.4. □

A very useful consequence is then:

Corollary 16.35. *Let $\pi : Y \to X$ be a covering space with k sheets. Suppose X is a cell complex. Then each cell in X lifts to k distinct cells in Y.*

Do note that this result relies on Proposition 16.16, which itself relies on Theorem 17.1, which we have not yet proven.

16.4 First steps towards the Galois correspondence

We now explore how holonomy can be exploited to associate to each covering space Y a subgroup $H \subset \pi_1(X, x)$. This will be accomplished fully in the next chapter.

16.4.1 The pushforward of a covering map is injective

First we observe that the fundamental group of a covering space sits naturally as a subgroup of the fundamental group of the base:

Proposition 16.36. *Let $\pi : (Y, y) \to (X, x)$ be a covering space. Then*

$$\pi_* : \pi_1(Y, y) \to \pi_1(X, x)$$

is injective.

Proof. Consider a loop $\tilde{\gamma} : (\mathbb{S}^1, 1) \to (Y, y)$ and write $\gamma := \pi \circ \tilde{\gamma}$ for its projection. By construction $\tilde{\gamma}$ lifts γ. The class $[\tilde{\gamma}]$ being in the kernel of π_* means that that $\pi_*[\tilde{\gamma}] = [\gamma]$ is the trivial class $[c_x]$ in $\pi_1(X, x)$.

Let Γ be a homotopy relative endpoints, between γ and c_x. According to the homotopy lifting property, Γ can be lifted to a homotopy $\tilde{\Gamma} : [0, 1] \times [0, 1] \to Y$ with $\tilde{\Gamma}(-, 0) = \tilde{\gamma}$. Using Lemma 16.30 we moreover see that $\tilde{\Gamma}$ is relative to endpoints as well. Lastly, we observe that $\tilde{\Gamma}(-, 1)$ is the constant map with value y, since $\Gamma(-, 1)$ was the constant loop at x. I.e. we have shown that $[\tilde{\gamma}] = [c_y]$, and thus shown that the kernel of π_* is trivial. □

16.4.2 Identification with a quotient of the source fibre

The relationship between π and $\pi_*(\pi_1(Y, y))$ allows us to relate (Y, y) with $\Pi_1(X)$ and therefore to think of (Y, y) in rather algebraic terms, as we now explain.

Recall the following notions: we write $\mathfrak{s} : \Pi_1(X) \to X$ for the source map and $\mathfrak{t} : \Pi_1(X) \to X$ for the target map. The former takes the class of a path $[\gamma]$ to its initial point $\gamma(0)$, the latter takes $[\gamma]$ to the final point $\gamma(1)$. This allows us to write $\mathfrak{s}^{-1}(x) \subset \Pi_1(X)$, the subset of homotopy classes of paths that begin at x.

Definition 16.37. *Let $\pi : (Y, y) \to (X, x)$ be a pointed covering space. We define a map of sets:*

$$\mathrm{hol}^y : \mathfrak{s}^{-1}(x) \to Y$$

by setting $\mathrm{hol}^y([\gamma]) := \mathrm{hol}_\gamma(y) \in \pi^{-1}(\gamma(1))$.

This is well-defined by Lemma 16.30. See Figure 16.5.

Lemma 16.38. *Let $\pi : (Y, y) \to (X, x)$ be a pointed covering space, with X and Y path-connected. Then, the following statements hold:*

- *The function* hol^y *is surjective.*

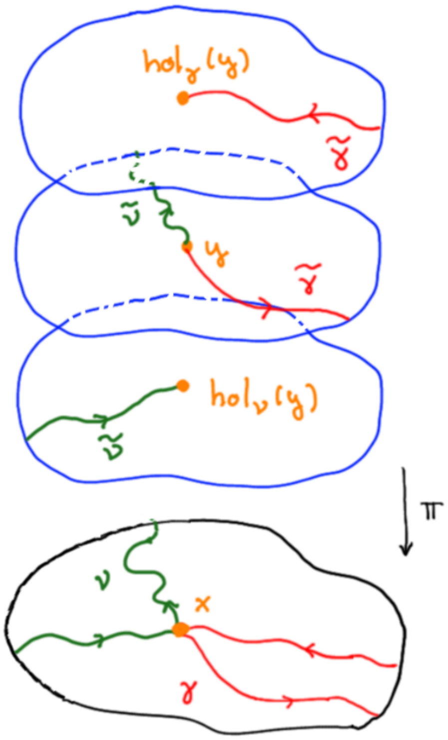


Figure 16.5: A pointed covering space $\pi : (Y, y) \to (X, x)$. Given loops ν and γ based at x, we lift them with starting point y. Their lifts $\tilde{\nu}$ and $\tilde{\gamma}$ finish at other points over x, which we denote $\mathrm{hol}_\nu(y)$ and $\mathrm{hol}_\gamma(y)$.

- *Its restriction to $\pi_1(X, x, a)$ is surjective onto the fibre $\pi^{-1}(a)$.*
- *Write $H \subset \pi_1(X, x)$ for the preimage of x via hol^y. Then, H is a subgroup.*
- *Two elements $[\gamma], [\nu] \in \pi_1(X, x, a)$ have the same image if and only if $[\bar{\gamma}][\nu] \in H$.*

Proof. For the first item, pick a point $z \in Y$. Since Y is path-connected, there is a path $\tilde{\gamma} : [0, 1] \to Y$ from y to z. It lifts its projection $\gamma := \pi \circ \tilde{\gamma}$, which is a path starting at x and finishing at some $a := \pi(z)$. It follows that $\mathrm{hol}^y([\gamma]) = z$. The second item is an immediate consequence.

For the third item, we must show that H contains the identity $[c_x]$, inverses, and compositions. To show $\mathrm{hol}^y([c_x]) = y$ we just observe that c_x lifts to the constant path at y. Regarding inverses, we assume $[\alpha] \in H$, meaning that $\mathrm{hol}^y([\alpha]) = y$. Then we use Lemma 16.31 and compute:

$$\begin{aligned} y = \mathrm{hol}^y([c_x]) &= \mathrm{hol}^y([\bar{\alpha}][\alpha]) = \mathrm{hol}_{\bar{\alpha}\cdot\alpha}(y) \\ &= \mathrm{hol}_{\bar{\alpha}} \circ \mathrm{hol}_{\alpha}(y) = \mathrm{hol}_{\bar{\alpha}}(y), \end{aligned}$$

which shows that $[\bar{\alpha}] \in H$. The proof for compositions is similar, if we assume $[\alpha], [\beta] \in H$ we get:

$$\mathrm{hol}^y([\beta][\alpha]) = \mathrm{hol}_{\beta} \circ \mathrm{hol}_{\alpha}(y) = \mathrm{hol}_{\beta} \circ \mathrm{hol}_{\alpha}(y) = \mathrm{hol}_{\beta}(y) = y,$$

which shows $[\beta][\alpha] \in H$.

For the last claim denote $z = \mathrm{hol}^y([\gamma])$. For the if direction we write:

$$\mathrm{hol}^y([\nu]) = \mathrm{hol}^y([\gamma][\bar{\gamma}][\nu]) = \mathrm{hol}_{\gamma} \circ \mathrm{hol}_{\bar{\gamma}\cdot\nu}(y) = \mathrm{hol}_{\gamma}(y) = \mathrm{hol}^y([\gamma]).$$

For the only if:

$$\mathrm{hol}^y([\bar{\gamma}][\nu]) = \mathrm{hol}_{\bar{\gamma}} \circ \mathrm{hol}_{\nu}(y) = \mathrm{hol}_{\bar{\gamma}}(z) = y.$$

□

And from these statements we obtain much more: the function hol^y is identifying Y with a quotient of the source fibre $\mathfrak{s}^{-1}(x)$. I.e. we can see elements in Y as equivalence classes of homotopy classes of paths. Consider the equivalence relation $[\gamma] \simeq [\nu] \in \mathfrak{s}^{-1}(x)$ if $[\bar{\gamma}][\nu] \in H$. Then:

Corollary 16.39. *Let $\pi : (Y, y) \to (X, x)$ be a pointed covering space, with X and Y path-connected. Write $H \subset \pi_1(X, x)$ for the preimage of x via hol^y. Then:*

- *$\mathrm{hol}^y : \mathfrak{s}^{-1}(x)/H \to Y$ is a bijection.*
- *H is the the image of $\pi_1(Y, y)$ via π_*.*

Proof. The first claim is immediate from the definition of H (see items (3) and (4) in Lemma 16.38).

For the second claim, we first show that H is contained in the image. Take then a class $[\gamma] \in H$ and let $\tilde{\gamma} : [0, 1] \to Y$ be its lift starting at y. By definition $y = \mathrm{hol}^y([\gamma]) = \tilde{\gamma}(1)$, meaning that $\tilde{\gamma}$ is a loop and thus $[\tilde{\gamma}] \in \pi_1(Y, y)$. It follows that $[\gamma] = \pi_*[\tilde{\gamma}]$.

