THE WHITNEY-GRAUSTEIN THEOREM AS AN INTRODUCTION TO DIFFERENTIAL TOPOLOGY

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Abstract. These are the notes for a course taught during the Utrecht Summer School on Geometry 2023 for undergrads. Their aim is to introduce the students to various concepts in Differential Topology (transversality, jet spaces, immersion theory). The gimmick is that the whole discussion is based around the Whitney-Graustein theorem, which says that the space of immersions of the circle into the plane has countably many path-components. Proving the result allows us to introduce various important definitions and ideas as we go along.

1. Introduction

The following notion is the main character in this minicourse:

Definition 1. *An* ***immersion*** *of the circle*[1] *into the plane is a C^1 map $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ such that $\gamma'(t) \neq 0$ for every $t \in \mathbb{S}^1$.*

We will spend a lot of time thinking about the set of all immersions:

$$\mathrm{Imm} := \{\gamma \in C^1(\mathbb{S}^1, \mathbb{R}^2) \mid \gamma \text{ is an immersion}\}.$$

Remark 2. *Given an interval I, we can analogously define an immersion $\nu : I \to \mathbb{R}^2$ as a map such that $\nu' \neq 0$. These will play an auxiliary role as we go along. More generally, one can talk about immersions in arbitrary dimension; for more information regarding this, refer to Appendix 11.1.* △

Let us consider a few examples; you should refer to Figure 1. The first curve γ_1 is the standard inclusion of $\mathbb{S}^1$ into $\mathbb{R}^2$. The second curve γ_2 is quite different: you may say that it turns twice counterclowise, instead of once. Similarly, the third curve γ_3 turns thrice counterclockwise. In contrast, the fourth curve γ_4 is very similar to γ_1: even though it is more wobbly, it also turns once counterclockwise. In fact, we can make a stronger statement: γ_1 and γ_4 are equivalent in the sense that we can **deform** one to the other; this is shown in Figure 2. You may want to check Figure 5, where the concept of rotating is explained (although it is not strictly necessary for now).

Deformations are defined rigorously in Section 2. It turns out that deformation defines an equivalence relation. This motivates us to ask the following question:

Key Question 3. Imm *is a pretty large set (it is uncountable, for instance). However, how many distinct immersions are there up to deformation? I.e. how large is the set*

$$\frac{\mathrm{Imm}}{\textit{deformation}}?$$

Here is the spoiler: $\dfrac{\mathrm{Imm}}{\mathrm{deformation}}$ can be identified with $\mathbb{Z}$: two immersions can be deformed to one another if and only if they "turn the same amount" (and the amount of turning is precisely an integer).

This question may seem rather elementary but making this answer rigorous is non-trivial. Doing so will allow us to discuss many important concepts in Differential Topology (which is a massive field within Mathematics).

[1] Recall that we can regard $\mathbb{S}^1$ as the subset of unit complex numbers. When we do this, we write z for the complex coordinate in $\mathbb{S}^1$. Alternatively, we can see $\mathbb{S}^1$ as the quotient $\mathbb{R}/\mathbb{Z}$ or, equivalently, as $[0,1]$ with endpoints identified. In this case we write t for our coordinate. $\gamma'(t) \in \mathbb{R}^2$ is then the velocity vector of γ at the point $\gamma(t)$.

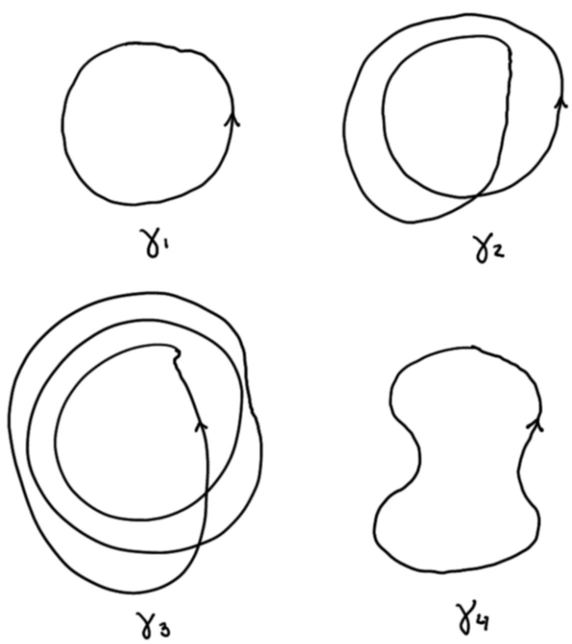

FIGURE 1. Four different immersions of the circle into $\mathbb{R}^2$.

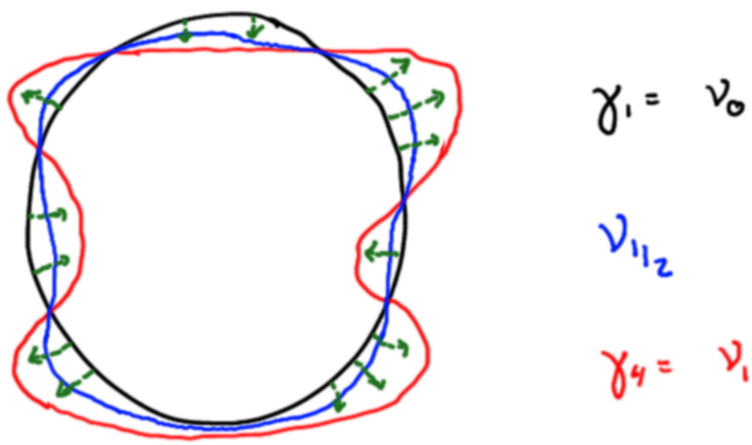

FIGURE 2. A deformation: a family of curves $(\nu_s)_{s\in[0,1]}$ interpolating between γ_1 and γ_4. The family is continuous in s and each ν_s is itself an immersion.

2. What does it mean to deform an immersion?

Our first goal is to explain rigorously the meaning of "deforming" an immersion to another immersion. The first step is to realise that Imm is more than just a set. It turns out that $C^1(\mathbb{S}^1, \mathbb{R}^2)$ is a metric space, Imm is a subspace, and deformations are simply paths in Imm. Let us elaborate.

2.1. **The Whitney distances.** Instead of focusing on C^1-curves, it is handy to work in more generality:

Definition 4. *Fix a pair of C^r-curves $\gamma, \nu : \mathbb{S}^1 \to \mathbb{R}^2$. The C^r-**distance** between γ and ν is defined as:*

$$d_{C^r}(\gamma, \nu) := \sum_{i=0}^{r} \sup_{t\in\mathbb{S}^1} |\partial^i\gamma(t) - \partial^i\nu(t)|.$$

I.e. d_{C^r} measures how far γ and ν and their derivatives up to order r are from one another. One can define also a C^∞-distance, but this is unnecessary for our purposes today.

Exercise 5. *Verify that that d_{C^r} satisfies the axioms of a distance.* △

Remark 6. *Even more generally: let $K \subset \mathbb{R}^n$ be a compact which is the closure of an open domain. Observe that one could define the C^r-distance between two C^r-maps $f, g : K \to \mathbb{R}^m$ by taking the sum*

$$\sum_{|I| \leq r} \sup_{p \in K} |\partial^I f(p) - \partial^I g(p)|$$

over all suprema of differences of partial derivatives up to order r. Here I denotes a multi-index $(i_1, \dots, i_n)$ so that ∂^I means the partial derivative $\partial^{i_1}_{x_1} \dots \partial^{i_n}_{x_n}$.

Even more generally, you could define the C^r-distance for functions between arbitrary closed manifolds. △

From the exercise we conclude:

Corollary 7. *For every integer $k \geq r$, the pair $(C^k(\mathbb{S}^1, \mathbb{R}^2), d_{C^r})$ is a metric space.*

When writing $C^r(\mathbb{S}^1, \mathbb{R}^2)$, without reference to a distance, it is customary to assume that it has been endowed with the C^r-distance.

Immersions are defined in terms of an strict inequality $\gamma(t)' \neq 0$ involving the first derivative. The solutions of this inequality (for each $t \in \mathbb{S}^1$) are the non-zero vectors, which is an open in $\mathbb{R}^2$. We use this observation to show that:

Proposition 8. $\mathrm{Imm} \subset C^1(\mathbb{S}^1, \mathbb{R}^2)$ *is an open subspace.*

Proof. Given $\gamma \in \mathrm{Imm}$, we are going to find $\varepsilon > 0$ such that the ε-ball around γ is contained in Imm. The key observation is that there is a number $C > 0$ such that $|\gamma'(t)| \geq C$ for all C. Indeed, $\mathbb{S}^1$ is compact and $|\gamma'| : \mathbb{S}^1 \to \mathbb{R}$ is a positive continuous function, so its infimum C is its minimum, which is then positive. You can now set $0 < \varepsilon < C$ and verify that any $\nu \in C^1(\mathbb{S}^1, \mathbb{R}^2)$ that is ε-close to γ satisfies $|\nu'| \geq C - \varepsilon > 0$, as desired. □

That is, every map close to an immersion is an immersion as well.