For the converse inclusion, take some $[\tilde{\gamma}] \in \pi_1(Y, y)$ and let $[\gamma] := \pi_*[\tilde{\gamma}] \in \pi_1(X, x)$ be its projection. It follows that $\mathrm{hol}^y([\gamma]) = y$, so $[\gamma] \in H$. □

A particularly important case is then:

Corollary 16.40. *Let $\pi : (Y, y) \to (X, x)$ be the universal cover. Then the map*

$$\mathrm{hol}^y : \mathfrak{s}^{-1}(x) \to Y$$

is a bijection.

That is, the map hol^y allows us to transfer the topology in Y to $\mathfrak{s}^{-1}(x)$. This explains how to construct covering spaces in the first place: We must put a topology in $\mathfrak{s}^{-1}(x)$ so the target map $\mathfrak{t} : \mathfrak{s}^{-1}(x) \to X$ becomes a covering map. Then $\mathfrak{s}^{-1}(x)$ will be the universal cover. Moreover, the quotient $\mathfrak{t} : \mathfrak{s}^{-1}(x)/H \to X$ will be the covering space associated to the subgroup H. This is explained in Section 17.3.

A useful consequence of the discussion is that:

Corollary 16.41. *Let $H = \pi_*(\pi_1(Y, y))$ be the subgroup associated to the pointed covering space $\pi : (Y, y) \to (X, x)$, with X path-connected. Then the cardinality of the fibre $\pi^{-1}(x)$ is the index of H in $\pi_1(X, x)$.*

Proof. Recall that the index is the cardinality of $\pi_1(X, x)/H$. The claim is then immediate from Corollary 16.39. □

16.5 Worked-out example

Cell structures and covering space theory can be combined to study (and in fact compute) the fundamental group. We now explain how this is done in practice. We also recommend that the reader takes a look at Theorems 17.26 and 17.27, where these techniques are used to compute (again) the fundamental groups of the spheres.

16.5.1 A 2-dimensional cell complex

Fix a positive integer k and recall the cell complex X given by the planar presentation with cyclic word a^k. I.e. X consists of a single vertex x, a single a, and a single face F attached according to a^k. In particular $\pi_1(X, x) = \langle a \mid a^k \rangle \simeq \mathbb{Z}/k\mathbb{Z}$.

We will now construct the universal cover $\pi : (Y, y) \to\to (X, x)$. Observe that it does exist because X is path connected and, as a cell complex, is also locally contractible and thus locally simply-connected. We will exploit the fact that Y must be isomorphic to $(\mathfrak{s}^{-1}(x), [c_x])$. See Figure 16.6 for a pictorial depiction of all the spaces involved.

Determining how many cells we need

On the one hand, Y inherits a cell complex structure from X (Proposition 16.16). In particular, its zeroeth skeleton Y_0 is the preimage $\pi^{-1}(x)$ of the zeroeth skeleton of X_0, which

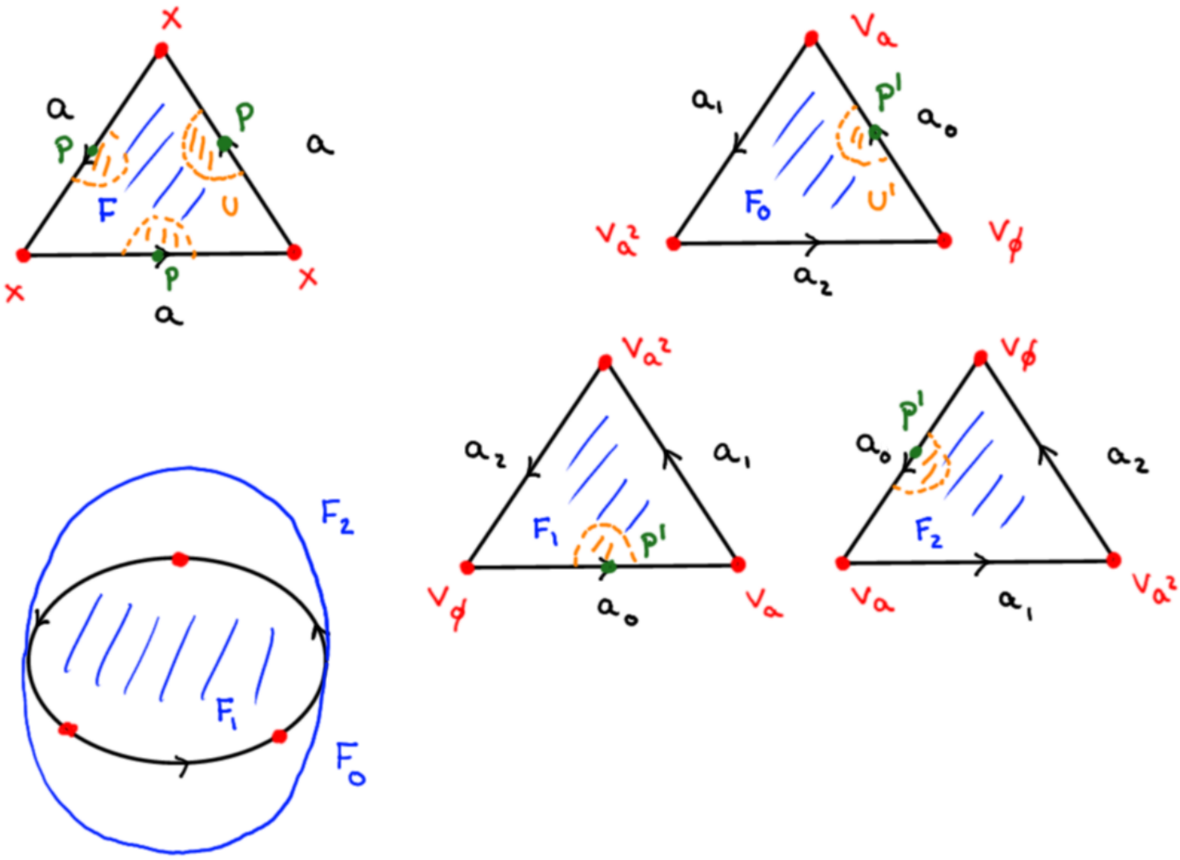


Figure 16.6: On the top left, the space X with $k = 3$. A point p in the edge is shown, together with a neighbourhood U which is seen not to be euclidean. On the right, the universal cover Y, exhibited by glueing three faces. A preimage p' of p is shown, together with a neighbourhood U' that is a sheet over U. On the bottom left, a schematic picture of Y glued together; it is homeomorphic to a sphere with an extra disc glued along the equator.

in this case is just the point x. On the other hand, since $Y \simeq \mathfrak{s}^{-1}(x)$, we know that $Y_0 = \pi^{-1}(x) \simeq \pi_1(X, x) \subset \mathfrak{s}^{-1}(x)$. Since $\pi_1(X, x) = \mathbb{Z}/k\mathbb{Z}$, it follows that Y_0 has k vertices, and it is natural to denote them by $v_\emptyset, v_a, v_{a^2}, \cdots, v_{a^{k-1}}$.

Moreover, since X is path-connected it follows that Y has k sheets, which in turn implies that we must be able to construct it by attaching k edges and k faces.

Determining how they are attached

Recall that each edge in Y is a lift of a. Moreover, using holonomy and the identification $Y \simeq \mathfrak{s}^{-1}(x)$, we see that the lift of a with starting point v_{a^i} must finish in $v_{a^{i+1}}$. It follows that we have to attach an edge a_i between v_{a^i} and $v_{a^{i+1}}$, for each $i = 0, \cdots, k-1$. Therefore, Y_1 is homeomorphic to $\mathbb{S}^1$, assembled from k edges glued cyclically. Its fundamental group is thus $\mathbb{Z}$, presented as $\langle (a_{k-1} \cdots a_0) \mid \rangle$, with a single generator.

With the same reasoning, each face in Y is a lift of F. Since F was attached along a^k, each lifted face F_i must be attached along k consecutive lifts of a. In this case, the only option is $a_{k-1} \cdots a_0$. I.e. all faces have the same attachment and the relation they introduce in π_1 is exactly the generator. We deduce that Y is simply-connected (this is an important reality check, since we are constructing the universal cover).

Defining the covering map

Now we define $\pi : Y \to X$. We map the vertices to x, and we therefore get a covering map $Y_0 \to X_0$. Each edge a_i is parameterised by $[-1, 1]$ and so is a. These identifications provide for us a unique map from a_i to a thus. This unique map sends endpoints to endpoints, so we can apply the gluing lemma and deduce that we have a continuous map $Y_1 \to X_1$.

Lastly, we address the faces. Each F_i is a copy of F, which is a polygon with k-sides. The parametrisation of F_i should be obtained from that of F as follows: we pick a point v_{a^i} over x and we let F_i be the lift of F passing through v_{a^i}. It follows that we must draw the F_i as copies of the polygon F, so the map between the two is the "identity". To label appropriately the sides of F_i, we pick one vertex in F and we label its preimage in F_i as v_{a^i}. Then we proceed counterclockwise, labelling the edges cyclically starting with a_i and finishing with a_{i-1}.