Exercise 9. *Fix an open subset $A \subset \mathbb{R}^2$. Prove that the subspace*

$$S := \{\gamma \in C^1(\mathbb{S}^1, \mathbb{R}^2) \mid \gamma'(t) \in A \text{ for all } t\} \subset C^1(\mathbb{S}^1, \mathbb{R}^2)$$

is open. △

Exercise 10. *Prove that* Imm *is not an open subspace of $(C^1(\mathbb{S}^1, \mathbb{R}^2), d_{C^0})$. Namely: You have to find an immersion γ and a sequence of maps $\gamma_i : \mathbb{S}^1 \to \mathbb{R}^2$ such that:*

- *$\gamma = \lim_{i \to \infty} \gamma_i$ with respect to the C^0-distance,*
- *none of the γ_i is an immersion.*

Hint: *You can pick γ to be any immersion, the usual inclusion of the circle will do. Moreover, writing the γ_i is very challenging and requires the use of bump functions. Instead of being completely formal, it is enough if you draw (or describe intuitively) what these γ_i must do.* △

2.2. **Paths of curves.** Being metric spaces, it follows that we can speak of continuous paths into $C^1(\mathbb{S}^1, \mathbb{R}^2)$ (and Imm). Let us discuss this a bit more carefully:

Proposition 11. *Let $\gamma, \nu \in C^1(\mathbb{S}^1, \mathbb{R}^2)$. Fix coordinates (s, t) in $[0, 1] \times \mathbb{S}^1$. There is a bijective correspondence between:*

- *Continuous paths $\Gamma : [0, 1] \to C^1(\mathbb{S}^1, \mathbb{R}^2)$ from γ to ν.*
- *Maps*

$$F : [0, 1] \times \mathbb{S}^1 \to \mathbb{R}^2$$

satisfying

- *$f_0 = \gamma$ and $f_1 = \nu$. Here $f_s := F(s, -)$,*
- *F is differentiable in the t-variable, meaning that each f_s is C^1,*
- *F and $\partial_t F$ are continuous.*

Proof. Given Γ we can define F (and viceversa) using the identity $F(s,t) = \Gamma(s)(t)$. Then we observe that the first property of F is equivalent to starting at γ and finishing at ν for Γ. Similarly, F being C^1 in t is equivalent to the fact that Γ takes values in $C^1(\mathbb{S}^1, \mathbb{R}^2)$. It remains to prove that Γ being continuous is equivalent to the continuity of F and $\partial_t F$.

Since $C^1(\mathbb{S}^1, \mathbb{R}^2)$ is a metric space, we see that G is continuous if and only if it is sequentially continuous, i.e. if and only if $s \to s_0$ implies that $\Gamma(s, -) = f_s$ approaches $\Gamma(s_0, -) = f_{s_0}$. The curve f_s approaches f_{s_0} if and only if $d_{C^1}(f_s, f_{s_0})$ goes to zero, which means that $f_s(t)$ is approaching $f_{s_0}(t)$ and $f'_s(t)$ is approaching $f'_{s_0}(t)$, uniformly for all t. Since this is true for all s_0 and all t, this is equivalent to F and $\partial_t F$ being continuous, as claimed. □

Remark 12. *Continuous paths into $C^r(\mathbb{S}^1, \mathbb{R}^2)$ are a more refined version of homotopies, taking into account higher derivatives. For more information on homotopies, refer to Appendix 11.2.* △

It turns out that the theory of paths in $C^1(\mathbb{S}^1, \mathbb{R}^2)$ is not very interesting. Namely, any two curves can be connected by a path:

Exercise 13. *Let $\gamma, \nu \in C^1(\mathbb{S}^1, \mathbb{R}^2)$. Prove that there is a continuous path between the two.* ***Hint:*** *It is sufficient if you prove it for ν equal to the constant zero curve.*

This can be restated as:

Corollary 14. *$\pi_0(C^1(\mathbb{S}^1, \mathbb{R}^2))$ has a single element.*

Proof. Recall the following facts: Two points in a metric (or, more generally, topological) space are path-connected if there is a continuous path between the two. This is an equivalence relation (exercise for you!). π_0 denotes the set of equivalence classes, which are usually called **path-components**. According to the previous example, in this concrete case there is a single equivalence class. □

2.3. **Deformations of immersions.** The story for immersions is much more interesting. Motivated by Proposition 11 we define:

Definition 15. *Let $\gamma, \nu \in \mathrm{Imm}$. A* ***deformation*** *(sometimes also called a* ***regular homotopy****) between γ and ν is a map*

$$F : [0,1] \times \mathbb{S}^1 \to \mathbb{R}^2$$

satisfying

- *$f_0 = \gamma$ and $f_1 = \nu$. Here $f_s := F(s, -)$.*
- *Each $f_s \in \mathrm{Imm}$.*
- *F and $\partial_t F$ are continuous.*

According to Proposition 11, deformations are precisely continuous paths in Imm. This means in particular (if you did the exercise suggested within the proof of Corollary 14) that:

Corollary 16. *Deformation defines an equivalence relation and*

$$\frac{\mathrm{Imm}}{\textit{deformation}} = \pi_0(\mathrm{Imm}).$$

Proof. Since deformations and paths in Imm are equivalent, the equivalence relations defining each side of the identity are the same. The claim follows. □

Using this language, we can rephrase Main Question 3 as follows:

Key Question 17. *What is the cardinality of the set $\pi_0(\mathrm{Imm})$?*

3. Examples of deformations

Let us play a bit with deformations and gain a bit more intuition by constructing examples.

3.1. Isotopies.

The first family of examples is based on the idea that moving $\mathbb{R}^2$ within itself in particular allows us to move every curve inside.

Definition 18. *A **diffeomorphism** $U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$ is a smooth homeomorphism whose inverse is also a smooth homeomorphism.*

*An **isotopy** of $\mathbb{R}^2$ is a smooth map*

$$F : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$$

such that

- *Each $f_s := F(s,-)$ is a diffeomorphism.*
- *f_0 is the identity.*

Diffeomorphisms are thus changes of coordinates and isotopies are like movies of coordinate changes, starting at the identity.

Exercise 19. *Let $(f_s : \mathbb{R}^2 \to \mathbb{R}^2)_{s\in[0,1]}$ be an isotopy. Let $\gamma \in \mathrm{Imm}$. Then $(f_s \circ \gamma)_{s\in[0,1]}$ is a deformation of γ.* △

The following exercise provides many examples of isotopies:

Exercise 20. *Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation-preserving linear isomorphism (i.e. a matrix with positive determinant). Prove that there is an isotopy $(A_s)_{s\in[0,1]}$ with $A_1 = A$. **Note**: the A_s can be chosen to be all linear too. **Hint**: Prove the statement for A in Jordan normal form first. Then, use the fact that any A is conjugate to one in normal form.* △

We can use the exercise to deduce:

Exercise 21. *Every $\gamma \in \mathrm{Imm}$ can be deformed to $\nu \in \mathrm{Imm}$ such that $\nu'(0)$ is of the form $(0,1)$ (i.e. it points upwards).* △

Remark 22. *The most common way of constructing an isotopy $F : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ is using a so-called **flow**. I.e. F is the solution of an ODE of the form*

$$(\partial_t F)(t,x,y) = G(t, F(t,x,y)),$$

where G is a function of the form $G : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$. Observe that $t \mapsto F(t,x,y)$ is the trajectory followed by the point $F(0,x,y)$ as the time t evolves. $(\partial_t F)(t,x,y)$ is then the velocity vector of such a trajectory. This means that G is prescribing the directions of the trajectories. A physical interpretation for this is that $\mathbb{R}^2$ contains a fluid that flows over time, with the fluid velocities given by G. Then F precisely describes how a concrete particle in said fluid moves.

Using the existence and uniqueness of ODEs, we can deduce that there are many isotopies. Indeed, we get an isotopy for each choice of (say, Lipschitz) function G. △

3.2. Reparameterisation by arc length.

Recall that a **reparametrisation** of a curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ is another curve of the form $\gamma \circ \rho$, where $\rho : \mathbb{S}^1 \to \mathbb{S}^1$ is a diffeomorphism. A reparametrisation is **orientation-preserving** if ρ' points counterclockwise.

Orientation-preserving reparametrisations are also examples of deformations:

Exercise 23. *Let $\gamma \in \mathrm{Imm}$ and let $\gamma \circ \rho$ be an orientation-preserving reparametrisation. Prove that the former can be deformed to the latter. **Note:** All immersions in the deformation can be assumed to be reparametrisations of γ. **Hint:** Try to deform ρ to the identity. To this end, you may want to lift ρ to a map $\tilde{\rho} : [0,1] \to \mathbb{R}$, as in Appendix 11.2.* △

Some parametrisations are particularly useful. Recall the following notion:

Definition 24. *The **length** of a curve γ is the quantity:*

$$\int_0^1 |\gamma'(t)|dt$$

I.e. the integral of the infinitesimal velocities $|\gamma'(t)|$.

Exercise 25. *Let γ be a curve and let $\gamma \circ \rho$ be an orientation-preserving reparametrisation. Prove that both have the same length.* △

We can now reparametrise an immersion so that it goes at constant speed:

Lemma 26. *Let $\gamma \in \mathrm{Imm}$ of length $C > 0$. Then, there is an orientation-preserving reparametrisation $\gamma \circ \rho$ such that $|(\gamma \circ \rho)'(t)| = C$ for all t.*

Proof. First, observe that the function $A : [0,1] \to [0,C]$ given by

$$A(s) = \int_0^s |\gamma'(a)|da$$

is strictly-increasing, differentiable with derivative $|\gamma'| > 0$, and takes 0 to 0 and 1 to C. It follows that it is a parametrisation of the interval $[0,C]$ using $[0,1]$.