Checking the covering property

We must verify that for each point p in X there is an evenly-covered neighbourhood. If p is in the interior of F, the result follows because any neighbourhood of p contained in the interior of F has k sheets, one in each F_i.

If p is in the interior of a, we know that a small contractible neighbourhood U of p can be assembled by taking k half-discs U_j in F, and glueing them along the identifications of the edges. We must show that U is evenly-covered. To do so, we will take a preimage p' of p in Y and show that there is a sheet $U' \subset Y$ homeomorphic to U passing through p'.

We first observe that there is an edge a_i such that p' is contained in its interior. Moreover, each face F_l of Y is incident to a_i in exactly one side. It follows that for each j there is a unique l such that U_j lifts to a half-disc U'_j in F_l incident to a_i. U' is then the union of these U'_j, proving the claim.

For the vertex p the proof is the same. A neighbourhood of p is assembled from k half-discs, corresponding to the corners of F. One can then argue identically.

Other covering spaces

Recall that there is a subgroup H_n of $\pi_1(X, x) \simeq \mathbb{Z}/k\mathbb{Z}$ for each integer n that divides k. Then H_n is generated by a^n, is isomorphic to the cyclic group of order $m = k/n$, and has index n. It follows (Corollary 16.41) that there must exist a covering space $\tau : (Y_n, z) \to\to (X, x)$ with n sheets such that $\tau_*(\pi_1(Y_n, z))$ is H_n.

The easiest way to construct Y_n is to quotient the universal cover Y. Namely, we identify a vertex v_{a^i} with v_{a^j} if $j - i$ is a multiple of n. Same for the edges and the faces.

16.6 Exercises

16.6.1 Covering spaces

Exercise 16.1: Find a covering map $p : (Y, y) \to (X, x)$ and an open $U \subset X$ that is not evenly-covered.

Exercise 16.2: Prove that $\mathbb{S}^2$ is the universal cover of $\mathbb{RP}^2$.

Exercise 16.3: Let K be the Klein bottle.

- Find a covering map of K to itself with more than one sheet.
- Find a covering map of $\mathbb{T}^2$ to K.

16.6.2 The Galois correspondence for covering spaces

Exercise 16.4: Consider, for $n > 1$, the path-connected covering spaces of $\mathbb{S}^1 \vee \mathbb{S}^n$. For each path-connected covering space, up to isomorphism:

- Endow it with a CW-structure.
- Compute its fundamental group.
- Describe the corresponding subgroup of the fundamental group of the base.

Exercise 16.5: Let M be the (open) Möbius band.

- Prove that every path-connected covering space of M with an even number of sheets is homeomorphic to a cylinder.
- Prove that every path-connected covering space of M with an odd number of sheets is homeomorphic to M.
- Prove that the universal cover of M is homeomorphic to $\mathbb{R}^2$.

Exercise 16.6: Consider, for $m, n > 1$, the path-connected covering spaces of $\mathbb{RP}^n \vee \mathbb{RP}^m$. For each covering space with 2-sheets, up to isomorphism:

- Endow it with a CW-structure.
- Compute its fundamental group.
- Describe the corresponding subgroup of the fundamental group of the base.

The Galois correspondence

Lecture 17

We continue with our study of covering spaces. In this final lecture we:

- Construct explicitly the covering space corresponding to a given subgroup (Section 17.3).
- Prove that there is a categorical equivalence between pointed covering spaces and subgroups, the so-called *Galois correspondence* (Section 17.2).
- Give a complete criterion for maps to lift to covering spaces (Section 17.1).
- Explore some applications of these results (Section 17.4).

Important: This chapter will not be part of the final exam, since it was not covered in class.

17.1 The lifting criterion

Our study of holonomy can now be used to provide a complete criterion for liftability of maps to covering spaces:

Theorem 17.1. *Let $\pi : (Y, y) \to (X, x)$ be a pointed covering map and $f : (A, a) \to (X, x)$ a pointed map. Suppose moreover that A is path-connected and locally path-connected. Then:*

- *f admits a lift $\tilde{f} : (A, a) \to (Y, y)$ if and only if $f_*(\pi_1(A, a)) \subset \pi_*(\pi_1(Y, y))$.*
- *This lift is unique.*

Proof. The only if direction follows from the identity $f = \pi \circ \tilde{f}$:

$$f_*(\pi_1(A, a)) = \pi_* \circ \tilde{f}_*(\pi_1(A, a)) \subset \pi_*(\pi_1(Y, y)).$$

For the if direction we lift paths. Observe first that we must set $\tilde{f}(a) = y$, due to the pointed condition. Given any other $a' \in A$, we pick $[\gamma] \in \pi_1(A, a, a')$, write $x' := f(a')$, consider the pushforward $[f \circ \gamma] \in \pi_1(X, x, x')$, and set

$$\tilde{f}(a') := \mathrm{hol}_{f \circ \gamma}(y).$$

We must now show this definition does not depend on our choice of γ. First note that, according to Lemma 16.30, $\text{hol}_{f\circ\gamma}$ does not depend on the representative $f\circ\gamma$, only on the class $f_*[\gamma]$. Secondly, the choice of class $[\gamma]$ also does not matter. If we choose some other $[\nu]\in\pi_1(A,a,a')$, we have that

$$\text{hol}_{f\circ\nu}(y)=\text{hol}_{f\circ(\gamma\cdot\bar{\gamma}\cdot\nu)}(y)=\text{hol}_{f\circ\gamma}\circ\text{hol}_{f\circ(\bar{\gamma}\cdot\nu)}(y)=\text{hol}_{f\circ\gamma}(y)$$

where we have used the assumption $f_*[\bar{\gamma}\cdot\nu]\in\pi_*(\pi_1(Y,y))$ and invoked Lemma 16.38 to see that the holonomy along it is trivial.

We have thus shown that $\tilde{f}$ is well-defined. We must now show that it is continuous. This follows from the evenly-covered property. Indeed, Y has a basis of neighbourhoods V such that V is a sheet over the evenly covered $\pi(V)$. Given $a'\in A$ with $\tilde{f}(a')\in V$ we have that $f^{-1}(\pi(V))$ is an open containing a', by continuity of A. It therefore contains an open $U\ni a'$ that is path-connected. We claim that $\tilde{f}|_U$ amounts to mapping U to $\pi(V)$ using f, and then lifting to V using the homeomorphism between the two. This will then establish continuity, since $\tilde{f}(U)\subset V$.

To prove the claim, observe that $a''\in U$ can be joined to a' using a path ν in U. Let γ be a path connecting a to a''. Then

$$\tilde{f}(a''):=\text{hol}_{f\circ(\nu\cdot\gamma)}(y)=\text{hol}_{f\circ\nu}\circ\text{hol}_{f\circ\gamma}(y)=\text{hol}_{f\circ\nu}(\tilde{f}(a'))$$

which proves the claim, since $f\circ\nu$ takes values in $\pi(V)$.

To establish uniqueness of $\tilde{f}$, do note that we did not have any choice in its construction, due to the uniqueness of the path lifting property for covering spaces. □

That is, maps lift when the loops in A, pushed to X, wrap less than the loops of Y, projected to X.

Corollary 17.2. *Let $p:(Y,y)\to(X,x)$ be a covering map. Let $f:(A,a)\to(X,x)$ be a map with A path-connected and locally path-connected. If A is simply-connected, f admits a lift.*

17.2 The Galois correspondence

We now state the correspondence between the category of (path-connected) pointed covering spaces $\text{Cover}(X,x)$ and the category of subgroups $\text{Subgrp}(\pi_1(X,x))$, assuming X is sufficiently nice. We first introduce these two categories.

17.2.1 The category of subgroups

Definition 17.3. *Given a group G, we can consider the category* $\text{Subgrp}(G)$ *defined by:*

- $\text{Ob}(\text{Subgrp}(G))$ *is the set of subgroups of G.*

- *For each two subgroups* $H, L \subset G$*, we say that* $\mathrm{Hom}_{\mathrm{Subgrp}(G)}(H, L) \simeq \{.\}$ *if* $H \subset L$*, and empty otherwise.*

This concept appeared in Exercise 1.6 and may remind you of the category of opens of a space (Exercise 1.5). The following is left as an exercise for the reader:

Lemma 17.4. *Prove that:*

- *The coproduct of* A, B *in* $\mathrm{Subgrp}(G)$ *is the (set theoretical) intersection.*
- *The product of* A, B *in* $\mathrm{Subgrp}(G)$ *is the subgroup of* G *generated by the elements of* A *and* B*.*
- *The pushout of a diagram* $A \leftarrow I \rightarrow B$ *is the product of* A *and* B *(i.e. the same as the product).*
- *Find a concrete example of* G*,* A *and* B *showing that the inclusion functor* $\mathrm{Subgrp}(G) \to \mathrm{Grp}$ *does not preserve products.*

17.2.2 The category of pointed covering spaces

In order to set up the category of pointed covering spaces we first introduce the morphisms:

Definition 17.5. *Let* $\pi : (Y, y) \to (X, x)$ *and* $\pi' : (Y', y') \to (X, x)$ *be pointed covering spaces. A* ***morphism of covering maps*** *from* π *to* π' *is a pointed map* $f : (Y, y) \to (Y', y')$ *such that* $\pi = \pi' \circ f$*.*

One can also say that f is a map fibered over X, since it takes fibres of π (i.e. preimages $\pi^{-1}(y)$ of points y) to fibres of π'. Identically, a morphism f preserves the projection to X, meaning that over an evenly covered subset $U \subset X$ it maps sheets to sheets. One can define the analogous notion in the unpointed setting, but we will not look into it further.