We claim that ρ is uniquely defined by the property $A \circ \rho(t) = Ct$. Indeed, differentiating this expression one deduces $|(\gamma \circ \rho)'(t)| = C$. Moreover, ρ is seen to be differentiable and an increasing reparametrisation from the identity $\rho = A^{-1}(Ct)$ and the fact that A and $t \mapsto Ct$ are themselves increasing C^1-parametrisations. □

The curve $\gamma \circ \rho \in \mathrm{Imm}$ is said to be the **reparametrisation by arclength** of γ.

3.3. **Little bumps.** Lastly, we observe that we can take a curve γ and "perturb it" slightly around any of its points. It is best to do this first in the following simplified situation:

Exercise 27. *Prove that:*

- *A curve $\nu_1 : [-1,1] \to \mathbb{R}^2$ of the form $t \mapsto (t, b(t))$, with $b : [-1,1] \to \mathbb{R}$ differentiable, is an immersion.*
- *ν_1 can be deformed to $\nu_0(t) = (t, 0)$.*

Note: *For the second item you should convince yourself that one can define deformations for curves parametrised by an interval.*

I.e. graphs over the x-axis are immersions and all of them can be deformed to one another.

Moreover, one can produce deformations that are "concentrated" around a point. The main ingredient needed to prove this is the existence of so-called **bump functions**:

Proposition 28. *Fix a constant $0 < \delta < 1$. Then, there exists a function $\chi : [-1,1] \to [0,1]$ such that*

- *$\chi = 0$ in $[-1, -1+\delta] \cup [1-\delta, 1]$*
- *$\chi = 1$ in $[-1+2\delta, 1-2\delta]$.*

We will not prove this statement (but you will probably see it in your introductory course to differentiable manifolds?). Refer to Figure 3 for a pictorial depiction.

Putting these ingredients together we obtain the following corollary:

Exercise 29. *Consider the immersions $\nu_0, \gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\nu_0(t) = (t, 0)$ and $\gamma(t) = (t, b(t))$. Fix constants $\varepsilon, \delta > 0$. Then, there is a curve $\nu_1(t) = (t, c(t))$ such that*

- *$c = b$ in $[-\varepsilon, \varepsilon]$.*
- *$c = 0$ in $(-\infty, -\varepsilon - \delta] \cup [\varepsilon + \delta, \infty)$.*

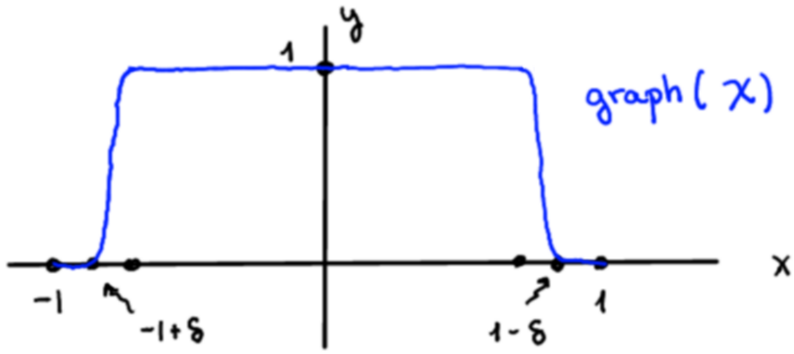

FIGURE 3. The bump function presented in Proposition 28.

Moreover, there is a deformation $(\nu_s(t) = (t, c_s(t))_{s\in[0,1]}$ *connecting* ν_0 *and* ν_1 *such that* $c_s(t) = 0$ *if* $|t| \geq \varepsilon + \delta$.

I.e. using the bump function χ, we are able to modify ν_0 and replace it, over the interval $[-\varepsilon, \varepsilon]$, by whatever graph we want. This means that there are many distinct immersions in the vicinity of ν_0. You should compare this to Proposition 8.

However, this is not a special fact about ν_0. We can argue more generally using the fact that any immersion looks almost straight at each of its points:

Lemma 30. *Let* $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ *be an immersion and let* $p \in \mathbb{S}^1$. *Then, there are*

- *a constant* $\varepsilon > 0$,
- *a map* $\rho : [-\varepsilon, \varepsilon] \to \mathbb{S}^1$,
- *an affine isomorphism* $A : \mathbb{R}^2 \to \mathbb{R}^2$,
- *a function* $f : [-\varepsilon, \varepsilon] \to \mathbb{R}$,

such that $\rho(0) = p$ *and* $A \circ \gamma \circ \rho(t) = (t, f(t))$.

Proof. We can argue as in Exercise 21 to find A. First, we let $C = \gamma(p)$. This means that $\gamma - C$ passes through zero at p. We then let B be a linear map such that $B \circ (\gamma - C)'(p) = (1, 0)$. Then A is the affine transformation $(x, y) \mapsto B((x, y) - C)$.

Once we do this, we can pick $\delta > 0$ small so that $(A \circ \gamma)(t) = (a(t), b(t))$ with $a'(t) \neq 0$ for every $t \in [p - \delta, p + \delta]$. Then ε is chosen to be any number small enough so that a^{-1} maps $[-\varepsilon, \varepsilon]$ into $[p - \delta, p + \delta]$. Lastly, we set $\rho := a^{-1}$ and $f := b \circ a^{-1}$. □

We can put the exercise together with the lemma to conclude:

Exercise 31. *Consider a point* $p \in \mathbb{S}^1$ *and a pair of immersions* γ, ν *such that* $\gamma'(p) = \nu'(p)$. *Then, there is a curve* $\tilde{\gamma} \in \mathrm{Imm}$ *such that:*

- $\tilde{\gamma}$ *is a deformation of* γ.
- $\tilde{\gamma}$ *agrees with* ν *in a neighbourhood of* p.

Hint: *Argue as in the lemma to rotate* γ *and* ν *in order to produce immersions whose derivative at* p *is* $(1, 0)$. *Then use the exercise to modify the rotated version of* γ *and make it look like the rotated version of* ν. *Lastly, undo the rotation.*

For a related example, refer to Figure 4.

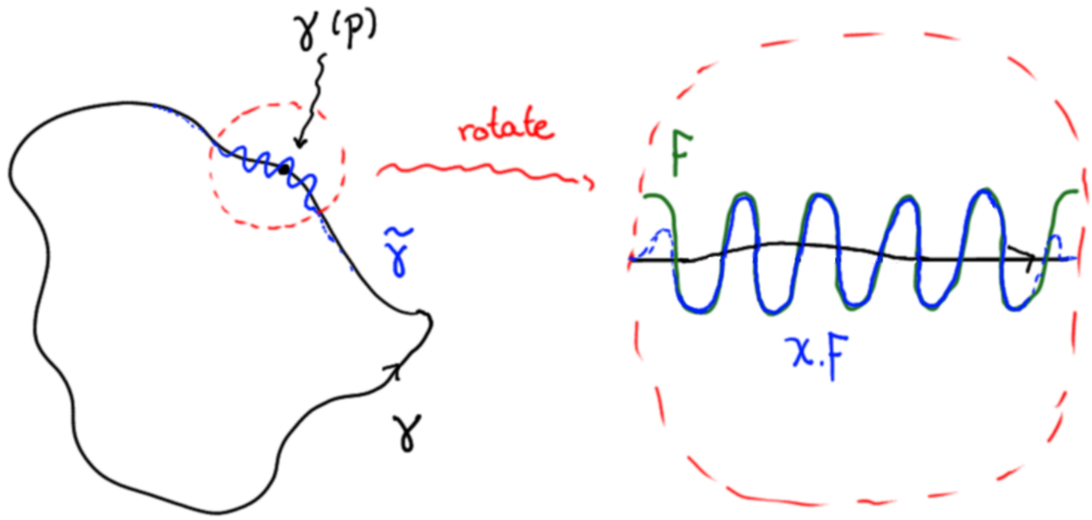

Figure 4. On the left, an immersion γ. We pick a point p and zoom-in around said point, making γ look somewhat straight (black curve on the right). We can now add some "wobble" to γ around p. To do so we take a function f that is wobbly (say, the sine times some small constant), we "cut if off" using our bump function χ, and we add χf to the local model of γ. We then translate this back to γ itself (undoing the zooming and rotating).

4. Stating the theorem of Whitney and Graustein

We are now ready to state our main result:

Theorem 32. *There is a bijection between* $\dfrac{\mathrm{Imm}}{\textit{deformation}}$ *and* $\mathbb{Z}$.

The claimed bijection takes a class $[\gamma]$ and sends it to "the number of times that γ' wraps around the origin". Our goal in this section is to make this bijection explicit.

4.1. **Maps between function spaces.** Being an immersion is a condition that is checked in terms of the first derivative. We have additionally made the claim that "how much γ' wraps" determines the deformation class of γ. This motivates us to take a small detour and talk about the relationship between the C^k-distances and taking derivatives.

In the following statement $C^r(\mathbb{S}^1, \mathbb{R}^2)$ is endowed with the C^r-distance.

Proposition 33. *Fix r a non-negative integer, possibly infinity, and an integer $0 \leq i \leq r$. The differential operator*

$$\partial^i : C^r(\mathbb{S}^1, \mathbb{R}^2) \to C^{r-i}(\mathbb{S}^1, \mathbb{R}^2)$$

that differentiates a function i times is continuous.

Proof. Observe that the following bound holds:

$$\begin{aligned} d_{C^{r-i}}(\partial^i\gamma, \partial^i\nu) &= \sum_{j=0}^{r-i} \sup_{t\in\mathbb{S}^1} |\partial^j\partial^i\gamma(t) - \partial^j\partial^i\nu(t)| \\ &= \sum_{j=i}^{r} \sup_{t\in\mathbb{S}^1} |\partial^j\gamma(t) - \partial^j\nu(t)| \leq d_{C^r}(\gamma, \nu) \end{aligned}$$

which implies that the preimage of the ε-ball centered at $\partial^i\gamma$ contains the ε-ball centered at γ, proving the claim. $\square$

The concrete case that is of interest to us is the following:

Definition 34. *The* ***scanning map*** *(for immersions of the circle into $\mathbb{R}^2$) is the continuous map*

$$\Psi : \mathrm{Imm} \to C^0(\mathbb{S}^1, \mathbb{R}^2 \setminus 0)$$

sending γ to the loop γ'.

All we have done here is restrict $\partial : C^1(\mathbb{S}^1, \mathbb{R}^2) \to C^0(\mathbb{S}^1, \mathbb{R}^2)$ to Imm which, by assumption, forces the image to takes values in $C^0(\mathbb{S}^1, \mathbb{R}^2 \setminus 0)$.

4.2. **Intermezzo: Exercises about differential operators.** In Remark 6 we defined the Whitney distances for maps $K \to \mathbb{R}^m$, with K a compact that is the closure of an open domain in $\mathbb{R}^n$. One can consider differential operators in this more general setting. We will pursue this in a series of exercises:

Exercise 35. *Prove that $C^r(K, \mathbb{R}^m)$ is a vector space over $\mathbb{R}$. Prove that the C^r-distance is compatible with the vector space structure, i.e. adding a vector preserves the C^r-distance and multiplying by a scalar dilates the distance.*

You may want to convince yourself that $C^r([0,1], \mathbb{R}^m)$ is infinite dimensional.

Definition 36. *A* ***differential operator*** *P of order i is a continuous map $C^r(K, \mathbb{R}^m) \to C^{r-i}(K, \mathbb{R}^m)$.*

Exercise 37. *Let $P : C^r(K, \mathbb{R}^m) \to C^{r-i}(K, \mathbb{R}^m)$ be the linear combination $\sum_{0 \leq |I| \leq i} a_I \partial^I$, where $a_I : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix.*

- *Prove that P is a linear map.*
- *Prove that P is a differential operator.*

Exercise 38. *In particular, the laplacian $\sum_i \partial^2_{x_i}$ is a linear differential operator of order 2.*

4.3. **The statement.** Fix a continuous path $\Gamma : [0,1] \to C^r(\mathbb{S}^1, \mathbb{R}^2)$. According to Proposition 33, we can compose Γ with ∂^i to yield another continuous path $\partial^i \circ \Gamma : [0,1] \to C^{r-i}(\mathbb{S}^1, \mathbb{R}^2)$. We can particularise this to the scanning map to deduce:

Corollary 39. *There is a well-defined map*

$$\pi_0(\Psi) : \pi_0(\mathrm{Imm}) \;\to\; \pi_0(C^0(\mathbb{S}^1, \mathbb{R}^2 \setminus 0))$$

sending $[\gamma]$ to $[\gamma']$.

Proof. All we have to verify is that $\pi_0(\Psi)$ does not depend on the representative of the class $[\gamma]$. Any other representative $\nu \in \mathrm{Imm}$ is, by definition, equivalent up to deformation to γ. This means that there is a path Γ in Imm from γ to ν. It follows, as stated above, that there is a path $\partial_t\Gamma$ (continuous in the C^0-topology) between γ' and ν', meaning that they represent the same class in $\pi_0(C^0(\mathbb{S}^1, \mathbb{R}^2 \setminus 0))$, as claimed. □

At this point you should refer to Appendix 11.2, where the notion of homotopy is explained. The result we need is the following:

Theorem 40. *There is a bijection*

$$\phi : \pi_0(C^0(\mathbb{S}^1, \mathbb{R}^2 \setminus 0)) \to \mathbb{Z}.$$

Namely, ϕ identifies the class $[\alpha]$ of a loop α with how much α wraps (i.e. "how many times it goes around the origin").

Proving this result is quite involved. It is one of the first results one builds towards when studying Algebraic Topology. As such, we will treat it as a blackbox, but some intuition about it is provided in the Appendix.

We can put the bijection ϕ together with the function $\pi_0(\Psi)$ to define:

Definition 41. *The **rotation** function for immersions*

$$\mathrm{rot} : \mathrm{Imm} \to \mathbb{Z}$$

is given by the expression

$$\mathrm{rot}(\gamma) := \phi([\gamma']).$$

By construction, rot only depends on γ up to deformation, so we often abuse notation and write $\mathrm{rot} : \pi_0(\mathrm{Imm}) \to \mathbb{Z}$. It follows that we can restate Theorem 32 as:

Theorem 42. *The rotation function* $\mathrm{rot} : \pi_0(\mathrm{Imm}) \to \mathbb{Z}$ *is a bijection.*

Theorem 42 will be proven in Sections 5 and 6. The proof we discuss there is the original one [1], which Whitney attributes it to Graustein. In subsequent sections we will look at alternate proofs.

5. The original proof. Surjectivity

We present now the proof of Theorem 42. Naturally, we can divide it into two parts. In this section we prove that

$$\mathrm{rot} : \pi_0(\mathrm{Imm}) \to \mathbb{Z}$$

is surjective. Injectivity is shown in Section 6.

Surjectivity is the easiest part of the proof. All we have to do is, for each $a \in \mathbb{Z}$, exhibit a curve $\gamma_a \in \mathrm{Imm}$ such that γ_a' goes a times around the origin. We provide concrete formulas below, but you may just want to look at Figure 5 and verify that the curves shown there do the job.

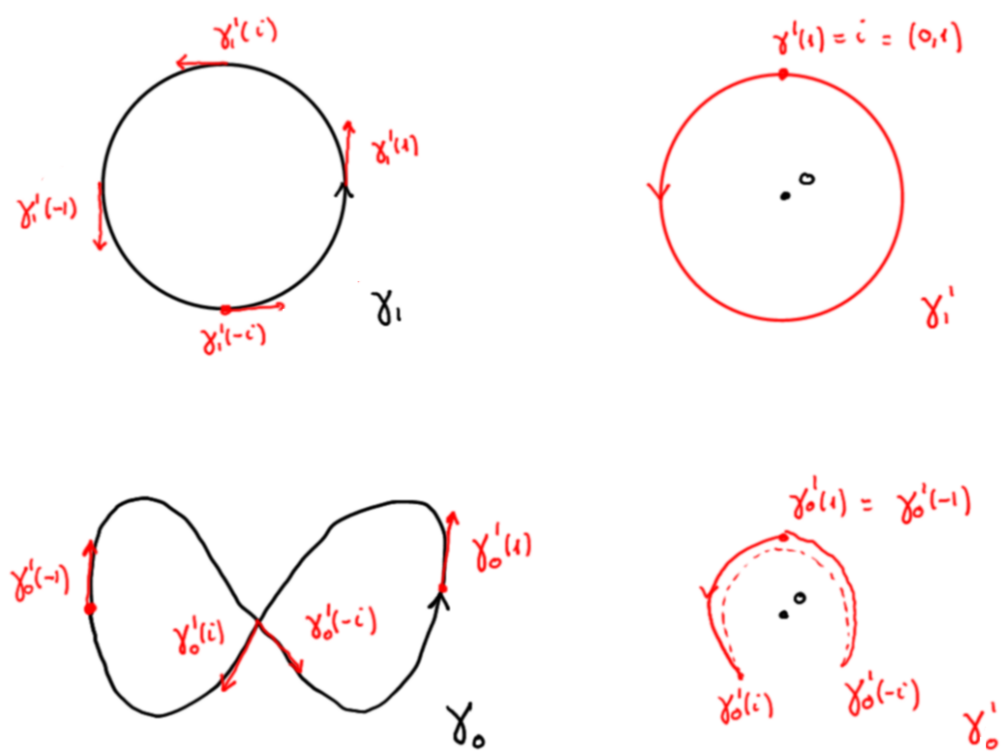

FIGURE 5. On the left, the curves γ_a such that γ_a' wraps a times around $\mathbb{S}^1$. We only show γ_1, as all γ_a with $a \neq 0$ are iterates of it, and γ_0. On the right, the derivatives γ_1' and γ_0' are drawn, showing that they go around the origin exactly one and zero times, respectively.

5.1. **First case: rotation different than zero.** For each $a \neq 0$, we set

$$\gamma_a(z) := z^a,$$

where we are also identifying $\mathbb{R}^2$ with the complex plane. Using standard coordinates $(x, y) \in \mathbb{R}^2$ this can equivalently be expressed as:

$$\gamma_a(t) = (\cos(2\pi at), \sin(2\pi at))$$
$$\gamma_a'(t) = a(-\sin(2\pi at), \cos(2\pi at)).$$

The derivative is readily seen to wrap a times, as claimed.

5.2. **Second case: rotation equal to zero.** For the case $a = 0$, consider:

$$\gamma_0(t) := (\cos(2\pi t), \sin(4\pi t))$$
$$\gamma_0'(t) = (-\sin(2\pi t), 2\cos(4\pi t)).$$

First, we observe that γ_0 is indeed immersed. The sine function vanishes only when $t \in \frac{1}{2}\mathbb{Z}$ and $\cos(2t)$ vanishes when $2t \in \frac{1}{2}\mathbb{Z} + 1/4$, i.e. when $t \in \frac{1}{4}\mathbb{Z} + 1/8$. That is, the two entries of γ_0' do not vanish simultaneously for any t.