Definition 17.6. *Let* (X, x) *be a path-connected pointed space. We define the* ***category of pointed covering spaces*** $\mathrm{Cover}(X, x)$ *of* (X, x) *as follows:*

- $\mathrm{Ob}(\mathrm{Cover}(X, x))$ *is the class of all pointed covering maps* $f : (Y, y) \to (X, x)$ *with* Y *path-connected.*
- *Given two pointed covering maps* $\pi : (Y, y) \to (X, x)$ *and* $\pi' : (Y', y') \to (X, x)$*, we let* $\mathrm{Hom}_{\mathrm{Cover}(X,x)}(\pi, \pi')$ *be the set of all morphisms from* π *to* π'*.*

Observe that X and Y are both assumed to be path-connected. This is done for convenience, since our goal is to relate covering spaces to the fundamental group, which is something computed path-component-wise. One can develop the theory of covering spaces allowing multiple path-components; we refer the reader to [Hat02].

17.2.3 The Galois correspondence

The Galois correspondence states:

Theorem 17.7. *Suppose* (X, x) *is path-connected and locally simply-connected. Then, the Galois functor*

$$\mathcal{G} : \mathrm{Cover}(X, x) \to \mathrm{Subgrp}(\pi_1(X, x))$$

is an equivalence of categories.

Let us comment first on the assumptions and then explain how $\mathcal{G}$ is defined. Suppose (X, x) is a space. In order to establish the Galois correspondence for $\text{Cover}(X, x)$ we must construct the universal cover $\pi : (Y, y) \to (X, x)$, since it corresponds to the trivial subgroup. Y must resemble X locally, due to the covering property, but also be simply-connected. It follows that X cannot be too wild. Recall:

Definition 17.8. *A space X is **locally simply-connected** if every point in X has a system of simply-connected neighbourhods.*

If X is not locally simply-connected (i.e. every small neighbourhood has non-trivial fundamental group), neither is Y. This does not quite contradict the simply-connectedness of Y (you could imagine a situation where these local non-trivial loops become trivial globally), but suggests we need to put some assumptions on X. For simplicity, we henceforth work under local simply-connectedness assumptions. For the (slightly more) general case, we refer the reader to [Hat02].

Then:

Definition 17.9. *The **Galois functor** $\mathcal{G} : \text{Cover}(X, x) \to \text{Subgrp}(\pi_1(X, x))$:*

- *Takes a pointed covering map $\pi : (Y, y) \to (X, x)$ to the subgroup $\pi_*(\pi_1(Y, y)) \subset \pi_1(X, x)$.*
- *Takes a morphism of pointed covering spaces $f : (Y, y) \to (Y', y')$ to the inclusion $\pi_*(\pi_1(Y, y)) \to \pi'_*(\pi_1(Y', y'))$.*

Theorem 17.7 can be broken down into a handful of concrete statements. The first says that every subgroup can be represented by a covering space:

Proposition 17.10. *Given a subgroup $H \subset \pi_1(X, x)$, there is a pointed covering map $\pi : (Y, y) \to (X, x)$ with $H = \pi_*(\pi_1(Y, y))$.*

This is proven in Section 17.3.

The second says that $\mathcal{G}$ is well-defined at the level of morphisms:

Proposition 17.11. *Fix elements $\pi : (Y, y) \to (X, x)$ and $\pi' : (Y', y') \to (X, x)$ in $\text{Cover}(X, x)$. Let $f : (Y, y) \to (Y', y')$ be a morphism. Then:*

- *f is itself a covering map.*
- *$f_* : \pi_1(Y, y) \to \pi_1(Y', y')$ is injective.*
- *$\pi'_* \circ f_* = \iota \circ \pi_*$, where ι is the inclusion $\pi_*(\pi_1(Y, y)) \to \pi'_*(\pi_1(Y', y'))$.*

Proof. Given a point $p \in X$, we can find a neighbourhood $U \subset X$ that is evenly covered for π. Similarly, there is a neighbourhood $U' \subset X$ that is evenly covered for π'. Their intersection $V := U \cap U'$ is evenly covered for both, thanks to Lemma 16.8. We therefore have that there are discrete spaces S and S' such that $\pi^{-1}(V) \simeq U \times S$ and $(\pi')^{-1}(V) \simeq U \times S'$. By continuity of f, we deduce that each $U \times \{s\}$ is mapped to some $U \times \{s'\}$, i.e. f maps sheets to sheets. Moreover, the morphism property $\pi = \pi' \circ f$ implies that the restricted map $f : U \times \{s\} \to U \times \{s'\}$ is a homeomorphism. This implies that each sheet in Y' is evenly covered by sheets of Y, proving the first claim.

The second item then follows from Proposition 16.36 and the third by the equality $\pi' \circ f = \pi$. □

The third one proves the equivalence of categories:

Proposition 17.12. *Fix elements $\pi : (Y, y) \to (X, x)$ and $\pi' : (Y', y') \to (X, x)$ in* $\mathrm{Cover}(X, x)$*. Then:*

- *There is at most one morphism from π to π'.*
- *There is a morphism from π to π' if and only if $\pi_*(\pi_1(Y, y))$ is a subgroup of $\pi'_*(\pi_1(Y', y'))$.*
- *In particular, the two covering spaces are isomorphic if and only if $\pi_*(\pi_1(Y, y)) = \pi'_*(\pi_1(Y', y'))$.*

Proof. Suppose f is a morphism from π to π'. Then f is a lift of the identity along π'. It follows that f is unique, due to the lifting criterion (Theorem 17.1).

The only if direction of the second item follows from Proposition 17.11. For the if direction we apply the lifting criterion. First note that the assumptions of the theorem hold: (1) elements of $\mathrm{Cover}(X, x)$ are path-connected and locally simply-connected, since X itself is locally simply-connected, and (2) we have the inclusion $\pi_*(\pi_1(Y, y)) \subset \pi'_*(\pi_1(Y', y'))$. It follows that there is a lift of π along π'. This is precisely what a morphism is.

For the last item, observe that the assumption means that there is a morphism f is a morphism from π to π' and a morphism g from π' to π. Then $g \circ f$ lifts π along itself. Since id_Y also lifts π along itself, the two must agree. The same reasoning shows that $f \circ g = \mathrm{id}_{Y'}$, so f and g are inverses. □

17.3 Constructing covering spaces

The only statement remaining in order to establish the Galois correspondence (Theorem 17.7) is Proposition 17.10, the existence of covering spaces. We tackle this now. The plan, as our discussion in Section 16.4 suggests, is to put a suitable topology in $\mathfrak{s}^{-1}(x)$ so it becomes the universal cover.

17.3.1 The fundamental groupoid as a topological space

It turns out that one can topologise the fundamental groupoid $\Pi_1(X)$ first, and the topology we want in $\mathfrak{s}^{-1}(x)$ will be the subspace topology.

Definition 17.13. *Let X be a locally simply-connected space. The topology in $\Pi_1(X)$ is generated by the basis:*

$$V_{[\gamma], U_\mathfrak{s}, U_\mathfrak{t}} := \{[\beta][\gamma][\alpha]^{-1} \mid [\alpha] \in \pi_1(U_\mathfrak{s}, \gamma(0), q), [\beta] \in \pi_1(U_\mathfrak{t}, \gamma(1), p)\}$$

where $[\gamma] \in \Pi_1(X)$, $U_\mathfrak{s}$ is a simply-connected neighbourhood of $\gamma(0)$, and $U_\mathfrak{t}$ is a simply-connected neighbourhood of $\gamma(1)$.

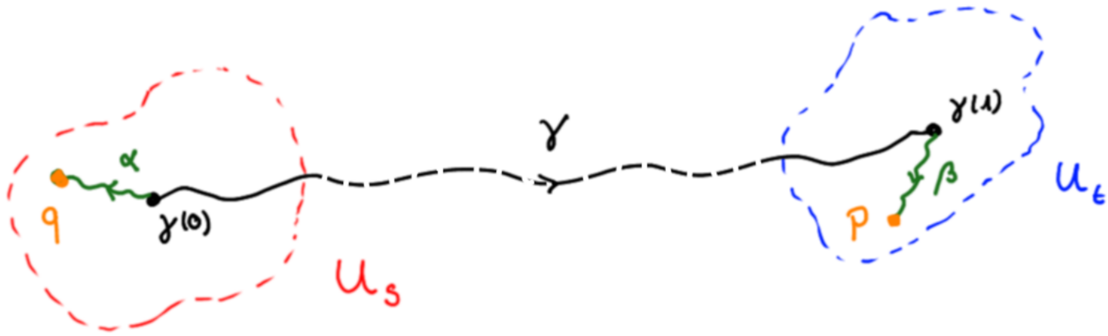


Figure 17.1: The data defining an open $V_{[\gamma],U_\mathfrak{s},U_\mathfrak{t}}$ in $\Pi_1(X)$. The open is readily seen to be homeomorphic to $U_\mathfrak{s} \times U_\mathfrak{t}$, showing that $\Pi_1(X)$ is a covering space of $X \times X$.

That is, we are defining a basis of opens around the element $[\gamma] \in \Pi_1(X)$, and each open $V_{[\gamma],U_\mathfrak{s},U_\mathfrak{t}}$ amounts to considering all the paths that can be obtained from $[\gamma]$ by attaching a little path $[\alpha]$ at the beginning and a little path $[\beta]$ at the end. This is shown in Figure 17.1.

We verify:

Lemma 17.14. *Suppose* $[\rho] \in V_{[\gamma],U_\mathfrak{s},U_\mathfrak{t}}$. *Then*

$$V_{[\rho],U_\mathfrak{s},U_\mathfrak{t}} = V_{[\gamma],U_\mathfrak{s},U_\mathfrak{t}}.$$

Proof. By assumption we can write $[\rho] = [\beta][\gamma][\alpha]^{-1}$ for some α and β. It follows that any other class $[\beta'][\gamma][\alpha']^{-1}$ in $V_{[\gamma],U_\mathfrak{s},U_\mathfrak{t}}$ can be written as

$$([\beta'][\beta]^{-1})([\beta][\gamma][\alpha])([\alpha]^{-1}[\alpha']^{-1}) = ([\beta'][\beta]^{-1})[\rho]([\alpha]^{-1}[\alpha']^{-1})$$

which proves the inclusion of the right hand side into the left hand side. The converse follows by switching the roles of $[\rho]$ and $[\gamma]$. □

Which allows us to confirm:

Lemma 17.15. *The collection of opens described in Definition 17.13 is indeed a basis.*

Proof. We just need to check that intersections of two opens $V_{[\gamma],U_\mathfrak{s},U_\mathfrak{t}}$ and $V_{[\gamma'],U'_\mathfrak{s},U'_\mathfrak{t}}$ are open. Indeed, given $[\rho]$ lying in both, we can write the two opens as $V_{[\rho],U_\mathfrak{s},U_\mathfrak{t}}$ and $V_{[\rho],U'_\mathfrak{s},U'_\mathfrak{t}}$, so we deduce that $V_{[\rho],U_\mathfrak{s}\cap U'_\mathfrak{s},U_\mathfrak{t}\cap U'_\mathfrak{t}}$ is in the intersection. □

By construction, the topology we have constructed makes $\Pi_1(X)$ resemble the pair groupoid $X \times X$. This means that:

Proposition 17.16. $\pi := (\mathfrak{s}, \mathfrak{t}) : \Pi_1(X) \to X \times X$ *is a covering map.*

Proof. Observe first that product opens $U_\mathfrak{s} \times U_\mathfrak{t}$, with both factors simply-connected, form a basis of the topology in $X \times X$. This allows us to check that the map π is continuous. Indeed, fix points $q \in U_\mathfrak{s}$ and $p \in U_\mathfrak{t}$. Then we have that:

$$\pi^{-1}(U_\mathfrak{s} \times U_\mathfrak{t}) \simeq \coprod_{[\gamma]\in\pi_1(X,q,p)} V_{[\gamma],U_\mathfrak{s},U_\mathfrak{t}},$$

and the right-hand side is a disjoint union of open sets $V_{[\gamma],U_\mathfrak{s},U_\mathfrak{t}}$, proving continuity.

In fact, the topology we have put in $\Pi_1(X)$ implies that the restriction

$$\pi : V_{[\gamma],U_\mathfrak{s},U_\mathfrak{t}} \to U_\mathfrak{s} \times U_\mathfrak{t}$$

is a homeomorphism. This implies that the evenly-covered property holds:

$$\begin{array}{ccc} (U_\mathfrak{s} \times U_\mathfrak{t}) \times \pi_1(X, q, p) & \xrightarrow[\simeq]{\psi} & \pi^{-1}(U_\mathfrak{s} \times U_\mathfrak{t}) \\ \pi_U \downarrow & & \downarrow \pi \\ U_\mathfrak{s} \times U_\mathfrak{t} & \xrightarrow[\mathrm{id}]{} & U_\mathfrak{s} \times U_\mathfrak{t} \end{array}$$

where the top arrow ψ is the obvious identification. □

One has to be careful with the lack of path-connectedness. Suppose that X has two path-components, X_0 and X_1. Then, the components $X_0 \times X_1$ and $X_1 \times X_0$ have empty preimage in $\Pi_1(X)$ (which was allowed in the definition of covering map).

17.3.2 The universal cover

Which allows us to deduce:

Corollary 17.17. *Let (X, x) be a pointed, path-connected, locally simply-connected space. Then*

$$\mathfrak{t} : (\mathfrak{s}^{-1}(x), [c_x]) \to (X, x)$$

is a pointed, surjective covering map. Here $[c_x]$ denotes the class of the constant loop at x.

Proof. We can regard X as the subspace $\{x\} \times X \subset X \times X$. Then, its preimage via $(\mathfrak{s}, \mathfrak{t})$ consists of those homotopy classes of paths that start at x. This is precisely what $\mathfrak{s}^{-1}(x)$ is. Observe that, since X is path-connected, every point in X does have a preimage in $\mathfrak{s}^{-1}(x)$. Lemma 16.8 applies. □

In Theorem 17.23 we will prove that $\mathfrak{s}^{-1}(x)$ is the universal cover of X, under these assumptions.

Corollary 17.18. *The topology in $\mathfrak{s}^{-1}(x)$ is generated by the basis*

$$V_{[\gamma],U} := \{[\beta][\gamma] \mid [\beta] \in \pi_1(U, \gamma(1), p)\}$$

with γ a path starting at x and U a simply-connected neighbourhood of $\gamma(1)$.

This is immediate from the description of the topology in $\Pi_1(X)$. Observe that $V_{[\gamma],U}$ is one of the sheets evenly covering U.

17.3.3 Covering spaces as quotients of the universal cover

We can now use Corollary 17.17 to produce covering spaces associated to subgroups of $\pi_1(X, x)$.

Definition 17.19. *Let (X, x) be a pointed space. Let H be a subgroup of $\pi_1(X, x)$. We say that the classes $[\gamma]$ and $[\eta]$ in $\mathfrak{s}^{-1}(x)$ are H-related if there is $[\nu] \in H$ such that $[\eta] = [\gamma][\nu]$. We write X_H for the quotient of $\mathfrak{s}^{-1}(x)$ under this equivalence relation.*

Since $[\nu] \in H$ is a the class of a loop based at X, we see that $[\gamma]$ and $[\eta]$ have the same endpoint. It follows that $\mathfrak{t}$ descends to the quotient as a well-defined map $\pi : X_H \to X$. Moreover, this map is pointed if we take $x \in X$ and $H \in X_H$ as basepoints. By $H \in X_H$ we mean the equivalence class of $[c_x]$, which is the equivalence class of any $[\nu] \in H$. Then:

Lemma 17.20. *The following is a covering map:*

$$\pi : (X_H, H) \to (X, x).$$

Proof. Given $y \in X$, consider a simply-connected neighbourhood U. As we saw earlier, it is evenly covered in $\mathfrak{s}^{-1}(x)$; its preimages are the $V_{[\gamma],U}$ with $[\gamma]$ ranging over the elements of $\pi_1(X, x, y)$. Given $[\nu] \in H$, we see that the sheet $V_{[\gamma],U}$ becomes identified with the sheet $V_{[\gamma][\nu],U}$, so they are the same open in X_H. I.e. the quotient map amounts to quotienting the set of sheets (very concretely $\pi_1(X, x, y) \to \pi_1(X, x, y)/H$). It follows that U is evenly-covered for π. □

17.3.4 Theorem 16.25 in some cases

Even though we did not prove Theorem 16.25 in full generality, we can verify it directly for the covering spaces we constructed by hand using the fundamental groupoid:

Lemma 17.21. *Let (X, x) be a pointed, path-connected, locally simply-connected space. Then its source fibre*

$$\mathfrak{t} : (\mathfrak{s}^{-1}(x), [c_x]) \to (X, x)$$

satisfies the unique homotopy lifting property.