Exercise 43. *Verify that γ_0' wraps zero times. As a preliminary step, check that γ_0' never points downwards.*

This concludes the surjectivity of rot. □

6. The original proof. Injectivity

The injectivity argument is the more difficult part of the proof. It is also the aspect that lends itself to many different approaches (each providing a different topological insight). Some future version of the notes will contain such alternative proofs.

Let us state the goal clearly: Given two immersions $\gamma, \nu \in \mathrm{Imm}$ with $\mathrm{rot}(\gamma) = \mathrm{rot}(\nu)$ (i.e. such that γ' and ν' wrap the same amount of times around the origin) we must prove that γ and ν can be deformed to one another.

6.1. **Simplification of the starting immersions.** Before we get to the argument itself, it is convenient to deform γ and ν to make them a bit nicer. You can prove the following statement combining some of our previous exercises:

Exercise 44. *There are immersions α and β such that:*

- *α is a deformation of γ, β is a deformation of ν,*
- *both have length 1 and are parameterised by arclength,*
- *both agree close to $1 \in \mathbb{S}^1$,*
- *α' and β' are not constant close to $1 \in \mathbb{S}^1$.*

The last two properties are a bit technical, but they will be used in our upcoming proof. In any case: the proof will be complete if we show that α can be deformed to β.

6.2. **Main ingredients of the proof.** Now we make two key observations. Here is the first:

Lemma 45. *There is a homotopy of curves*

$$(\eta_s : [0, 1] \to \mathbb{S}^1)_{s\in[0,1]}$$

beginning at $\eta_0 = \alpha'$ and finishing at $\eta_1 = \beta'$ such that:

- *η_s agrees with α and β close to $1 \in \mathbb{S}^1$.*
- *In particular, η_s is not constant close to $1 \in \mathbb{S}^1$.*

Proof. Since we parametrised by arclength, α' and β' have unit length and are therefore maps $\mathbb{S}^1 \to \mathbb{S}^1$. Moreover, α' and β' agree with each other close to 1.

Separately, according to Theorem 40, our original derivatives γ' and ν' were homotopic to each other as maps $\mathbb{S}^1 \to \mathbb{R}^2 \setminus \{0\}$, since they wrap around the origin the same amount of times. This is equivalent to the fact that α' and β' are homotopic as maps $\mathbb{S}^1 \to \mathbb{S}^1$. This means that a homotopy η_s exists between the two. The two additional properties follow by inspecting the proof of Theorem 40 and making small adaptations (but, as we said, this will remain a blackbox). □

The second observation is that α can easily be recovered from α'. Indeed, according to the fundamental theorem of calculus

Lemma 46. $\alpha(t) = \alpha(0) + \int_0^t \alpha'(a)da$.

Which, together with the fact that $\alpha(0) = \alpha(1)$, forces the added constraint

$$A(\alpha') := \int_0^1 \alpha'(s)ds = 0, \tag{1}$$

meaning that the average of α' vanishes. The same is true for β.

These observations suggest that we consider the curves

$$f_s(t) := \eta_s(0) + \int_0^t \eta_s(a)da.$$

However, these are not necessarily closed. This is because the average

$$A(\eta_s) := \int_0^1 \eta_s(a)da \tag{2}$$

may not be zero. As such, f_s is a continuous curve $[0,1] \to \mathbb{R}^2$, instead of having $\mathbb{S}^1$ as domain.

In order to conclude the proof, we have to modify each η_s, for all s simultaneously, in order to change the integral $A(\eta_s)$ and equate it to zero.

6.3. **Concluding the proof.** The final piece of the puzzle is the following observation:

Lemma 47. *The average $A(\eta_s)$ appearing in Equation 2 has norm strictly smaller than* 1.

Proof. Indeed, we readily see that

$$|A(\eta_s)| \leq \int_0^1 |\eta_s(a)|da = \int_0^1 da = 1$$

and the equality holds if and only if η_s is constant. However, not being constant is precisely one of the (technical) properties we obtained in Lemma 45. □

Remark 48. *One can ignore the technical properties of Lemma 45 in the case* $\mathrm{rot}(\gamma) \neq 0$. *Indeed, under this assumption we have that γ', ν', α', β' and all η_s wrap around a non-zero amount of times around the origin so they cannot be constant. It follows that we do not have to force such a property "by hand".* △

The lemma implies, in particular, that $A(\eta_s) \neq \eta_s(t)$ for all t and s, since the later has norm exactly 1. It follows that the curve $\tilde{\eta}_s := \eta_s - A(\eta_s)$ takes values in $\mathbb{R}^2 \setminus 0$ and verifies $A(\tilde{\eta}_s) = 0$. Therefore, the curves

$$g_s(t) := \tilde{\eta}_s(0) + \int_0^t \tilde{\eta}_s(a)da$$

are immersions $\mathbb{S}^1 \to \mathbb{R}^2$, providing the desired deformation between α and β. □

6.4. **Intermezzo: convex curves.** Using the ideas we have seen so far, it turns out that we can also classify the convex curves in $\mathbb{R}^2$:

Definition 49. *A C^2 curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ is said to be* ***convex*** *if $\{\gamma'(t), \gamma''(t)\}$ is a positive basis of $\mathbb{R}^2$, for each $t \in \mathbb{S}^1$.*

The set of all convex curves is denoted by Convex and is a subspace of $C^2(\mathbb{S}^1, \mathbb{R}^2)$.

Exercise 50. *Prove that* Convex *is a subset of* Imm.

Exercise 51. *Prove that* Convex *is an open subset of $C^2(\mathbb{S}^1, \mathbb{R}^2)$.*

Exercise 52. *Prove that* Convex *is not an open subset of $(C^2(\mathbb{S}^1, \mathbb{R}^2), d_{C^1})$.*

Exercise 53. *Prove that* $\mathrm{rot} : \mathrm{Convex} \to \mathbb{Z}$ *takes values in $\mathbb{Z}^+$, the positive integers.*

Exercise 54. *Prove that* rot *defines a map* $\pi_0(\mathrm{Convex}) \to \mathbb{Z}^+$*.*

Exercise 55. *Prove that* $\mathrm{rot} : \pi_0(\mathrm{Convex}) \to \mathbb{Z}^+$ *is surjective.*

The following is the most difficult exercise. Basically, you should take the full proof of injectivity we did above and check step by step that you can adapt it. Something to keep in mind as you go along is that, in fact, many of the steps become simpler:

Exercise 56. *Prove that* $\mathrm{rot} : \pi_0(\mathrm{Convex}) \to \mathbb{Z}^+$ *is injective.*

7. Transversality

Other proofs of Theorem 42 rely on the idea of so-called transversality. Transversality is also interesting by itself, since it will tell us a lot about the structure of the space of all curves.

The basic motivation behind it is the following question: *How do "generic" curves behave?* Roughly speaking, the answer is that you can produce some pretty ugly curves by hand, but "most" curves behave extremely nicely. This insight turns out to be extremely powerful: whenever you want to do something with a curve, you first deform it a bit to be "generic" and, now that it is much nicer, it is easier to argue.

Remark 57. *More generally one can study "generic" maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ or between manifolds. For simplicity we keep focusing on curves.* △

7.1. **Affine transversality.** First we consider the linear case:

Definition 58. *Consider the vector space $\mathbb{R}^n$, a vector subspace A of dimension $a < n$, and a vector subspace L of dimension 1. We say that A and L are* ***transverse*** *if $A + L$ has dimension $a + 1$.*

Recall that $\mathbb{RP}^{n-1}$, the $(n-1)$-dimensional projective space, is the space of all lines in $\mathbb{R}^n$. Identically, it is the set of equivalence classes of vectors in $\mathbb{R}^n \setminus \{0\}$, up to scaling. We can use it to formalise the fact that transversality occurs generically:

Lemma 59. *Fix a linear subspace $A \subset \mathbb{R}^n$ of dimension a. Write* $\mathrm{Trans}(A) \subset \mathbb{RP}^{n-1}$ *for the set of lines transverse to L. Then,* $\mathrm{Trans}(A)$ *is open and dense.*

Proof. As stated above, $\mathbb{RP}^{n-1}$ is a quotient of $\mathbb{R}^n \setminus \{0\}$. Under this quotient, A is mapped to $\mathbb{RP}(A)$, the set of lines contained in A. Due to the properties of the quotient topology, $\mathbb{RP}(A)$ is closed because A is closed. $\mathrm{Trans}(A)$ is the complement of $\mathbb{RP}(A)$, so it is open.

Moreover, given a line in $\mathbb{RP}(A)$, you can push it slightly and yield a curve not in A, proving that $\mathrm{Trans}(A)$ is dense. □

A more quantitative way of thinking about this is that $\mathbb{RP}^{n-1}$ has dimension $n-1$ but the set of non-transverse lines $\mathbb{RP}(A)$ has dimension $a-1$ (because it is homeomorphic to $\mathbb{RP}^{a-1}$, meaning that it is much smaller.

We can take this a step further and look into the affine case:

Definition 60. *Consider the affine space $\mathbb{R}^n$, an affine subspace A of dimension $a < n$, and an affine line L. We say that A and L are* ***transverse*** *if either:*

- *$a < n-1$ and they are disjoint.*
- *$a = n-1$ and L is not contained in A.*

The claim is that the set of affine lines transverse to A is open and dense. Here are a couple of examples:

Example 61. *In the planar case ($n = 2$ and $a = 1$) being transversal means not being A. This is true for all curves except A itself.* $\triangle$

Example 62. *Let $n = 3$ and $a = 1$. A curve is transverse to A if it is disjoint from it. In order to prove openess observe that if A and L are disjoint, they are separated by some non-zero distance $C > 0$. It follows that any line at distance at most $C/2$ from L is a least $C/2$ away from A. To prove density, note that there are two cases: If $L = A$ then you can push L slightly in the orthogonal direction to produce a curve that is disjoint. If L is not A, but they share a point, then together they determine an affine plane P. Then you can push L orthogonally to P.* $\triangle$

Now you can prove the more general statement:

Exercise 63. *Fix an affine subspace $A \subset \mathbb{R}^n$ of dimension $a < n$.*

- *Verify that the space of all affine lines can be identified with:*
$$\{(L, p) \in \mathbb{RP}^{(n-1)} \times \mathbb{R}^n \mid p \text{ and } L \text{ are orthogonal}\}.$$
- *Using this, describe the set of affine lines transverse to A.*
- *Using the previous descriptions, show that the set of affine lines transverse to A is open and dense.*

7.2. **Transversal curves.** In the previous subsection we looked at linear/affine curves and how they interact with linear/affine subspaces. These ideas can be translated to the setting of general curves. First, we generalise linear subspaces to the non-linear setting:

Definition 64. *A subset $N \subset \mathbb{R}^n$ is a* ***submanifold*** *of dimension $n - m$ if there is a differentiable function $F : \mathbb{R}^n \to \mathbb{R}^m$ such that*

- *$N = F^{-1}(q)$ for some $q \in \mathbb{R}^m$,*
- *d_pF has rank m at all points $p \in N$.*

Let us unpack the definition: For each $q \in \mathbb{R}^m$, we can consider the level set $N = F^{-1}(q)$. The way to think about this is that F is a collection of m (non-linear) equations and N is the set of solutions associated to q. Suppose then that d_pF has rank m at all points $p \in N$. If F were linear, this would mean that F is a set of m linearly independent equations, meaning that its solution space is $(n-m)$-dimensional. We are not requiring F to be linear. However, under our rank m assumption, we can apply the implicit function theorem and show that, around a given $p \in N$, there are coordinates so that N is $\mathbb{R}^{n-m}$ and F is the projection to the last m coordinates. I.e. there are coordinates in which F and N do look linear!

Definition 65. *Fix a submanifold $N = F^{-1}(q) \subset \mathbb{R}^n$ of dimension $n-m$ and a C^1-curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$.*

We say that γ ***is transverse to*** *N if:*

- *$m > 1$ and $\gamma(\mathbb{S}^1)$ is disjoint from N.*
- *$m = 1$ and for every $\gamma(t) \in N$, it holds that $\gamma'(t)$ is not in the kernel of $d_{\gamma(t)}F$.*

The second condition means that, in the coordinates in which $N \cong \mathbb{R}^{n-m}$, whenever γ intersects N, γ' has to be linearly independent from it. The situation is depicted in Figure 6.

The following result is (a concrete case) of Thom transversality, which is one of the fundamental results in Differential Topology:

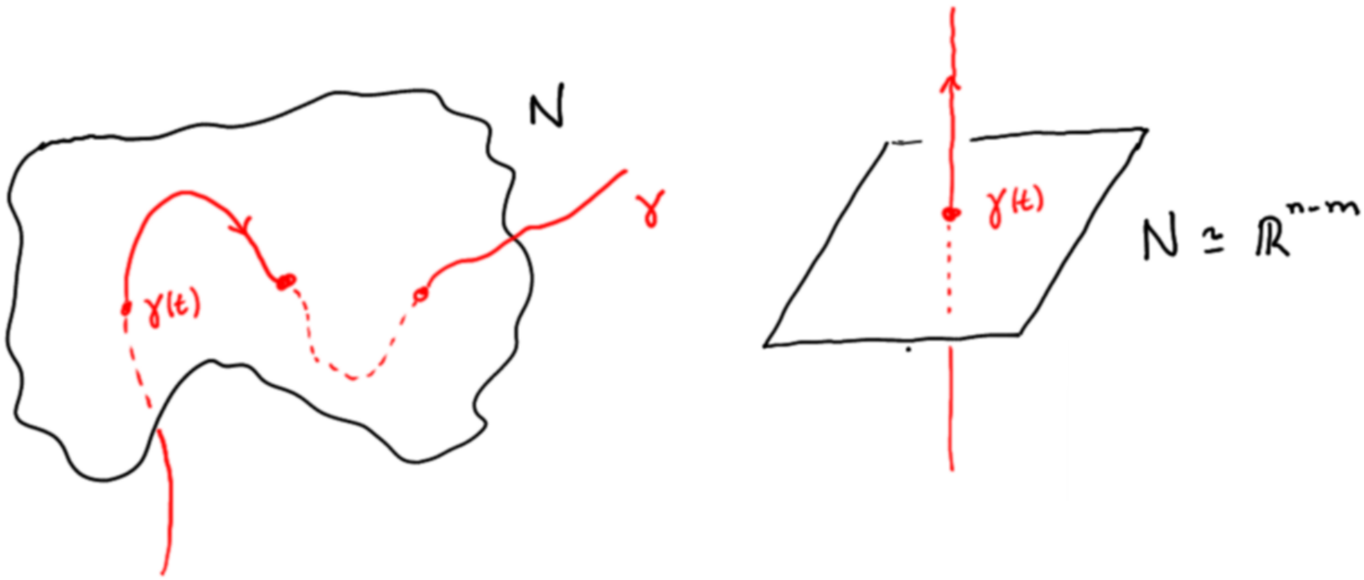

FIGURE 6. On the left, a submanifold $N \subset \mathbb{R}^3$ of dimension 2 (i.e. a surface) and an immersion γ. They intersect transversely in three points. Transversality means that we can change coordinates around the first intersection point (right picture) so that N looks like horizontal $\mathbb{R}^2$ and γ like the vertical axis. I.e. the intersection resembles the affine case.

Theorem 66. *Fix*

- *a constant* $\varepsilon > 0$,
- *a compact* $A \subset \mathbb{R}^k$ *given as the closure of some open domain,*
- *a continuous map* $E : A \to C^r(\mathbb{S}^1, \mathbb{R}^n)$,
- *a submanifold* $N \subset \mathbb{R}^n$ *of dimension* $n - m$.

Suppose that $k + 1 < m$. *Then, there exists another continuous map* $\tilde{E} : A \to C^r(\mathbb{S}^1, \mathbb{R}^n)$ *such that:*

- $d_{C^r}(E(a), \tilde{E}(a)) < \varepsilon$ *for all* $a \in A$.
- *All* $E(a)$ *are disjoint from* N.

Proving this is beyond the scope of these notes. The intuitive reasoning (building on what we said in the affine case) is that: we are looking at a k-dimensional family of curves E, each of which has dimension 1, and at a submanifold $N \subset \mathbb{R}^n$ that has dimension $n - m$. Since $(n+1-m)+k$ is smaller than n, there is enough space for E not to meet N if we perturb it a bit.

Corollary 67. *A generic curve avoids any submanifold of dimension at most* $n - 2$.

7.3. **Thom transversality in jet spaces.** Theorem 66 provides a partial answer to the question "how does a generic curve behave?" Namely, the theorem says that a family of curves in $\mathbb{R}^n$ will avoid, generically, all subsets of $\mathbb{R}^n$ whose dimension is sufficiently small. This is a rather specific statement, but we can improve the theorem a bit in order to obtain a much more general result. We need the following framework:

Definition 68. *Consider a* C^r *curve* $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$. *Its* r***-jet*** *is the curve*

$$j^r\gamma : \mathbb{S}^1 \to \mathbb{S}^1 \times \coprod^{r+1} \mathbb{R}^n$$

given by $j^r\gamma(t) := (t, \gamma(t), \gamma'(t), \gamma''(t), \cdots, \partial^r\gamma(t))$.

For simplicity of notation we will write $J^r(\mathbb{S}^1, \mathbb{R}^n) := \mathbb{S}^1 \times \coprod^{r+1} \mathbb{R}^n$, and we call it the r**-jet space**. Points in this space are of the form $(t, z_0, \cdots, z_r)$, where z_0 represents a point in $\mathbb{R}^n$, z_1 represents a velocity vector, z_2 a second derivative and so on. Altogether, $(t, z_0, \cdots, z_r)$ represents an r-order Taylor polynomial of curve in $\mathbb{R}^n$.

Exercise 69. *Define a subset* $\mathcal{R}_{\mathrm{Imm}} \subset J^1(\mathbb{S}^1, \mathbb{R}^2)$ *as* $\{z_1 \neq 0\}$. *Observe that:*

- $\mathcal{R}_{\mathrm{Imm}}$ *is an open subset.*
- $J^1(\mathbb{S}^1, \mathbb{R}^2)$ *has dimension* 5 *and the complement of* $\mathcal{R}_{\mathrm{Imm}}$ *is a submanifold of dimension* 3.
- *A curve* $\gamma \in C^1(\mathbb{S}^1, \mathbb{R}^2)$ *belongs to* Imm *if and only if* $j^1\gamma$ *takes values in* $\mathcal{R}_{\mathrm{Imm}}$.