Proof. Suppose we are given a homotopy $F : A \times [0, 1] \to X$ starting at $f : A \to X$ and a lift $\tilde{f} : A \to \mathfrak{s}^{-1}(x)$. Let us consider the paths $\gamma_{a,s}(t) := F(a, st) : [0, 1] \to X$. They allow us to define a function $\tilde{F} : A \times [0, 1] \to \mathfrak{s}^{-1}(x)$ by setting $(a, s) \mapsto [\gamma_{a,s}]$. By construction, $\mathfrak{t}([\gamma_{a,s}]) = \gamma_{a,s}(1) = F(a, s)$, so $\tilde{F}$ is a lift of F.

We must verify that $\tilde{F}$ is continuous. Recall that the topology in X is generated by evenly-covered opens, so the topology in $(\mathfrak{s}^{-1}(x)$ is generated by opens U which are sheets over evenly covered opens $\mathfrak{t}(U)$ of X. Then we see that $\tilde{F}^{-1}(U) = F^{-1}(\mathfrak{t}(U))$ is open by continuity of F. □

We can prove the same result for X_H, the space constructed in Definition 17.19. This is a consequence of Lemma 17.21: one can lift to the universal cover $\mathfrak{s}^{-1}(x)$ and then project down to X_H:

Corollary 17.22. *Let* (X, x) *be a pointed, path-connected, locally simply-connected space. Then the covering map*

$$\mathfrak{t} : (X_H, H) \to (X, x)$$

satisfies the unique homotopy lifting property.

17.3.5 The Galois correspondence

We have thus established the existence of a covering space for each subgroup H of $\pi_1(X, x)$:

Theorem 17.23. *Let* (X, x) *be a pointed, path-connected, locally simply-connected space. Then the covering map*

$$\mathfrak{t} : (X_H, H) \to (X, x)$$

satisfies $\mathfrak{t}_*(\pi_1(X_H, H)) = H$.

In particular,

$$\mathfrak{t} : (\mathfrak{s}^{-1}(x), [c_x]) \to (X, x)$$

is simply-connected and is thus the universal cover.

Proof. This was already proven in Corollary 16.39. □

17.4 Applications

We now work out some interesting corollaries of the theory we have developed.

17.4.1 Group theory

Proposition 16.16 applied to graphs reads:

Corollary 17.24. *Let* $\pi : Y \to X$ *be a covering space and suppose* X *is a* 1*-dimensional cell complex. Then so is* Y*.*

Which has the following consequence about the structure of free groups:

Corollary 17.25. *Every subgroup of a free group is also free.*

Proof. Recall that G being free means that it is isomorphic to $*_I Z$ is free, for some set I. We can consider the 1-dimensional cell complex $(X, x) := \vee_I(\mathbb{S}^1, 1)$, whose fundamental group is G. Since (X, x) is path-connected and locally simply-connected, the Galois correspondence (Theorem 17.7) applies. It follows that any subgroup $H \subset G$ is the image of $\pi_1(Y, y)$ via π, for some path-connected covering space $\pi_1 : (Y, y) \to (X, x)$. The argument follows from the fact that the fundamental group $\pi_1(Y, y)$ is also free, since Y is also a 1-dimensional cell complex. □

17.4.2 Spheres

One can compute the fundamental group of the spheres using covering space theory, without invoking van Kampen:

Theorem 17.26. *Let $\pi : (Y, y) \to (\mathbb{S}^n, N)$ be a path-connected covering space of the n-sphere, with $n \geq 2$. Then π is a homeomorphism. In particular, $\mathbb{S}^n$ is simply-connected.*

Proof. Recall that the standard cell structure of $\mathbb{S}^n$ consists of a single vertex N and a single n-cell. Using Proposition 16.16 we deduce that Y is obtained from Y_0 by attaching n-cells. Attaching n-cells does not change π_0, if $n \geq 2$, so the path-connected assumption for Y implies that Y_0 consists of a single vertex. Moreover, since X is path-connected, the number of sheets is constant, so in this case it must be one. If there is a single sheet, π is a homeomorphism, proving the first claim.

$(\mathbb{S}^n, N)$ is path-connected and locally simply-connected, so it has a universal cover. However, we have just shown that every path-connected covering space is homeomorphic to $\mathbb{S}^n$. It follows that $\mathbb{S}^n$ is its own universal cover and is thus simply-connected. □

In the case of the circle, we can recover our favourite result, without using van Kampen:

Theorem 17.27. *$\pi_1(\mathbb{S}^1, 1) \simeq \mathbb{Z}$.*

Proof. Identify $\mathbb{S}^1$ with $[0, 1]/(0 \simeq 1)$ and endow with the cell structure with a single vertex 0 and a single edge a. We have seen that the universal cover is $\mathbb{R}$, with covering map $\pi(t) = [t]$. We can use π to lift the cell structure to $\mathbb{R}$. It has infinitely many vertices (namely, the integers $\mathbb{Z} = \pi^{-1}(0)$) and infinitely many edges (which we denote $a_i := [i, i + 1]$).

We can now use the fact that the universal cover must be isomorphic to the source fibre $\mathfrak{s}^{-1}(0) \subset \Pi_1(\mathbb{S}^1)$. Under this identification, $\mathbb{Z}$ is identified with $\pi_1(\mathbb{S}^1, 1)$, since both are the fibres over 0 via the covering map. This tells us already that $\pi_1(\mathbb{S}^1, 1)$ is countable infinite. However, this is just set theoretically. We want to figure out that what the group operation is.

In order to do so, we let $0 \in \mathbb{Z}$ be identified with $[c_0] \in \pi_1(\mathbb{S}^1, 0)$. Consider then the edge a. Its lift starting at 0 is a_1, which finishes at $1 \in \mathbb{Z}$. This implies that the point $1 \in \mathbb{Z}$ is identified with $[a] \in \pi_1(\mathbb{S}^1, 0)$. Iterating, we deduce that $k \in \mathbb{Z}$ must be identified with $[a]^k$. This tells us that the group operation is given by the usual addition in $\mathbb{Z}$, as claimed. □

17.4.3 The circle has no higher holes

Consider the following general statement:

Lemma 17.28. *Let $\pi : (Y, y) \to (X, x)$ be the universal cover and assume that Y deformation retracts to y. Then any map $f : (A, a) \to (X, x)$, with A simply-connected and locally path-connected, is nullhomotopic.*

Proof. Since A is path-connected and locally path-connected we can apply the lifting criterion (Theorem 17.1) and produce a lift $\tilde{f} : (A, a) \to (Y, y)$. Using the deformation retraction of Y to y we deduce that $\tilde{f}$ is pointed homotopic to the constant map c_y. Applying π we deduce that f is pointed homotopic to c_x. □

As an immediate application we deduce that $\mathbb{S}^1$ has no holes bounded by higher dimensional spheres:

Corollary 17.29. *Every map $f : (\mathbb{S}^n, N) \to (\mathbb{S}^1, 1)$ with $n \geq 2$ is nullhomotopic.*

Proof. $\mathbb{S}^n$ is simply-connected and locally path-connected and the universal cover of $\mathbb{S}^1$ is $\mathbb{R}$, which deformation retracts to any point. □

17.5 Exercises

17.5.1 Lifting criterion

Exercise 17.1: Let $p : (Y, y) \to (X, x)$ be a covering map. Let $f : (A, a) \to (X, x)$ be a map with A path-connected and locally path-connected. Show that if f is null-homotopic (as an unpointed map), it admits a lift.

Exercise 17.2: Let $n \geq 2$. Prove that $[\mathbb{S}^n, \mathbb{T}^m] = \{.\}$, for all m.

Exercise 17.3: Show that if a path-connected, locally path-connected, space X has $\pi_1(X, x)$ finite, then every map $X \to \mathbb{S}^1$ is nullhomotopic.

Exercise 17.4: Let $K \subset A$ be a deformation retract, with A locally path-connected. Let $p : (Y, y) \to (X, x)$ be a covering map and let $f : (A, a) \to (X, x)$ be a map. Assume that $g = f|_K$ admits a lift $\tilde{g} : K \to Y$. Show that f admits a unique lift $\tilde{f} : A \to Y$ such that $\tilde{f}|_K = \tilde{g}$.

17.5.2 The fundamental groupoid as a space

Exercise 17.5: Show that $\Pi_1(X)$ is homeomorphic to $X \times X$ if and only if X is simply-connected.

Exercise 17.6: Show that $\Pi_1(X)$ is contractible if and only if X is contractible. **Hint:** You may want to prove and use that X is a retract of $\Pi_1(X)$.

Exercise 17.7: Show that $\Pi_1(\mathbb{S}^1)$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. Describe the structure maps in terms of this identification.

Exercise 17.8: Let $f : X \to Y$ be a map. Show that $f_* : \Pi_1(X) \to \Pi(Y)$ is a continuous map. Show that f_* is a homeomorphism if and only if f is a homeomorphism.