Exercise 70. *Define* $\mathcal{R}_{\mathrm{Convex}} \subset J^2(\mathbb{S}^1, \mathbb{R}^2)$ *as the set of points* (t, z_0, z_1, z_2) *such that* $\{z_1, z_2\}$ *form a positive basis of* $\mathbb{R}^2$. *Observe that:*

- $\mathcal{R}_{\mathrm{Convex}}$ *is an open subset.*
- $J^2(\mathbb{S}^1, \mathbb{R}^2)$ *has dimension* 7 *and the complement of* $\mathcal{R}_{\mathrm{Imm}}$ *is a submanifold of dimension* 6.
- *A curve* $\gamma \in C^2(\mathbb{S}^1, \mathbb{R}^2)$ *belongs to* Convex *if and only if* $j^2\gamma$ *takes values in* $\mathcal{R}_{\mathrm{Convex}}$.

Exercise 71. *Define* $\mathcal{R}_r \subset J^r(\mathbb{S}^1, \mathbb{R}^2)$ *as the set of points* $\{z_1 = z_2 = \cdots = z_r = 0\}$. *Observe that:*

- $J^2(\mathbb{S}^1, \mathbb{R}^2)$ *has dimension* $2r + 3$ *and* $\mathcal{R}_r$ *is a submanifold of dimension* 3.
- *For all* $r > 0$, *a curve* $\gamma \in C^r(\mathbb{S}^1, \mathbb{R}^2)$ *is constant if and only if* $j^r\gamma$ *takes values in* $\mathcal{R}_r$.

The point about all of these exercises is that they relate a subspace of curves (say Imm) to a subspace in jet space (say $\mathcal{R}_{\mathrm{Imm}}$). What we have gained in this way is that, whereas Imm is infinite dimensional, $\mathcal{R}_{\mathrm{Imm}}$ is not. The following theorem, which is a more general version of Thom transversality, says that a lot of the properties of $\mathcal{R}_{\mathrm{Imm}}$ transfer to Imm:

Theorem 72. *Fix*

- *a constant* $\varepsilon > 0$,
- *a compact* $A \subset \mathbb{R}^k$ *given as the closure of some open domain,*
- *a continuous map* $E : A \to C^r(\mathbb{S}^1, \mathbb{R}^n)$,
- *a submanifold* $N \subset J^r(\mathbb{S}^1, \mathbb{R}^n)$ *of dimension* $n(r + 1) + 1 - m$.

Suppose that $k + 1 < m$. *Then, there exists another continuous map* $\tilde{E} : A \to C^r(\mathbb{S}^1, \mathbb{R}^n)$ *such that:*

- $d_{C^r}(E(a), \tilde{E}(a)) < \varepsilon$ *for all* $a \in A$.
- *All* $j^rE(a)$ *are disjoint from* N.

I.e. now the jets of our functions are able to avoid submanifolds of jet space, generically.

7.4. **Applications.** We can now combine Theorem 72 with the previous exercises:

Exercise 73. *A generic curve* $\gamma \in C^1(\mathbb{S}^1, \mathbb{R}^2)$ *is immersed.*

Saying "a generic curve satisfies X" is interpreted as "there is an open and dense subspace in the space of curves such that X applies".

Exercise 74. *A generic path of curves* $E : [0, 1] \to C^1(\mathbb{S}^1, \mathbb{R}^2)$ *satisfies that* $\Gamma(s) \in \mathrm{Imm}$ *for all* s *except a finite collection.*

One may then wonder what happens precisely at the times $s_0 \in [0, 1]$ in which $\Gamma(s_0)$ is not immersed. This is explained in Figure 7.

Exercise 75. *A generic curve* $\gamma \in C^2(\mathbb{S}^1, \mathbb{R}^2)$ *is convex except in a finite collection of points.*

Exercise 76. *Let* $0 \leq k < 2r - 1$ *and let* $A \subset \mathbb{R}^r$ *be a compact given as the closure of an open domain. Prove that:*

- *A generic family of curves* $E : A \subset \mathbb{R}^k \to C^r(\mathbb{S}^1, \mathbb{R}^2)$ *satisfies* $j^rE(a)(t) \notin \mathcal{R}_r$ *for all* a *and* t.
- *In particular, no* $E(a)$ *is constant in an open interval.*

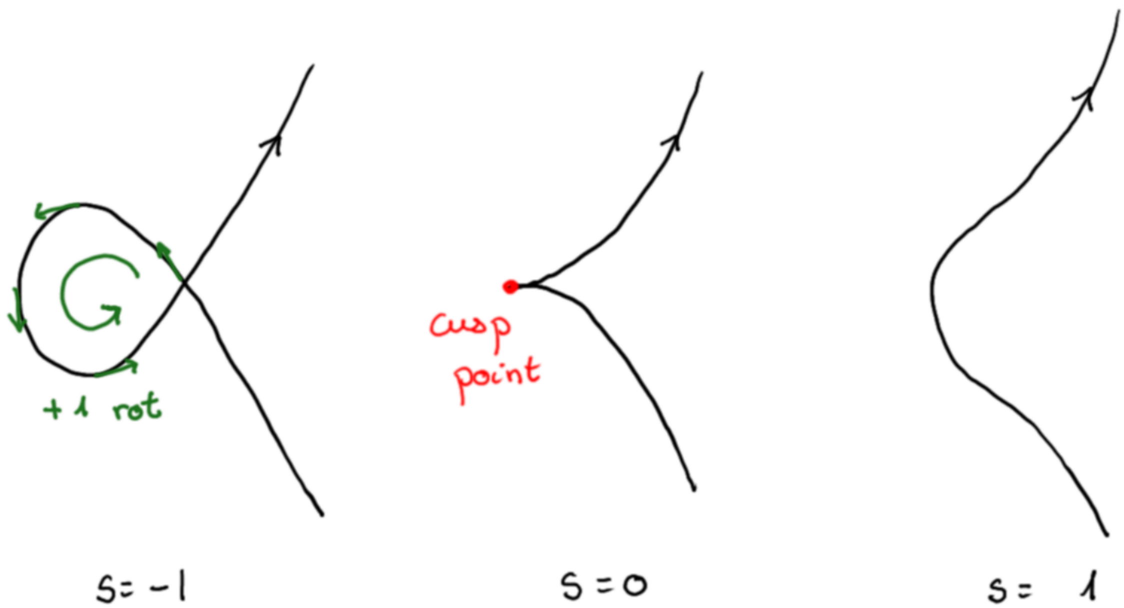

FIGURE 7. A path of curves parametrised by $s \in [-1, 1]$. All the curves shown are immersed, except the one at $s = 0$, which has a so-called **cusp**. I.e. close to the "cusp point" we can find coordinates so that the curve is given by the expression $t \mapsto (t^2, t^3)$. You may want to verify that the path of curves $(s, t) \mapsto (t^2, t^3 + st)$ consists of immersions for all $s \neq 0$. You should draw these curves and observe that the rotation of the immersion decreases by 1 as we cross $s = 0$. It turns out that cusps are the only events that can occur for a generic path of curves. We can use this reasoning to draw an schematic picture of how Imm sits inside $C^1(\mathbb{S}^1, \mathbb{R}^2)$; this is shown in Figure 8.

8. Removal of singularities

To be done.

For the reader that made it all the way here: the plan here would be to reprove the theorem of Whitney and Graustein (Theorem 42) as follows: Given immersions γ and ν with $\mathrm{rot}(\gamma) = \mathrm{rot}(\nu)$, we can use Exercise 13 to produce a path Γ of C^1 curves between them. This path will probably not consist of immersions, but at least we can apply Exercise 74 so that only finitely many of them are not. Furthermore, according to Figure 7, we know exactly the issue that these curves present: they have one cuspidal point. The proof then amounts to using the identity $\mathrm{rot}(\gamma) = \mathrm{rot}(\nu)$ to "cancel" these cusps carefully. Namely, the rotation number changes by ± 1 at each cusp, meaning that, altogether, we must be able to group the cusps into pairs in which one of the cusps increases the rotation and the other decreases it. Once they are paired up, they can be removed together.

9. Convex integration

To be done.

10. Holonomic approximation

To be done.

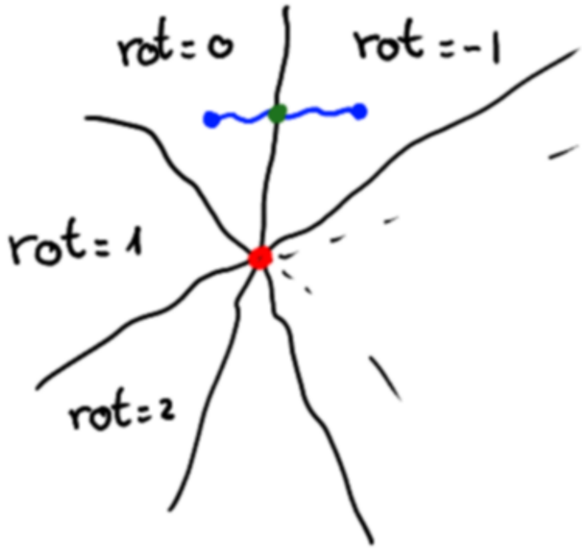

Figure 8. According to Theorem 42, Imm has $\mathbb{Z}$ components, each one corresponding to a different amount of rotation. Imm is dense in $C^1(\mathbb{S}^1, \mathbb{R}^2)$ and its components are separated by a "hypersurface" H consisting of non-immersed curves; this is shown in black. H consists mostly of curves that have a single cusp. This is the part of the hypersurface that a generic path will cross. A generic path is shown in blue, its only non-immersed curve is marked in green.
However, H also contains curves that behave worse than that. For instance, within it there is a further hypersurface consisting of those curves with two cusps. There is an even smaller subspace consisting of curves with 27 cusps. There is another subspace consisting of curves that have a single non-immersed point but which is worse than a cusp. All the way down, there is the subspace containing the constant curves, which is finite-dimensional. Using transversality it is possible to study all these different pieces.

11. Appendix

In this appendix we collect various important facts that we will use but not prove. You will (probably?) see all of them by the time you finish your bachelor studies.

11.1. **Immersions.**

Immersions can be defined more generally, in arbitrary dimension:

Definition 77. *Fix integers $n \leq m$ and a domain $\Omega \subset \mathbb{R}^n$. A C^1-map $f : \Omega \to \mathbb{R}^m$ is an* ***immersion*** *if, for every $p \in \Omega$, its differential $d_p f$ at p has trivial kernel.*

Having trivial kernel means that the n columns of $d_p f$ are linearly independent. That is, $T_p f := \text{image}(d_p f) + f(p)$ is an n-dimensional affine subspace of $\mathbb{R}^m$. We say that $T_p f$ is the **tangent space** of $f(\Omega)$ at the point $f(p)$. It is the best affine approximation to $f(\Omega)$ at $f(p)$.

In particular, we are saying that $f(\Omega)$ sits nicely within $\mathbb{R}^m$, since close to $f(p)$ it resembles an affine subspace. In fact, we can apply the implicit function theorem to deduce:

Proposition 78. *Given $p \in \Omega$, there are:*

- *a neighbourhood $U \subset \mathbb{R}^m$ of $f(p)$ and*
- *a change of coordinates $\psi : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^m$,*

such that $\psi(f(\Omega) \cap U) = \mathbb{R}^n \cap V$.

I.e. $f(\Omega)$ sits so nicely in $\mathbb{R}^m$ that, at least close to $f(p)$, we can change coordinates and make it look like the vector subspace $\mathbb{R}^n \subset \mathbb{R}^m$.

11.2. **Homotopies.** In these notes we are considering curve immersions, which are C^1 maps with non-zero derivative. Our focus has been on deformations, which we think of as "movies of immersions". A more elementary idea to consider would be a "movie" of continuous maps (more elementary since it involves no conditions on derivatives):

Definition 79. *Fix topological spaces X and Y. A* ***homotopy*** *between two maps $f, g \in C^0(X, Y)$ is a continuous map*

$$F : [0, 1] \times X \to Y$$

satisfying $f_0 = g$ and $f_1 = g$. Here $f_s := F(s, -)$. We then say that f and g are ***homotopic****.*

You may want to verify that

Exercise 80. *Being homotopic is an equivalence relation.* △

We write

$$[X, Y] = \frac{C^0(X, Y)}{\text{homotopy}}$$

for the set of equivalence classes, which are often called homotopy classes. The first result one can prove in this direction is:

Exercise 81. *Let X be a topological space. Then $[X, \mathbb{R}^n]$ has a single element.* ***Hint:*** *Every map $f : X \to \mathbb{R}^n$ is homotopic to the constant map with value 0.* △

And the following is the first non-trivial result (which appeared earlier already as Theorem 40):

Theorem 82. $[\mathbb{S}^1, \mathbb{S}^1] \equiv \mathbb{Z}$. *The same holds for $[\mathbb{S}^1, \mathbb{R}^2 \setminus \{0\}]$.*

We will not provide a proof. The idea is that each map $f : \mathbb{S}^1 \to \mathbb{S}^1$ can be assigned an integer measuring how much f wraps around the origin. Two maps are homotopic if and only if they wrap the same amount. See Figure 9. The same is true for maps $\mathbb{S}^1 \to \mathbb{R}^2 \setminus \{0\}$.

Exercise 83. *Think of $C^0(\mathbb{S}^1, \mathbb{R}^2)$ as a metric space with the C^0-distance. Verify that:*

- *The resulting topology is equivalent to the compact-open topology (if you know what that is).*
- *A homotopy of loops $\mathbb{S}^1 \to \mathbb{R}^2$ is the same as a continuous path into $C^0(\mathbb{S}^1, \mathbb{R}^2)$.*
- *Given any subspace $A \subset \mathbb{R}^2$, it holds that $[\mathbb{S}^1, A] = \pi_0(C^0(\mathbb{S}^1, A))$.* △

References

[1] H. Whitney. *On regular closed curves in the plane*. Compositio Mathematica, Volume 4 (1937), 276–284.

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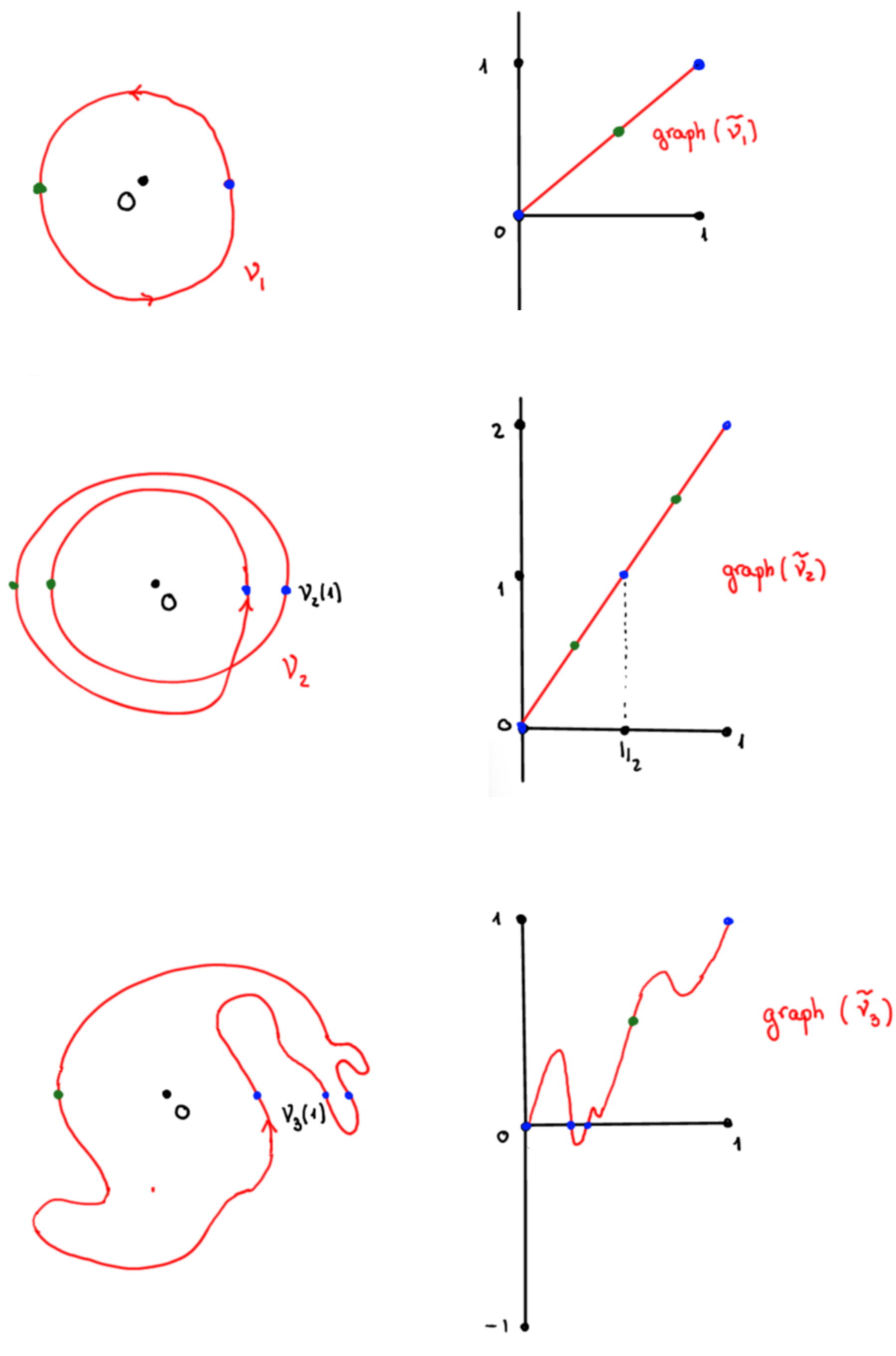

FIGURE 9. Given a map $\gamma : \mathbb{S}^1 \to \mathbb{S}^1$, we can lift it to a map $\tilde{\gamma} : [0,1] \to \mathbb{R}$. The amount that γ wraps around zero corresponds exactly to $\tilde{\gamma}(1) - \tilde{\gamma}(0)$. If two curves γ and ν rotate the same, linearly interpolating $s\tilde{\gamma} + (1-s)\tilde{\nu}$ will produce a homotopy from γ to ν. The figure shows three examples of curves γ_1 (rotating once), γ_2 (rotating once), and γ_3 (rotating once as well). We draw them with values in $\mathbb{R}^2 \setminus \{0\}$ for simplicity, but you should imagine that they are projected radially to $\mathbb{S}^1$. On the right, we see their lifts, which are precisely seen to go up once, twice, and once, respectively (i.e this is measuring exactly the wrapping around the origin).