Exams from previous years

Appendix

18.1 Instructions

- The exam is closed-book.
- Write your name and student number in all the pages of the exam.
- You may write your solutions in either Dutch or English.
- You must justify the claims you make.
- You may use results from the lectures or the dictaat, but you must provide a clear statement (with complete hypothesis and conclusion).
- Try to write with clear handwriting. Structure your explanations clearly, using one paragraph for each new idea and one sentence for each particular claim.
- **Advice**: Make as many pictures as possible to clarify the nature of the spaces you work with. Correct drawings demonstrate to the grader that you understand what you are trying to justify.

18.2 Exam 2024

Exercise 18.1 (1 point)**:** Find a space M and subspaces $A, B \subset M$ such that A is homeomorphic to B, M retracts to A, but M does not retract to B.

Exercise 18.2 (0.75 points)**:** Let (A, a) be a pointed space. Suppose that $\pi_1(A, a) \neq 0$. Prove that there is a space B such that $[B, A] \not\simeq \{.\}$.

Exercise 18.3 (1 point)**:** Find a pointed space (P, p) and a class $[\gamma] \in \pi_1(P, p)$ such that conjugation by γ

$$\beta_\gamma : \pi_1(P, p) \to \pi_1(P, p)$$

is a non-trivial group isomorphism (non-trivial means different from the identity).

Exercise 18.4 (1.25 points)**:** Let (X, x) be a pointed space. Let $A \subset X$ be a path-connected subspace containing x. Assume that the pushforward of the inclusion

$$\iota_* : \pi_1(A, x) \to \pi_1(X, x)$$

is surjective. Show that:

- For every $a, b \in A$, each path γ from a to b in X is homotopic relative endpoints to a path from a to b in A.
- The map $\iota_* : \pi_1(A, a, b) \to \pi_1(X, a, b)$ is surjective for every $a, b \in A$.

Find an example where the previous statements hold but $\iota_* : \Pi_1(A) \to \Pi_1(X)$ is not surjective.

Exercise 18.5 (1 point)**:** Suppose that A and B are compact, path-connected surfaces. Prove that A is homeomorphic to B if and only if $A\#\mathbb{T}^2$ is homotopy equivalent to $B\#\mathbb{T}^2$. Recall that $\#$ means connected sum of surfaces.

Exercise 18.6 (3 points)**:** Let $\mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$ be the torus. Write $f : \mathbb{S}^1 \to \mathbb{T}^2$ for the map $z \mapsto (z, 1)$. Let $\mathbb{RP}^2$ be the projective plane and $p \in \mathbb{RP}^2$ the vertex in its standard planar presentation. Write $g : \mathbb{S}^1 \to \mathbb{RP}^2$ for the constant map with image p. Consider the space

$$X = \text{pushout}(\mathbb{T}^2 \xleftarrow{f} \mathbb{S}^1 \xrightarrow{g} \mathbb{RP}^2).$$

- Endow X with the structure of a 2-dimensional cell-complex.
- Compute the fundamental group of X at (the class of) p.
- Observe that there is a canonical map $\phi : \mathbb{T}^2 \to X$. Compute the pushforward $\phi_* : \pi_1(\mathbb{T}^2, (1, 1)) \to \pi_1(X, p)$ and prove that it is not injective (recall that computing means that you should write explicitly how ϕ_* maps generators to generators).
- Is the subspace $L := \phi(\mathbb{S}^1 \times \{-1\}) \subset X$ a retract of X?

Exercise 18.7 (2 points)**:** Let $\mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$ be the torus. Fix an integer k. Consider maps $f, g : \mathbb{S}^1 \to \mathbb{T}^2$ defined as $f(z) := (z^k, 1)$ and $g(z) := (1, z^2)$. Let Y be the space obtained from T^2 by attaching two 2-cells, one with attaching map f and another with attaching map g. Fix a basepoint $y := (1, 1) \in \mathbb{T}^2 \subset Y$.

- Compute the fundamental group of (Y, y).
- Produce a 2-sheeted covering map $\pi : (A, a) \to (Y, y)$, with A path-connected. You should justify that π is indeed covering.
- Compute the fundamental group of (A, a). Compute its image $\mathrm{im}(\pi_*) \subset \pi_1(Y, y)$.
- Is there a covering map $\tau : (B, b) \to (Y, y)$ with B path-connected and not homeomorphic to A?

18.3 Retake 2024

Exercise 18.8 (1 point)**:** Find a space X and two maps $\gamma, \nu : \mathbb{S}^1 \to X$ such that, for all maps $\rho : \mathbb{S}^1 \to \mathbb{S}^1$, it holds that $[\gamma] \neq [\nu \circ \rho] \in [\mathbb{S}^1, X]$.

Exercise 18.9 (2 points)**:** Construct a pointed space (A, a) whose fundamental group is isomorphic to

$$G := \langle g_1, g_2, g_3, g_4 \mid g_1^3 g_2 g_3^2, [g_3, g_4], g_4^3 \rangle.$$

- Compute the first homologies of A.
- Is there a compact surface whose fundamental group is isomorphic to G?

Exercise 18.10 (1.5 points)**:** Find an example of

- a pointed space (X, x),
- a subspace $A \subset X$ containing x,
- and a non-nullhomotopic loop $\gamma : (\mathbb{S}^1, 1) \to (A, x)$

such that conjugation by $[\gamma]$ is trivial in $\pi_1(A, x)$ but is non-trivial in $\pi_1(X, x)$.

Exercise 18.11 (3 points)**:** Fix an integer $k \geq 1$. We use complex coordinates $z \in \mathbb{S}^1$. Let X be the result of attaching a 2-cell to $\mathbb{S}^1$ using the map $f : \mathbb{S}^1 \to \mathbb{S}^1$ given by $f(z) = z^k$.

- Draw X, endow it with a cell structure, compute its fundamental group.

Since X is given by attaching a cell to $\mathbb{S}^1$, we have a natural inclusion $\iota : \mathbb{S}^1 \to X$. Consider also the map $g : \mathbb{S}^1 \to \mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$ given by $g(z) = (z, z)$. Define

$$Y = \text{pushout}(\mathbb{T}^2 \xleftarrow{g} \mathbb{S}^1 \xrightarrow{\iota} X).$$

- Endow Y with a cell structure, compute its fundamental group.
- Find a subspace $A \subset Y$, homeomorphic to the circle, such that Y retracts to A.

Exercise 18.12 (2.5 points)**:** Construct spaces A, B, C and covering maps $\pi : B \to A$ and $\tau : C \to A$ such that:

- B and C are not homeomorphic.
- B and C are not simply-connected.

Then:

- Fix basepoints $a \in A$, $b \in B$, $c \in C$ so that the maps π and τ are pointed. Compute the fundamental groups of (A, a), (B, b) and (C, c).
- Compute the pushforwards of π and τ at the level of π_1.

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Exercise 18.13 (1 point)**:** Let $A \subset B$ be a retract.

- Suppose that B is contractible. Prove that A is also contractible.
- Find an example in which A is contractible but B is not.

Exercise 18.14 (1 point)**:** Fix a set S. We let $\mathcal{C}$ be its pair groupoid $S \times S \rightrightarrows S$. Given elements $x, y \in S = \text{Ob}(\mathcal{C})$:

- Do they have a product in the category $\mathcal{C}$? If so, determine it.
- Do they have a coproduct in the category $\mathcal{C}$? If so, determine it.

Exercise 18.15 (2.5 points)**:** Let $f : \mathbb{S}^1 \to \mathbb{S}^2$ be the inclusion of the equator and $g : \mathbb{S}^1 \to \mathbb{RP}^2$ a constant map. Consider the space

$$X = \text{pushout}(\mathbb{S}^2 \xleftarrow{f} \mathbb{S}^1 \xrightarrow{g} \mathbb{RP}^2).$$

- Endow X with the structure of a 2-dimensional cell-complex.
- Compute the fundamental group of X.
- Compute the first homologies of X.
- Is X a surface?

Exercise 18.16 (1 point)**:** Let (A, a) be a pointed space with fundamental group isomorphic to

$$G = \langle g_1, \ldots, g_n \mid r_1, \cdots, r_m \rangle.$$

Find a pointed space (B, b) and an inclusion $i : (A, a) \to (B, b)$ such that:

- $\pi_1(B, b)$ is isomorphic to the abelianisation of $\pi_1(A, a)$,
- $i_* : \pi_1(A, a) \longrightarrow \pi_1(B, b)$ is precisely the canonical homomorphism from $\pi_1(A, a)$ to its abelianisation.

Exercise 18.17 (1 point)**:** Construct a 2-dimensional cell complex whose fundamental group is not isomorphic to the fundamental group of a closed surface.

Exercise 18.18 (1 point)**:** Let Y be a path-connected manifold with finite fundamental group. Prove that every map $Y \to \mathbb{S}^1$ is nullhomotopic.

Exercise 18.19 (2.5 points)**:** Let $(A, a) = (\mathbb{RP}^2, q) \vee (\mathbb{RP}^2, q)$.

- Produce a 2-sheeted covering map $\pi : (B, b) \to (A, a)$, with B path-connected. You should justify that π is indeed covering.
- Compute the fundamental group of (B, b). Compute its image $\pi_*(\pi_1(B, b)) \subset \pi_1(A, a)$.
- Let X be the space introduced in Exercise 18.15. Is X a covering space of A?

18.5 Retake 2023

Exercise 18.20 (1 point)**:** Find a space A and a subspace $B \subset A$ such that:

- B is a retract of A,
- B is not contractible,
- B is not a deformation retract of A.

Exercise 18.21 (1.5 points)**:** Let X be a space. We define its category of opens $\mathrm{SO}(X)$ as follows:

- Objects in $\mathrm{SO}(X)$ are open subsets $U \subset X$.

- For each pair of objects $U \subset V$ in $\mathrm{SO}(X)$, $\mathrm{Hom}(U, V)$ contains a single element, the inclusion $i_{UV} : U \to V$. Otherwise, if U is not a subset of V, $\mathrm{Hom}(U, V)$ is empty.

Then:

- Verify that $\mathrm{SO}(X)$ is a category.
- Prove that $U \cap V$ is the product of U and V, as elements of $\mathrm{SO}(X)$.
- Prove that $U \cup V$ is the coproduct of U and V, as elements of $\mathrm{SO}(X)$.

Exercise 18.22 (1 point)**:** Let $\gamma_k : \mathbb{S}^1 \to \mathbb{S}^1$ be the map $\gamma_k(z) = z^k$. Prove that the pushforward

$$(\gamma_k)_* : \Pi_1(\mathbb{S}^1) \to \Pi_1(\mathbb{S}^1)$$

is a groupoid isomorphism if and only if $k = \pm 1$.

Exercise 18.23 (1 point)**:** Construct a 2-dimensional cell complex, homotopy equivalent to the 2-torus, but which is not a surface.

Exercise 18.24 (2 points)**:** Let $f : \mathbb{S}^1 \to \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ be the map $z \mapsto (z, z^2)$ and $g : \mathbb{S}^1 \to \mathbb{S}^2$ the inclusion of the equator. Consider the space

$$X := \text{pushout}(\mathbb{T}^2 \xleftarrow{f} \mathbb{S}^1 \xrightarrow{g} \mathbb{S}^2).$$

- Endow X with the structure of a 2-dimensional cell-complex.
- Compute the fundamental group of X.
- Compute the first homologies of X.

Exercise 18.25 (3.5 points)**:** Consider the space $A := \mathbb{RP}^2 \vee \mathbb{T}^2$. Fix a basepoint a.

- Endow A with a cell structure.
- Compute the fundamental group of (A, a).
- Produce a 2-sheeted covering map $\pi : (B, b) \to (A, a)$, with B path-connected.
- Compute the fundamental group of (B, b). Compute its image $\pi_*(\pi_1(B, b)) \subset \pi_1(A, a)$.
- Produce a 2-sheeted covering map $\tau : (C, c) \to (A, a)$, not isomorphic to π, with C path-connected.

You have to justify that π and τ are indeed covering.

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Exercise 18.26 (1.5 points)**:** Find two inclusions $i, j : \mathbb{S}^1 \to \mathbb{S}^1 \vee \mathbb{S}^1$ such that:

- $i(\mathbb{S}^1)$ is a retract.
- $j(\mathbb{S}^1)$ is a retract.
- i and j are not homotopic.

Then, find two non-homotopic retractions $f, g : \mathbb{S}^1 \vee \mathbb{S}^1 \to i(\mathbb{S}^1)$.

Note: In order to define the wedge $\mathbb{S}^1 \vee \mathbb{S}^1$ you have to pick basepoints but, in this particular case, the resulting space does not depend on these choices (up to homeomorphism).

Exercise 18.27 (0.75 points)**:** Let (X, p) be a pointed topological space. Let $\psi : \pi_1(\mathbb{S}^1, 1) \to \pi_1(X, p)$ be a group homomorphism. Show that there is a map $f : (\mathbb{S}^1, 1) \to (X, p)$ such that $f_* = \psi$.

Exercise 18.28 (0.75 points)**:** Consider T^2 given by its standard planar representation with one vertex p, two edges a and b, and one (square) face D. Let A be its 1-skeleton and let $\iota : A \to T^2$ be the inclusion.

Let $f : A \to \mathbb{S}^1$ be a map. Prove that f can be extended to T^2; that is, there is $g : T^2 \to \mathbb{S}^1$ such that $f = g \circ \iota$.

Exercise 18.29 (1 point)**:** Prove that $[\mathbb{S}^2 \vee \mathbb{S}^1, \mathbb{S}^1] \cong \mathbb{Z}$ (as sets). **Hint:** You may want to use $[\mathbb{S}^1, \mathbb{S}^1] \cong \mathbb{Z}$, as proven in class.

Exercise 18.30 (2 points)**:** The suspension of a space Z is defined as:

$$\Sigma Z := (Z \times [0, 1])/\sim,$$

where $(z, 0) \sim (z', 0)$ and $(z, 1) \sim (z', 1)$ for every $z, z' \in Z$.

Let $(X, p) := \vee^k(\mathbb{S}^1, 1)$.

- Endow ΣX with a cell structure; be explicit about the number of cells used and their attaching maps.
- Prove that ΣX is simply-connected.

Hint: You can use the cell structure on X as a guide to produce the cell structure of ΣX. You may want to think about the case $k = 1$ first.

Exercise 18.31 (2 points)**:** Let A, B be two copies of the torus $T^2 := \mathbb{S}^1 \times \mathbb{S}^1$. Let $a \in \mathbb{Z}$. Define the space

$$C := (A \coprod B)/(A \ni (z, 0) \cong (z, z^a) \in B).$$

- Compute the fundamental group of C.
- Compute the first homology of C.
- Prove that C is not a surface.

Exercise 18.32 (2 points)**:** Let K be the Klein bottle. Fix a basepoint $p \in K$. Prove the following statements:

- All the covering spaces of K with finitely many sheets have Euler characteristic zero.
- There is a covering space of K that is orientable.
- There are two path-connected, 2-sheeted, pointed covering spaces of (K, p) that are not isomorphic (as elements in $\mathrm{Cover}(K, p)$).

18.7 Retake 2022

Exercise 18.33 (3 points)**:** Prove, or provide a counterexample to, the following statements:

- Let A and B be homotopy equivalent spaces. Then A is compact if and only if B is compact.
- For all positive integers n, there is a cell structure on $\mathbb{S}^1$ with n vertices.
- Let Σ be a closed, path-connected, non-orientable surface. Then, Σ is not simply-connected.
- There is a (pointed) surface (C, c) whose fundamental group is not abelian.

Exercise 18.34 (1 point)**:** Fix spaces A, B, and K. Suppose that $K \subset A$ is a deformation retract. Let $\pi : \widetilde{B} \to B$ be a covering space. Assume that all these spaces are path-connected and locally contractible.

Prove that the following statements are equivalent:

- $f : A \to B$ lifts to a map $\widetilde{f} : A \to \widetilde{B}$.
- $g := f|_K : K \to B$ lifts to a map $\widetilde{g} : K \to \widetilde{B}$.

Exercise 18.35 (2.5 points)**:** Let a be a positive integer. Consider a copies $\{S_i\}_{i=1,\cdots,a}$ of the 2-sphere, with north poles $\{n_i \in S_i\}_{i=1,\cdots,a}$ and south poles $\{s_i \in S_i\}_{i=1,\cdots,a}$. Define

$$X := \left(\coprod_{i=1,\cdots,a} S_i \right) / \{n_i \cong s_{i+1} \text{ for every } i < a \text{ and } n_a \cong s_1\}.$$

- Endow X with a CW-structure. Be explicit about the number of cells you use and their attaching maps.
- Is X a surface?
- Compute the fundamental group of X.

Exercise 18.36 (2.5 points)**:** Let X be as in the previous exercise. Given a positive integer b:

- Find a path-connected covering space $\pi : Y_b \to X$ with b sheets.
- Compute the fundamental group of Y_b.
- Fix a basepoint $p \in X$ and a lift $q \in Y_b$. Describe the map $\pi_* : \pi_1(Y_b, q) \to \pi_1(X, p)$ (by explaining what it does on generators).

Then, provide a complete list of all the pointed, path-connected covering spaces of (X, p), up to isomorphism.

Exercise 18.37 (1 point)**:** Let $(W, p) := (\mathbb{S}^1, 1) \vee (\mathbb{S}^1, 1)$. Fix a positive integer $k \geq 2$. Prove that there is a path-connected covering space of (W, p) with fundamental group isomorphic to $*_k\mathbb{Z}$. **Hint:** Find examples with $k = 2, 3, 4$ and try to see what the pattern is.

Bibliography

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